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ZAREMBA'S BOUNDARY VALUE PROBLEM IN THE SMIRNOV CLASS OF HARMONIC FUNCTIONS IN DOMAINS WITH PIECEWISE-SMOOTH BOUNDARIES

Abstract. Zaremba's problem is studied in weighted Smirnov classes of harmonic functions in domains bounded by arbitrary simple smooth curves as well as in some domains with piecewise-smooth boundaries. The conditions of solvability are obtained and the solutions are written in quadratures.

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რეზიუმე. ზარემბას სასაზღვრო ამოცანა შესწავლილია ჰარმონიულ ფუნქციათა სმირნოვის წონიან კლასებში გლუვი წირებით შემოსაზღვრულ არეებში და აგრეთვე ზოგიერთ უბან-უბან გლუვ საზღვრიან არეებში. დადგენილია ამოხსნადობის პირობები და აგებულია ამონახსნები კვადრატურებში.

Boundary value problems for harmonic functions of two variables are well-studied under various assumptions regarding the unknown functions and the domains in which they are considered. In particular, problems are studied for harmonic functions of the class $e^p(D)$ being real parts of analytic in a simlpy connected domain D functions of the Smirnov class $E^p(D)$ (for their definition see, e.g., [1, Ch. IX–X], or [2]). In these classes the Dirichlet, Neumann and Riemann–Hilbert problems are investigeted in domains with piecewise-smooth boundaries (see, e.g., [3]–[7]). The boundary value problems are considered in some analogous classes, as well ([7]–[9]).

Of special interest is the investigation of a mixed boundary value problem of Smirnov type, when values of unknown functions are prescribed on a part L_1 of the boundary L of the domain D, while the values of its normal derivative are given on the supplementary part $L_2 = L \setminus L_1$.

S. Zaremba was the first who studied this problem ([10]) and hence in literature it frequently is called Zaremba's problem (see, e.g., [11]).

In [12] we have introduced the weighted Smirnov classes of harmonic functions $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ and investigated Zaremba's problem in the abovementioned classes when D is a bounded domain with Lyapunov boundary L, and ρ_1 and ρ_2 are power functions. The same problem has been considered in [13] for some domains with piecewise-Lyapunov boundaries. However, we did not succeed in covering the case of domains with smooth boundaries because when reducing, by means of a conformal mapping, the problem to the case of a circle, we obtain a problem in the class $e(L_{1p}(\omega_1), L'_{2q}(\omega_2))$, where ω_1 and ω_2 are not power functions, and hence the emerged Smirnov classes need further investigation.

In the present work we show that the method of investigation of Zaremba's problem suggested by us in [12] and [13] allows us to obtain a picture of solvability of the problem in domains with arbitrary smooth boundaries and also in some domains with piecewise-smooth boundaries. Towards this end, we use properties of the conformal mapping of a unit circle onto the domain with a piecewise-smooth boundary and of its derivative (see, e.g., [14] and [5, Ch. III]). On the basis of these properties we manage to show that the functions of the class $e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ for $p > 1$, $q > 1$, are representable by the Poisson integral. We also succeed in extending to the case of the emerged nonpower weights ω_1 and ω_2 some needed for investigation properties of the Smirnov class stated in [12] for power weights. Next, using Stein's interpolation theorem on weight functions ([15]) for singular integrals with Cauchy kernel, we reveal such properties of the functions ω_1 and ω_2 which allow us to solve the characteristic Cauchy singular integral equation in the weighted Lebesgue classes with the weight ω_2, rather important for investigation of Zaremba's problem.

1^0. Definitions, Notation and Some Auxiliary Statements

Let D be a simply connected domain with Jordan smooth oriented boundary L. Let $\mathcal{L}_k = [A_k, B_k]$, $k = \overline{1,n}$, be arcs lying separately on L (the points

$A_1, B_1, A_2, B_2, \dots, A_m B_m$ lie separately on L following each other in the positive direction), and let $[A_k', B_k']$ be the arcs lying on $\mathcal{L}_k$. Assume

$$L_1 = \bigcup_{k=1}^{m} \mathcal{L}_k, \quad \widetilde{L} = \bigcup_{k=1}^{m} \big([A_k, A_k'] \cup [B_k', B_k]\big), \quad L_2 = L \setminus L_1. \tag{1}$$

Let $z = z(w)$ be a conformal mapping of the circle $\cup = \{w: \ |w| < 1\}$ onto the domain D, and $w = w(z)$ be the inverse mapping. Assume

$$\Gamma_1 = w(L_1), \quad \widetilde{\Gamma} = w(\widetilde{L}), \quad \Gamma_2 = w(L_2), \quad \gamma = \{w: \ |w| = 1\}, \tag{2}$$

$$\Theta(E) = \big\{\vartheta: \ 0 \le \vartheta \le 2\pi, \ e^{i\vartheta} \in E, \ E \subset \gamma\big\},$$

$$\Gamma_j(r) = \big\{w: \ w = re^{i\vartheta}, \ \vartheta \in \Theta(\Gamma_j)\big\}, \ \ j = 1, 2, \quad L_j(r) = z(\Gamma_j(r)).$$

Let $C_1, C_2, \dots, C_{2m}$ be the points $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m$ taken arbitrarily, and $D_1, D_2, \dots, D_n$ be points on $L \setminus \widetilde{L}$, different from C_k. Note that the points $D_1, D_2, \dots, D_{n_1}$ lie on L_1 and the points $D_{n_1+1}, \dots, D_n$ lie on L_2.

Let p and q be numbers from the interval $(1, \infty)$, and we assume that

$$\rho_1(z) = \prod_{k=1}^{n_1} (z - D_k)^{\alpha_k}, \ \ -\frac{1}{p} < \alpha_k < \frac{1}{p'}\,, \ \ p' = \frac{p}{p-1}\,, \tag{3}$$

$$\rho_2(z) = \prod_{k=1}^{m_1} (z - C_k)^{\nu_k} \prod_{k=m_1+1}^{2m} (z - C_k)^{\lambda_k} \prod_{k=n_1+1}^{n} (z - D_k)^{\beta_k}, \tag{4}$$

$$-\frac{1}{q} < \nu_k \le 0, \ \ 0 \le \lambda_k < \frac{1}{q'}\,, \ \ -\frac{1}{q} < \beta_k < \frac{1}{q'}\,, \ \ q' = \frac{q}{q-1}\,.$$

Definition 1 ([12])**.** Let $r_1(z)$, $r_2(z)$ be analytic functions given in D. We say that the function $u(z)$, $z = x + iy$, harmonic in the domain D, belongs to the class $e(L_{1p}(r_1), L_{2q}'(r_2))$, if

$$\sup_r \Bigg[\int\limits_{L_1(r)} \big|u(z) r_1(z)\big|^p \, |dz| + \int\limits_{L_2(r)} \Big(\Big|\frac{\partial u}{\partial x}\Big|^q + \frac{\partial u}{\partial y}\Big|^q\Big) |r_2(z)|^q \, |dz|\Bigg] < \infty. \tag{5}$$

Assume $e(L_{1p}, L_{2q}'(r_2)) \equiv e(L_{1p}(1), L_{2q}'(r_2))$. If $L = L_1 = \gamma = \gamma_1$, then the class $e(\gamma_{1p}(1))$ coincides with the class of harmonic functions h_p. For $p > 1$, the functions of that class are representable by the Poisson integral (see, e.g., [1, Ch. IX]).

Definition 2. Let E be a finite union of closed intervals lying on the real straight line. By $A(E)$ we denote the set of functions $f(t)$ absolutely continuous on E, that is, the functions f for which for an arbitrary $\varepsilon > 0$ there is a number $\tau > 0$ such that if $\cup(\alpha_k, \beta_k)$ is an arbitrary finite union of nonintersecting intervals from E such that $\sum(\beta_k - \alpha_k) < \delta$, then the inequality $\sum |f(\beta_k) - f(\alpha_k)| < \varepsilon$ is fulfilled.

If $f(z)$ is a function defined on the subset E of the curve L and $z = z(s)$ is the equation of the curve L with respect to the arc coordinate, then we say that $f(z)$ is absolutely continuous on E and write $f \in A(E)$, if the function $f(z(s))$ is absolutely continuous on the set $\{s:\ z(s) \in E\}$.

Statement 1 ([12, Lemma 9]). *If $f(t) \in A(L_2 \cup \widetilde{L})$, then the function $f(z(\tau))$, where $z(\tau)$ is the restriction on γ of the conformal mapping of $\overline{\cup}$ onto $\overline{D}$, belongs to $A(\Gamma_2 \cup \widetilde{\Gamma})$, and vice versa, if $\varphi \in A(\Gamma_2 \cup \widetilde{\Gamma})$, then $\varphi(w(t)) \in A(L_2 \cup \widetilde{L})$.*

Statement 2 ([12, Lemma 8]). *If $U(z) = U(x,y)$ belongs to the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$, then the function $u(w) = U(z(w)) = U(x(\xi,\eta), y(\xi,\eta))$ belongs to the class $e\big(\Gamma_{1p}(\rho_1(z(w))\sqrt[p]{z'(w)}), \Gamma'_{2q}(\rho_2(z(w))\sqrt[q]{z'(w)})\big)$.*

Thus by means of substitution $z = z(w)$, where $z = z(w)$ is the conformal mapping of $\cup$ onto D, the function $U(z)$ of the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ transforms into the function $u(w)$ of the class $e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, where

$$\omega_1(w) = \rho_1(z(w))\sqrt[p]{z'(w)}, \tag{6}$$

$$\omega_2(w) = \rho_2(z(w))\sqrt[q]{z'(w)}. \tag{7}$$

2^0. Formulation of a Mixed Problem and Its Reduction to a Problem in the Circle

Consider the following mixed problem (Zaremba's problem in Smirnov class of harmonic functions): Find a function $U(z)$ satisfying the conditions

$$\begin{cases} \Delta U = 0, \ \ U \in e(L_{1p}(\rho_1), L'_{2q}(\rho_2)), \ \ p > 1, \ \ q > 1, \\ U^+\big|_{L_1 \setminus \widetilde{L}} = F, \ \ F \in L^p(L_1 \setminus \widetilde{L}, \rho_1), \ \ U^+ \in A(L_2 \cup \widetilde{L}), \\ U^+\big|_{\widetilde{L}} = \Psi, \ \ \Psi' \in L^q(\widetilde{L}, \rho_2), \ \ \Big(\dfrac{\partial U}{\partial n}\Big)^+\Big|_{L_2} = G, \ \ G \in L^q(L_2, \rho_2). \end{cases} \tag{8}$$

Relying on Statements 1 and 2, the following theorem is valid.

Theorem 1. *Let ρ_1, ρ_2, ω_1, ω_2 be the functions given by the equalities* (3)–(4) *and* (6)–(7).

If $U = U(z)$ is a solution of the problem (8) *and*

$$f(\tau) = F(z(\tau)), \quad \psi(\tau) = \Psi(z(\tau)), \quad g(\tau) = G(z(\tau)), \tag{9}$$

then the function $u(w) = U(z(w))$ is a solution of the problem

$$\begin{cases} \Delta u = 0, \ \ u \in e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2)), \\ u^+\big|_{\Gamma_1 \setminus \widetilde{\Gamma}} = f, \ \ f \in L^p(\Gamma_1 \setminus \widetilde{\Gamma}, \omega_1), \ \ u^+ \in A(\Gamma_2 \cup \widetilde{\Gamma}), \\ u^+\big|_{\widetilde{\Gamma}} = \psi, \ \ \psi' \in L^q(\widetilde{L}, \rho_2), \ \ \Big(\dfrac{\partial u}{\partial u}\Big)^+\Big|_{\Gamma_2} = g, \ \ g \in L^q(\Gamma_2; \omega_2). \end{cases} \tag{10}$$

Conversely, if $u = u(w)$ is a solution of the problem (10), *then $U(z) = u(w(z))$ is a solution of the problem* (8).

3^0. The Weight Properties of the Functions ω_1 and ω_2

By $W^p(\Gamma)$ we denote the set of all functions $r(t)$ given on the set Γ which is a finite union of simple rectifiable curves for which the operator

$$T: f \to Tf, \quad (Tf)(t) = \frac{r(t)}{\pi i} \int\limits_{\Gamma} \frac{1}{r(\zeta)} \frac{f(\zeta)}{\zeta - t}\, d\zeta, \quad t \in \Gamma,$$

is bounded in $L^p(\Gamma)$. Assume that $W^p = W^p(\gamma)$. Obviously, if Γ is a finite union of nonintersecting closed arcs on Γ and $r \in W^p$, then the restriction on Γ of the functions r (i.e., $\chi_\Gamma(t) r(t)$) belongs to $W^p(\Gamma)$.

We will need the following results.

Statement 3 (see, e.g., [5, p. 104]). *If $G(t)$ is a continuous on γ function such that $[\operatorname{ind} G]_\gamma = \frac{1}{2\pi} [\arg G(t)]_\gamma = 0$, then the function*

$$r(\tau) = \exp \left\{ \frac{1}{2\pi i} \int\limits_{\gamma} \frac{\ln G(\zeta)}{\zeta - \tau}\, d\zeta \right\}, \quad \tau \in \gamma, \tag{11}$$

belongs to the set $\bigcap\limits_{\delta>1} W^\delta$.

Corollary 1. *For any real number a we have*

$$r^a(\tau) \in \bigcap_{\delta>1} W^\delta. \tag{12}$$

Corollary 2. *If μ is a real continuous on γ function, then*

$$\exp \left\{ \frac{1}{2\pi} \int\limits_{\gamma} \frac{\mu(\zeta)}{\zeta - \tau}\, d\zeta \right\} = r(\tau) \in \bigcap_{\delta>1} W^\delta. \tag{13}$$

Statement 4. *If the domain D is bounded by a simple closed smooth curve L and $z(w)$ is a conformal mapping of U onto D, then:*

(a) *$z'(\tau) \in \bigcap\limits_{\delta>1} W^\delta$, and $[z'(w)]^{\pm 1} \in \bigcap\limits_{\delta>1} H^\delta$, where H^δ is the class of Hardy;*

(b) *if $c \in \gamma$, then $z(w) - z(c) = (w-c) z_c(w)$, $[z_c(w)]^{\pm 1} \in \bigcap\limits_{\delta>1} H^\delta$ and*

$$z(\tau) - z(c) = (\tau - c) z_c(\tau), \quad \textit{where } z_c(\tau) \in \bigcap_{\delta>1} W^\delta. \tag{14}$$

Statements (a) and (b) are particular cases of theorems stated in [14] (see also [5, Ch. III]). In particular, Statement (a) can be found in [5, Ch. III, Theorem 1.1, Corollary 1], and Statement (b) is also therein, Ch. III, Theorem 3.1. In this connection, as it follows from the proofs, both $z'(\tau)$ and $z_c(\tau)$ are representable by equalities of the type (11) (see, respectively, [5, p. 139, the equality (1.14) and p. 154, the equalities (3.16) and (3.18)]).

By virtue of Corollaries 1 and 2 of Statement 3, for any $a \in \mathbb{R}$ we have

$$[z'(\tau)]^a,\ [z_c(\tau)]^a,\ z_0(\tau) = \prod_{k=1}^{n} [z_{c_k}(\tau)]^a \in \bigcap_{\delta>1} W^\delta, \tag{15}$$
$$c_k \in \gamma, \ \ c_j \neq c_k, \ \ j \neq k.$$

Consequently, we also have

$$[\sqrt[p]{z'(\tau)}]^a,\ [\sqrt[q]{z'(\tau)}]^a \in \bigcap_{\delta>1} W^\delta. \tag{16}$$

Theorem 2. *If the functions* ρ_1 *and* ρ_2 *are given by the equalities* (3) *and* (4)*, then the functions* ω_1 *and* ω_2 *defined by the equalities* (6) *and* (7) *belong, respectively, to* W^p *and* W^q*.*

Proof. We have

$$\rho_1(z(\tau)) = \prod_{k=1}^{n} \big(z(\tau) - z(d_k)\big)^{\alpha_k},$$

where $d_k = w(D_k)$, $-\frac{1}{p} < \alpha_k < \frac{1}{p'}$. From the equalities (14) it follows that

$$\rho_1(z(\tau)) = \prod_{k=1}^{n_1} (\tau - d_k)^{\alpha_k} \prod_{k=1}^{n_1} z_{d_k}(\tau) = r_1(\tau) r_2(\tau).$$

By means of the above assumptions regarding α_k, we can find numbers a, $b \in (0,1)$ such that

$$r_1^{\frac{1}{(1-a)(1-b)}} = \left[\prod_{k=1}^{n_1} (z - d_k)^{\alpha_k}\right]^{\frac{1}{(1-a)(1-b)}} \in W^p.$$

Moreover, by virtue of (15) we have $r_2^{\frac{1}{a(1-b)}} \in \bigcap\limits_{\delta>1} W^\delta$.

Here we use the following Stein's theorem ([15]).

Let M be a linear operator acting from one space of measurable functions to the other,

$$1 \le l_1, l_2, s_1, s_2 \le \infty, \quad l^{-1} = (1-a) l_1^{-1} + a l_2^{-1},$$
$$s^{-1} = (1-a) s_1^{-1} + a s_2^{-1}, \ \ 0 \le a \le 1,$$
$$\big\|(Mf)k_i\big\|_{s_i} \le C_i \|f u_i\|_{l_i}.$$

Then

$$\|(Mf)k\|_s \le C \|fu\|_l,$$

where

$$k = k_1^{1-a} k_2^a, \quad u = u_1^{1-a} u_2^a, \quad C = C_1^{1-a} C_2^a.$$

Assuming in this theorem

$$k_1(\tau) = u_1(\tau) = \left[\prod_{k=1}^{n_1} (\tau - d_k)^{\alpha_k}\right]^{\frac{1}{(1-a)(1-b)}}, \ \ k_2(\tau) = u_2(\tau) = \prod_{k=1}^{n_1} [z_{d_k}(\tau)]^{\frac{1}{a(1-b)}},$$
$$s_1 = s_2 = s = p > 1,$$

we find that the function

$$r_1^{1-a} r_2^a = \Big[\prod_{k=1}^{n_1} (\tau - d_k)^{\alpha_k} \prod_{k=1}^{n_1} z_{d_k}\Big]^{\frac{1}{1-b}} \Big(= \rho_1^{\frac{1}{1-b}}(z(\tau))\Big)$$

belongs to W^p.

Further, taking in the above theorem

$$k_1(\tau) = u_1(\tau) = \rho_1^{\frac{1}{1-b}}(z(\tau)), \quad k_2(\tau) = u_2(\tau) = \big(\sqrt[p]{z'}\big)^{\frac{1}{b}}, \quad s_1 = s_2 = s = p,$$

we find that

$$\Big(\big[\rho_1(z(\tau))\big]^{\frac{1}{1-b}}\Big)^{1-b} \Big(\big[\sqrt[p]{z'}\big]^{\frac{1}{b}}\Big)^b = \rho_1(z(\tau))\sqrt[p]{z'(\tau)} = \omega_1(\tau) \in W^p.$$

Taking into account that $-\frac{1}{q} < \beta_k < \frac{1}{q'}$, $-\frac{1}{q} < \nu_k \le 0$, $0 \le \lambda_k < \frac{1}{q'}$, we analogously see that $\omega_2(\tau) \in W^q$. □

4^0. One Property of Functions of the Class $e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ for $p > 1$, $q > 1$

Theorem 3. *If $u \in e(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, where $p > 1$, $q > 1$, then:*

(i) *if $p < q$, then $u \in h_p$;*
(ii) *if $p > q$, then $u \in h_{q_1}$ for any $q_1 \in [0, q]$;*
(iii) *if $p = q$, then $u \in h_{p_1}$ for any $p_1 \in (0, p)$.*

Proof. (i) Let

$$I(r) = \int_0^{2\pi} \big|u(re^{i\vartheta})\big|^p \, d\vartheta.$$

We have

$$\begin{aligned}
I(r) &= \int_{\Theta(\Gamma_1)} \big|u(re^{i\vartheta})\big|^p \, d\vartheta + \int_{\Theta(\Gamma_2)} \big|u(re^{i\vartheta})\big|^p \, d\vartheta \le \\
&\le \sup_r \int_{\Theta(\Gamma_1)} \big|u(re^{i\vartheta})\big|^p \, d\vartheta + \int_{\Theta(\Gamma_2)} \Big| \int_0^r \frac{\partial u}{\partial r} \, dr - u(0)\Big|^p d\vartheta \le \\
&\le M_1 + 2^p \Big(\int_{\Theta(\Gamma_2)} \Big| \int_0^r \frac{\partial u}{\partial r} \, dr\Big|^p d\vartheta + |u(0)|^p \, 2\pi \Big) = \\
&= M_2 + 2^p \int_{\Theta(\Gamma_2)} \Big(\int_0^r \Big|\frac{\partial u}{\partial r}\Big| \, dr\Big)^p d\vartheta = \\
&= M_2 + 2^p I_1(r).
\end{aligned} \tag{17}$$

Since $\left|\frac{\partial u}{\partial r}\right| \le \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right|$, we have

$$I_1(r) = \int\limits_{\Theta(\Gamma_2)} \Big|\int\limits_0^r \Big|\frac{\partial u}{\partial r}\Big|\, dr\Big|^p d\vartheta \le \int\limits_{\Theta(\Gamma_2)} \Big[\int\limits_0^r \Big|\Big(\frac{\partial u}{\partial x}\Big| + \Big|\frac{\partial u}{\partial y}\Big|\Big)\, dr\Big]^p d\vartheta =$$

$$= \int\limits_{\Theta(\Gamma_2)} \Big[\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big| + \Big|\frac{\partial u}{\partial y}\Big|\Big)|\omega_2|\,\frac{1}{|\omega_2|}\, dr\Big]^p d\vartheta \le$$

$$\le \int\limits_{\Theta(\Gamma_2)} \Big[\Big(\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big| + \Big|\frac{\partial u}{\partial y}\Big|\Big)^q |\omega_2|^q\, dr\Big)^{\frac{p}{q}} \Big(\int\limits_0^r \frac{dr}{|\omega_2|^{q'}}\Big)^{\frac{p}{q'}}\Big] d\vartheta \le$$

$$\le (2^q)^{\frac{p}{q}} \int\limits_{\Theta(\Gamma_2)} \Big(\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big|^q + \Big|\frac{\partial u}{\partial y}\Big|^q\Big)|\omega_2|^q\, dr\Big)^{\frac{p}{q}} \Big(\int\limits_0^r \frac{dr}{|\omega_2|^{q'}}\Big)^{\frac{p}{q'}} d\vartheta =$$

$$= 2^p \int\limits_{\Theta(\Gamma_2)} \Big(\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big|^q + \Big|\frac{\partial u}{\partial y}\Big|^q\Big)|\omega_2|^q\, dr\Big)^{\frac{p}{q}} (J(\vartheta))^{\frac{p}{q'}}\, d\vartheta, \tag{18}$$

where we have put

$$J(\vartheta) = \int\limits_0^r \frac{dr}{|\omega_2(re^{i\vartheta})|^{q'}}\,.$$

Estimate the value $\sup\limits_{\Theta(\Gamma_2)} J(\vartheta)$. We have

$$\frac{1}{|\omega_2(re^{i\vartheta})|^{q'}} \le \frac{M_3}{\prod\limits_{k=m_1+1}^{2m} |re^{i\vartheta} - c_k|^{\lambda_k q'} \prod\limits_{\beta_k>0} |re^{i\vartheta} - d_k|^{\beta_k q'} |z_0(re^{i\vartheta})|^{q'}}$$

(for definition of z_0, see (15)).

Assume that $\alpha = \sup\limits_k(\lambda_k q', \beta_k q')$. By virtue of the inequalities (7), we have $0 \le \alpha < 1$. Since $|re^{i\vartheta} - c_k| \ge 1 - r$, $|re^{i\vartheta} - d_k| \ge 1 - r$, $\Theta(\Gamma_2) = \bigcup_{k=1}^m (\beta_k, \alpha_{k+1})$ with $\alpha_{m+1} = \alpha_1$, and on every interval (β_k, α_{k+1}) there are no more than three points from the set $\cup\{z_k\} = \cup\{c_k\}\cup\cup\{d_k\}$, then dividing the corresponding intervals into three or two parts, we will represent $\Theta(\Gamma_2)$ as the union of no more than $6m$ intervals, and on every interval

$$\frac{1}{|\omega_2(re^{i\vartheta})|^{q'}} \le \frac{M_4}{(1-r)^\alpha |z_0(re^{i\vartheta})|^{q'}}\,, \quad M_4 = \max_{k\ne j} \frac{3}{|z_k - z_j|}\,.$$

Thus

$$\sup_{\Theta(\Gamma_2)} J(\vartheta) \le (6m) M_4 \sup_{\Theta(\Gamma_2)} \int\limits_0^r \frac{dr}{(1-r)^\alpha |z_0(re^{i\vartheta})|^{q'}} =$$

$$= M_5 \sup_{\Theta(\Gamma_2)} \int_0^r \frac{dr}{(1-r)^\alpha |z_0(re^{i\vartheta})|^{q'}}. \tag{19}$$

Applying in the last integral Hölder's inequality with exponent $\frac{1+\alpha}{2\alpha}$, we obtain

$$(J(\vartheta))^{\frac{p}{q'}} \le M_6 \Bigg(\int_0^r \frac{dr}{(1-r)^{\frac{1+\alpha}{2}}} \Bigg)^{\frac{p}{q'}\,\frac{2\alpha}{1+\alpha}} \Bigg(\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{q'\,\frac{1+\alpha}{1-\alpha}}} \Bigg)^{\frac{p}{q'}\,\frac{1-\alpha}{1+\alpha}} \le$$
$$\le M_7 \Bigg(\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{q'\,\frac{1+\alpha}{1-\alpha}}} \Bigg)^{\frac{p}{q'}\,\frac{1-\alpha}{1+\alpha}}. \tag{20}$$

Show that the integral

$$J_1(\vartheta) = \Bigg(\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{q'\,\frac{1+\alpha}{1-\alpha}}} \Bigg)^{\frac{p}{q'}\,\frac{1-\alpha}{1+\alpha}}$$

is a function integrable in any degree on γ and hence on $\Theta(\Gamma_2)$.

Towards this end, we note that if $\frac{p}{q'}\,\frac{1-\alpha}{1+\alpha} \le 1$, then

$$J_1(\vartheta) \le 1 + \int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{q'\,\frac{1+\alpha}{1-\alpha}}}.$$

If, however, $\frac{p}{q'}\,\frac{1-\alpha}{1+\alpha} > 1$, then using Hölder's inequality with the above exponent, we have

$$J_1(\vartheta) \le \int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{p}}.$$

From the above estimates we can see that $J_1(\vartheta) \in \bigcap_{\delta>1} L^\delta([0,2\pi])$ if we prove that for arbitrary $\delta > 1$ the function $\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^\mu}$ is integrable in the δ-th degree for any μ.

We have

$$\int_0^{2\pi} \Bigg(\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^\mu} \Bigg)^\delta d\vartheta \le \int_0^{2\pi}\int_0^r \frac{dr}{|z_0(re^{i\vartheta})|^{\mu\delta}}\, d\vartheta \le$$
$$\le \int_0^1\int_0^{2\pi} \frac{d\vartheta}{|z_0(re^{i\vartheta})|^{\mu\delta}}\, dr = M_8 < \infty.$$

This inequality is valid since $\frac{1}{z_0} \in \bigcap_{\delta>1} H^\delta$ (see Statement 4). Thus we have proved that $J(\vartheta) \in \bigcap_{\delta>1} L^\delta[0,2\pi]$.

Applying now to the right-hand side of (18) Hölder's inequality with exponent $\frac{q}{p} > 1$, we obtain

$$I_1(r) \le 2^p \int\limits_{\Theta(\Gamma_2)} \Big[\int\limits_0^r \Big(\Big|\frac{\partial u}{\partial x}\Big|^q + \Big|\frac{\partial u}{\partial y}\Big|^q \Big) |\omega_2|^q \, dr \Big] \, d\vartheta \Big(\int\limits_{\Theta(\Gamma_2)} |J(\vartheta)|^{\frac{p}{q'}\,\frac{q}{q-p}} \Big)^{\frac{q-p}{q}}.$$

But $u \in e(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, whence it follows that $\sup\limits_r I_1(r) < \infty$, and from (17) we can conclude that $\sup\limits_r I(r) < \infty$ and hence $u \in h_p$.

(ii) It can be easily verified that if $p_1 < p_2$, then $u \in e(\Gamma_{1p_2}, \Gamma'_{2q}(\omega_2)) \subset u \in e(\Gamma_{1p_1}, \Gamma'_{2q}(\omega_2))$. Therefore if $p > q > q_1$ and $u \in e(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, then $u \in e(\Gamma_{1q_1}, \Gamma'_{2q}(\omega_2))$ and $u \in h_{q_1}$.

(iii) If $u \in e(\Gamma_{1p}, \Gamma'_{2p}(\omega_2))$, then for any $1 < p_1 < p$, we have $u \in e(\Gamma_{1p_1}, \Gamma'_{2q}(\omega_2))$, and hence $u \in h_{p_1}$. □

Let now $u \in e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$. Since $\frac{1}{\omega_1} \in H^{p'+\varepsilon}$, $\varepsilon > 0$, there exists $\eta > 0$ such that $u \in e(L_{1+\eta}, L'_{2q}(\omega_2))$, and therefore $\sup\limits_r \int\limits_{\Theta(\Gamma_1)} |u(re^{i\vartheta})|^{1+\eta} \, d\vartheta < \infty$. Assuming $1 + \eta < q$, by Theorem 3 we can conclude that $u \in h_{1+\eta}$. As far as the functions of the class $h_{1+\eta}$ are representable by the Poisson integral, we state the following

Theorem 4. *If $u \in e(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$, then u is likewise representable by the Poisson integral.*

5^0. Reduction of the Problem (10) to a Singular Integral Equation

If $w(A_k) = a_k$, $w(B_k) = b_k$, $w(A'_k) = a'_k$, $w(B'_k) = b'_k$, we have

$$\Gamma_1 = w(L_1) = \bigcup_{k=1}^m (a_k, b_k), \quad \widetilde{\Gamma} = \bigcup [a_k, a'_k] \cup [b'_k, b_k], \quad \Gamma_2 = \gamma \setminus \Gamma_1.$$

Following the way of investigation of the problem (10) carried out in Section 3^0 of [12], we can state that if u is a solution of the problem (10) and $u^+(e^{i\vartheta})$ is its boundary function, then the function $\frac{\partial u^+}{\partial \vartheta}$ is a solution of the integral equation

$$\frac{1}{2\pi} \int\limits_{\Theta(\Gamma_2)} \frac{\partial u^+}{\partial \vartheta} \operatorname{ctg} \frac{\vartheta - \varphi}{2} \, d\vartheta = \mu(\varphi), \quad e^{i\varphi} \in \Gamma_2, \tag{21}$$

where

$$\mu(\varphi) = -g(\varphi) - \frac{1}{2\pi} \int\limits_{\Theta(\Gamma_1 \setminus \widetilde{\Gamma})} f(\vartheta) \frac{d\vartheta}{\sin^2 \frac{\vartheta - \varphi}{2}} - \frac{1}{2\pi} \int\limits_{\Theta(\Gamma_2)} \psi(\vartheta) \frac{d\vartheta}{\sin^2 \frac{\vartheta - \varphi}{2}} +$$

$$+ \sum_{k=1}^m \Big[\psi(a'_k) \operatorname{ctg} \frac{\alpha'_k - \varphi}{2} - \psi(b'_k) \operatorname{ctg} \frac{\beta'_k - \varphi}{2} \Big] - \widetilde{\psi}(\varphi), \tag{22}$$

$$\widetilde{\psi}(\varphi)=\frac{1}{2\pi}\int\limits_{\gamma}\chi_{\widetilde{\Gamma}}(\vartheta)\,\frac{\partial\psi}{\partial\vartheta}\,\mathrm{ctg}\,\frac{\vartheta-\varphi}{2}\,d\vartheta. \tag{23}$$

Here $\chi_{\widetilde{\Gamma}}$ is the characteristic function of the set $\widetilde{\Gamma}$, we write $f(\vartheta)$, $\psi(\vartheta)$, $g(\varphi)$ instead of $f(e^{i\vartheta})$, $\psi(e^{i\vartheta})$, $g(e^{i\varphi})$ and put $a'_k=e^{i\alpha_k}$, $b'_k=e^{i\beta'_k}$.

Let us show that under the adopted assumptions the functions $\frac{\partial u^+}{\partial\vartheta}$ and μ belong to the class $L^q(\Gamma_2;\omega_2)$.

We start with the function $\frac{\partial u^+}{\partial\vartheta}$. Tracing the proof of Lemma 1 in [12], we establish that the condition $u\in e(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2))$ is equivalent to the condition

$$\sup_r\Bigg[\int\limits_{\Theta(\Gamma_1)}\big|u(re^{i\vartheta})\omega_1(re^{i\vartheta})\big|^p\,d\vartheta+ \\ +\int\limits_{\Theta(\Gamma_2)}\bigg|\sqrt{\Big(\frac{\partial u}{\partial x}\,(re^{i\vartheta})\Big)^2+\Big(\frac{\partial u}{\partial y}\,(re^{i\vartheta})\Big)^2}\,\omega_2(r^{i\vartheta})\bigg|^q\,d\vartheta\Bigg]<\infty \tag{24}$$

for the functions ω_1, ω_2, as well (and not only for the power functions). It is now not difficult to see that the statement below is valid.

Statement 5. *If $u\in e(\Gamma_{1p}(\omega_1),\Gamma'_{2q}(\omega_2))$, $p>1$, $q>1$, and $u^+\in A(\Gamma_2)$ (in particular, if u is a solution of the problem* (10)*), then $\big(\frac{\partial u}{\partial\vartheta}\big)^+$ and $\frac{\partial u^+}{\partial\vartheta}$ belong to $L^q(\Gamma_2;\omega_2)$.*

The proof of the above statement is analogous to that of Lemma 5 in [12] if in the appropriate place we take advantage of the fact that the condition (5) is equivalent to the condition (24).

For the function μ to belong to $L^q(\Gamma_2;\omega_2)$, as is seen from the equality (22), it suffices to show that $\widetilde{\psi}\in L^q(\Gamma_2;\omega_2)$. This follows from Theorem 2 since $\lambda(\vartheta)=\chi_{\widetilde{\Gamma}}(\vartheta)\,\frac{\partial\psi}{\partial\vartheta}\in L^q(\gamma,\omega_2)$ (because $\frac{\partial\psi}{\partial\vartheta}\in L^q(\Gamma_2;\omega_2)$), while the operator

$$\lambda\to\widetilde{\lambda},\quad \widetilde{\lambda}(\varphi)=\frac{1}{\pi}\int\limits_0^{2\pi}\lambda(\vartheta)\,\mathrm{ctg}\,\frac{\vartheta-\varphi}{2}\,d\vartheta$$

is bounded in $L^q(\gamma,\omega_2)$ if the singular Cauchy operator is bounded in it, and latter is bounded in $L^q(\gamma,\omega_2)$ since $\omega_2\in W^q$ by Theorem 2.

6^0. The Solution of the Equation (21) in the Space $L^q(\Gamma_2;\omega_2)$

Assuming $\tau=e^{i\vartheta}$, $t=e^{i\varphi}$ and taking into account that

$$\frac{d\tau}{\tau-t}=\Big(\frac{1}{2}\,\mathrm{ctg}\,\frac{\vartheta-\varphi}{2}+\frac{i}{2}\Big)\,d\vartheta,$$

the equation (21) can be written in the form

$$\frac{1}{\pi i}\int\limits_{\Gamma_2}\frac{\partial u^+}{\partial\vartheta}\,\frac{d\tau}{\tau-e^{i\varphi}}=i\mu(\varphi)+a,\quad a=\frac{1}{2\pi}\int\limits_{\Gamma_2}\frac{\partial u^+}{\partial\vartheta}\,d\vartheta. \tag{25}$$

Since u^+ is a boundary value of a solution of the problem (10), we see

$$a=\frac{1}{2\pi}\sum_{k=1}^{m}\big(u(a_{k+1})-u(b_k)\big)=\frac{1}{2\pi}\sum_{k=1}^{m}\big[\psi(a_{k+1})-\psi(b_k)\big],\quad a_{m+1}=a_1.$$

Thus the function $\frac{\partial u^+}{\partial\vartheta}$ is a solution of the singular integral equation

$$\frac{1}{\pi i}\int\limits_{\Gamma_2}\frac{\partial u^+}{\partial\vartheta}\,\frac{d\tau}{\tau-e^{i\varphi}}=i\mu(\varphi)+\frac{1}{2\pi}\sum_{k=1}^{m}\big[\psi(a_{k+1})-\psi(b_k)\big] \tag{26}$$

belonging to $L^q(\Gamma_2;\omega_2)$.

Let Γ be a finite union of arcs $[a_k,b_k]\subset\gamma$, ρ be a weight function from W^q and

$$S_\Gamma:\varphi\to S_\Gamma\varphi,\quad (S_\Gamma\varphi)(t)=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\tau)}{\tau-t}\,d\tau\,,\quad t\in\Gamma.$$

Let $\lambda\in L^q(\Gamma;\rho)$. Consider the singular integral equation

$$S_\Gamma\varphi=\lambda \tag{27}$$

in the class $L^q(\Gamma;\rho)$.

This equation has been investigated in different classes of functions by many authors. In our formulation, when ρ is a power function of definite type, it is solved in [16] (see also [17, Ch. III, § 7, pp. 103–109]; a history of the question can be found therein). In connection with investigation of Zaremba's problem, in [12] we showed that this result from [17] was generalized to a general case of power weight functions. We will now show that the property of solvability of the equation (27) in the classes $L^q(\Gamma;\rho)$ for power weights preserves for wider classes of weights, as well.

The points $a_1a_2,\dots,a_m;b_1,b_2,\dots,b_m$ taken arbitrarily are denoted below by $c_1,c_2,\dots,c_{2m}$. Let

$$\Pi_1(z)=\sqrt{\prod_{k=1}^{m_1}(z-c_k)},\quad \Pi_2(z)=\sqrt{\prod_{k=m_1+1}^{2m}(z-c_k)},\quad R(z)=\frac{\Pi_1(z)}{\Pi_2(z)}\,,$$

where the branch of the first function is taken arbitrarily and that of the second one is selected in such a way that the function $R(z)$ in the neighborhood of $z=\infty$ expands in the series $z^{m-m_1}+A_1z^{m-m_1-1}+\cdots$.

Assume

$$R(\tau)=\frac{\Pi_1(\tau)}{\Pi_2(\tau)}\,,\quad \tau\in\Gamma. \tag{28}$$

Let

$$\rho(\tau)=\prod_{k=1}^{m_1}(\tau-c_k)^{\nu_k}\prod_{k=m_1+1}^{2m}(\tau-c_k)^{\lambda_k},\ -\frac{1}{q}<\nu_k\le 0,\ \ 0\le\lambda_k<\frac{1}{q'}\,. \quad (29)$$

Moreover, we assume that $-\frac{1}{q}<\frac{1}{2}+\nu_k<\frac{1}{q'}$, $-\frac{1}{q}<\lambda_k-\frac{1}{2}<\frac{1}{q'}$, i.e.,

$$-\frac{1}{q}<\nu_k<\min\Big(0;\frac{1}{q'}-\frac{1}{2}\Big),\quad \max\Big(0;\frac{1}{2}-\frac{1}{q}\Big)\le\lambda_k<\frac{1}{q'}\,. \quad (30)$$

Finally, let

$$\omega(\tau)=\rho(z(\tau))\rho_0(\tau), \quad (31)$$

where $\rho_0\in\bigcap\limits_{\delta>1}W^\delta$.

Suppose

$$U_\Gamma\varphi=RS_\Gamma\,\frac{1}{R}\,\varphi. \quad (32)$$

If $\varphi(\tau)\equiv q(\tau)$ is an arbitrary polynomial, then $q\in L^p(\Gamma;\Pi_1^{-1}\Pi_2^{p-1})$, and by Lemma 1 of [17, p. 105] we obtain

$$(U_\Gamma S_\Gamma q)(\tau)=q(\tau),\ \text{ when }\ m_1\ge m. \quad (33)$$

However, if $m>m_1$, then

$$(U_\Gamma S_\Gamma q)(\tau)=q(\tau)+R(\tau)Q_{r-1}(\tau), \quad (34)$$

where $Q_{r-1}(\tau)$ is a polynomial of degree not higher than $r-1$, $r=m-m_1-1$.

Since $\omega(\tau)=\rho(z(\tau))\rho_0(\tau)$, according to Theorem 2 $\omega\in W^q$. Moreover, since the conditions (30) are fulfilled, the function $\widetilde{\omega}(\tau)=R(\tau)\omega(\tau)$ belongs to W^q. Since the set of polynomials $\{q_n\}$ is dense in $L^q(\Gamma;\widetilde{\omega})$ for any $\widetilde{\omega}\in W^q$, passing in the equalities (33) and (34) to the limit as $q=q_n\to\varphi\in L^q(\Gamma;\omega)$, we find that

$$(U_\Gamma S_\Gamma\varphi)=\varphi\ \text{ for }\ m\le m_1,\ \text{ and }\ (U_\Gamma S\varphi)=\varphi+RQ_{r-1}\ \text{ for }\ m>m_1. \quad (35)$$

On the basis of the above equalities, just as in [17, pp. 107–108] (see also [12, p. 46]) we prove

Theorem 5. *Let for the weight ρ given by the equality* (29) *the conditions* (30) *be fulfilled, and $\omega(\tau)=\rho(z(\tau))\sqrt[q]{z'(\tau)}$, where $z=z(w)$ is a conformal mapping of the circle $\cup$ onto a simply connected domain bounded by a simple closed smooth curve L, and let Γ be a finite union of arcs from γ. Then the equation*

$$S_\Gamma\varphi=\lambda$$

(i) *is solvable for $m_1\le m$ and all its solutions are given by the equality*

$$\varphi(\tau)=(U_\Gamma\lambda)(\tau)+RQ_{r-1}(\tau), \quad (36)$$

where $Q_{r-1}(\tau)$ is an arbitrary polynomial of order $r-1$, $r=m-m_1$ $(Q_{-1}(\tau)\equiv 0)$.

(ii) *for $m_1 > m$, the equality is solvable if and only if*

$$\int_{\Gamma} \tau^k R(\tau)\lambda(\tau)\,d\tau = 0, \quad k = \overline{0, l-1}, \quad l = m_1 - m, \tag{37}$$

and if these conditions are fulfilled it is uniquely solvable and the solution is given by the equality (36), *where $Q_{r-1}(\tau) \equiv 0$.*

7^0. The Solution of the Problem (8)

Having at hand Theorem 5, we are able to investigate the equation (21): find the conditions of its solvability and write out all solutions. By virtue of the same theorem, solving the equation (26) and hence (21), we can find the function $\frac{\partial u^+}{\partial \vartheta}$ on Γ_2; integrating it, we find $u^+(\tau)$ on Γ_2. There appear arbitrary constants which (or a part of which) are defined by the conditions of absolute continuity of u^+ on $\Gamma_2 \cup \widetilde{\Gamma}$ (see (10)). Having found the values u^+ on Γ_2, we will have u^+ on the entire neighborhood, because it was given on Γ_1 beforehand. By virtue of Theorem 4, all the above-said allows us to find $u(w)$ by using the Poisson formula with density $u^+(e^{i\vartheta})$. Having known $u(w)$, by Theorem 1 we find a solution $U(z) = u(w(z))$, $z \in D$, of the problem (8).

Detailed calculations are analogous to those carried out in [12] (Sections 5^0–7^0). Omitting them, we can formulate the final result.

Theorem 6. *Let:*

(a) *the domain D, the curve L, and its parts L_1, $\widetilde{L}$, L_2 be defined according to Section 1^0 and the equalities* (1), *while the weight functions $\rho_1(z)$, $\rho_2(z)$ be defined by the conditions* (3)–(4)*;*

(b) *$z = z(w)$ be a conformal mapping of the unit circle $\cup$ onto D; $w = w(z)$ be the inverse mapping; the sets Γ_1, $\widetilde{\Gamma}$, Γ_2 be defined by* (2) *and the functions ω_1, ω_2 by the equalities* (6)–(7)*;*

(c) *$a_k = w(A_k) = e^{i\alpha_k}$, $b_k = w(B_k) = e^{i\beta_k}$, $a'_k = w(A'_k) = e^{i\alpha'_k}$, $b'_k = w(B'_k) = e^{i\beta'_k}$, $0 \le m_1 \le 2m$, $c_k = w(C_k)$, $d_k = w(D_k)$;*

(d) *the function $R(\tau)$ be defined by the equality* (28).

If the problem (8) *is considered in the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$, $p > 1$, $q > 1$, the functions f, ψ, g are defined by the equalities* (9) *and we assume that for the exponents of the weights the conditions* (30) *are fulfilled, then:*

I. *If $m_1 \le m$, then for the solvability of the problem* (8) *it is necessary and sufficient that the conditions*

$$\int_{\beta_k}^{\alpha_{k+1}} \operatorname{Re}\left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Theta(\Gamma_2)} \frac{i\mu(\tau) + a}{R(\tau)(\tau)(\tau - e^{i\alpha})}\,d\tau\right] d\alpha =$$
$$= \psi(e^{i\alpha_{k+1}}) - \psi(e^{i\beta_k}), \quad k = \overline{1, m}, \tag{38}$$

be fulfilled, where

$$\mu(\tau)=\mu(e^{i\varphi})\equiv\mu(\varphi)=$$

$$=-g(\varphi)+\frac{1}{2\pi}\sum_{k=1}^{m}\Big[\psi(e^{i\alpha_{k+1}})\operatorname{ctg}\frac{\alpha_{k+1}-\varphi}{2}-\psi(e^{i\alpha_k})\operatorname{ctg}\frac{\beta_k-\varphi}{2}\Big]-$$

$$-\frac{1}{2\pi}\int\limits_{\Theta(\Gamma\setminus\widetilde{\Gamma})} f(\vartheta)\,\frac{d\vartheta}{2\sin^2\frac{\vartheta-\varphi}{2}}-\frac{1}{2\pi}\int\limits_{\Theta(\widetilde{\Gamma})}\psi(\vartheta)\,\frac{d\vartheta}{2\sin^2\frac{\vartheta-\varphi}{2}}\,, \tag{39}$$

$$a=\frac{1}{2\pi}\sum_{k=1}^{m}\big[\psi(e^{i\alpha_{k+1}})-\psi(e^{i\beta_k})\big],\quad \alpha_{m+1}=\alpha_1. \tag{40}$$

II. *If $m_1>m$, then for the solvabilty of the problem* (8) *it is necessary and sufficient that the conditions* (38) *and*

$$\int\limits_{\Gamma_2}\frac{i\mu(\tau)+a}{R(\tau)}\,\tau^k\,d\tau=0,\quad k=\overline{0,l-1},\quad l=m_1-m, \tag{41}$$

be fulfilled.

III. *If the above conditions are fulfilled, then a solution of the problem* (8) *is given by the equality*

$$U(z)=u^*(w(z))+u_0(w(z)), \tag{42}$$

where

$$u^*(w)=u(re^{i\vartheta})=\frac{1}{2\pi}\int\limits_{\Theta(\widetilde{\Gamma})}\psi(\vartheta)P(r,\vartheta-\varphi)\,d\vartheta+$$

$$+\frac{1}{2\pi}\int\limits_{\Theta(\Gamma_1\setminus\widetilde{\Gamma})} f(\vartheta)P(r,\vartheta-\varphi)\,d\vartheta+\frac{1}{2\pi}\int\limits_{\Theta(\Gamma_2)}W_{\Gamma_2}(\vartheta)P(r,\vartheta-\varphi)\,d\vartheta \tag{43}$$

in which

$$P(r,x)=\frac{1-r^2}{1+r^2-2r\cos x}\,,$$

$$W_{\Gamma_2}(\vartheta)=\int\limits_{\beta_1}^{\vartheta}\chi_{\Theta(\Gamma_2)}(\alpha)\Big[\operatorname{Re}\,\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Gamma_2}\frac{i\mu(\tau)+a}{R(\tau)(\tau-e^{i\alpha})}\,d\tau\Big]\,d\alpha+B_k,$$

$\Theta(E)=\{\vartheta:\ e^{i\vartheta}\in E\}$, *and χ_{E} denotes characteristic function of the set E,*

$$B_k=\psi(e^{i\alpha_{k+1}})-\int\limits_{\beta_i}^{\alpha_{k+1}}\chi_{\Theta(\Gamma_2)}(\alpha)\operatorname{Re}\Big[\frac{R(e^{i\alpha})}{\pi i}\int\limits_{\Gamma_2}\frac{i\mu(\tau)+a}{R(\tau)(\tau-e^{i\alpha})}\,d\tau\Big]\,d\alpha, \tag{44}$$

and

$$u_0(re^{i\vartheta}) = \begin{cases} 0, \quad when \;\; m_1 > m, \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} W^*_{\Gamma_2}(\vartheta)P(r,\vartheta-\varphi)\,d\vartheta, \\ W^*_{\Gamma_2}(\vartheta) = \displaystyle\int_{\beta_1}^{\vartheta} \chi_{\Theta(\Gamma_2)}(\alpha)\operatorname{Re}\big[R(e^{i\alpha})Q_{r-1}(e^{i\alpha})\big]\,d\alpha + A_k, \\ \qquad\qquad\qquad e^{i\vartheta} \in (b_k, a_{k+1}), \end{cases} \tag{45}$$

$$A_k = -\int_{\beta_k}^{\alpha_{k+1}} \operatorname{Re}\big[R(e^{i\alpha})Q_{r-1}(e^{i\alpha})\big]\,d\alpha, \tag{46}$$

$Q_{r-1}(\tau) \equiv 0$, *and for* $m_1 < m$,

$$Q_{r-1}(e^{i\vartheta}) = \sum_{j=0}^{r-1}(x_j + iy_j)e^{ij\vartheta}, \tag{47}$$

where the coefficients x_j, y_j, $j = \overline{0, r-1}$, *are defined from the system*

$$\begin{cases} \displaystyle\sum_{j=0}^{r-1}\int_{\beta_k}^{\alpha_{k+1}} \Big[x_j R_1(e^{i\vartheta})\cos j\vartheta - y_j R_2(e^{i\vartheta})\sin j\vartheta\Big]\,d\vartheta = 0, \\ \displaystyle\sum_{j=0}^{r-1}\int_{\beta_k}^{\alpha_{k+1}} \Big[x_j R_2(e^{i\vartheta})\cos j\vartheta + y_j R_1(e^{i\vartheta})\sin j\vartheta\Big]\,d\vartheta = 0 \end{cases} \tag{48}$$

and we put $R_1(e^{i\vartheta}) = \operatorname{Re} R(e^{i\vartheta})$, $R_2(e^{i\vartheta}) = \operatorname{Im} R(\vartheta)$.

If the rank of the matrix composed by the coefficients of the system (48) *is equal to* ν, *then among the numbers* $x_0, x_1, \dots, x_{r-1}, y_0, y_1, \dots, y_{r-1}$ *there are* $2(m-m_1)-\nu$ *arbitrary constants, and hence the general solution of the problem* (8) *contains* $2(m-m_1)-\nu$ *arbitrary real parameters.*

8^0. On a Mixed Problem in Domains with Piecewise-Smooth Boundaries

In [13], the problem (8) is investigated in domains with piecewise-Lyapunov boundaries. For curves with arbitrary nonzero angles there is Theorem 1 in [13] which shows relations between the values p, q, α_k, β_k, ν_k, λ_k, and if they are fulfilled, the statements of type I–III in Theorem 6 of the present work remain valid. A detailed analysis of cases where the above relations are realized is given. When investigating the problem we have used the results obtained by S. Warschawskiĭ ([18]) on conformal mappings of a circle on a domain with piecewise-Lyapunov boundary.

Consider the case where L is a piecewise-smooth curve. Assume that on L there are angular points $t_1, t_2, \dots, t_s$ with the values $\mu_k\pi$, $0 < \mu_k \le 2$, $k = \overline{1,s}$ of the interior angles at these points. In this case, for conformal mapping we use some results from [14] (see also [5, Ch. III]) according to which

$$z'(w) = \prod_{k=1}^{s} (w - \tau_k)^{\mu_k - 1} z_1(w), \quad \tau_k = w(t_k),$$
$$z(w) = \prod_{k=1}^{s} (w - \tau_k)^{\mu_k} z_2(w),$$

where $[z_1(w)]^{\pm 1}$, $[z_2(w)]^{\pm 1}$ belong to $\bigcap\limits_{\delta>0} H^\delta$, while the functions $z_1(\tau)$, $z_2(\tau)$ have the form (11) and hence belong to $\bigcap\limits_{\delta>1} W^\delta$.

On the basis of the above-said, taking into account the results of Sections 3^0 and 4^0 of the present work and following the reasoning from [13], we can see that the basic result obtained in [13, Theorem 1] for the problem (8), when L is a piecewise-smooth curve, remains valid.

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