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ON AN INTEGRAL EQUATION WITH MONOTONIC NONLINEARITY

Abstract. We prove the existence of a nonnegative and bounded solution of a type of homogeneous integral equations with monotonic nonlinearity. Under certain assumptions on the kernel, the properties of the obtained solutions are investigated. Some particular examples which arise in applications are demonstrated.

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1. Introduction

We consider the following nonlinear integral equation:

$$\varphi^p(x) = \int_0^\infty K(x,t)\varphi(t)\,dt, \quad x > 0, \tag{1}$$

in regard to unknown function $\varphi(x) \geq 0$. Here $p > 1$ is a real number, $0 \leq K(x,t)$ is a measurable function defined on $(0,+\infty)\times(0,+\infty)$ satisfying the condition

$$\sup_{x>0} \int_0^\infty K(x,t)\,dt = 1. \tag{2}$$

We will also consider the general integral equation of Hammerstein type:

$$f(x) = \int_0^\infty K(x,t)Q(f(t))\,dt, \tag{1*}$$

where the function $Q(x)$ is defined on $(-\infty,+\infty)$ and satisfies some additional conditions (see Theorem 6).

The problems (1), (2) and (1*), (2) are of considerable interest not only in mathematics, but also in the theory of nonlocal interactions, string filed theory, cosmology, kinetic theory of gases (see [1]–[6]).

In the present paper, under certain assumptions on the kernel $K(x,t)$ we prove the existence of a nontrivial, nonnegative and bounded solution of nonlinear homogenous equations (1) and (1*). The properties of the obtained solutions are investigated (see Theorems 1–3, 6). We also undertake mathematical investigation of a special case which arises in applications, particularly in the dynamics of P-adic closed string field theory (see Theorems 4–5). Some particular examples of the function $Q(x)$ are listed.

2. Convolution type nonlinear integral equation

2.1. **Symmetric kernel.** First, we consider the equation (1), in particular, the case where

$$K(x,t) = k_0(x-t); \quad 0 \leq k_0 \in L_1(-\infty,+\infty).$$

We have

$$\psi^p(x) = \int_0^\infty k_0(x-t)\psi(t)\,dt, \quad x > 0, \quad p > 1. \tag{3}$$

The condition (2) takes the form of

$$\int_{-\infty}^{+\infty} k_0(x)\,dx = 1. \tag{4}$$

We also assume that

$$k_0(-x) = k_0(x), \quad \forall\, x > 0. \tag{5}$$

Denoting $f(x) = \psi^p(x)$, we have

$$f(x) = \int_0^\infty k_0(x-t)\sqrt[p]{f(t)}\,dt, \quad x > 0, \quad p > 1. \tag{6}$$

We will consider the following iteration process

$$f^{(n+1)}(x) = \int_0^\infty k_0(x-t)\sqrt[p]{f^{(n)}(t)}\,dt, \quad f^{(0)}(x) \equiv 1, \quad n = 0, 1, 2 \ldots . \tag{7}$$

The following statements are valid.

Statement 1. *The sequence of functions $\{f^{(n)}(x)\}_0^\infty$ is monotonously decreasing as n increases.*

Proof. Indeed, for $n = 0$ we have

$$f^{(1)}(x) \leq \int_{-\infty}^{+\infty} k_0(t)dt = 1 \equiv f^{(0)}(x).$$

Assuming that the analogous inequality holds for n and using the monotonicity of the function $y = \sqrt[p]{x}$ on $(0, +\infty)$, from (7) we obtain

$$f^{(n+1)}(x) \leq f^{(n)}(x). \qquad \square$$

Statement 2. *The following inequality is valid*

$$f^{(n)}(x) \geq \left(\frac{1}{2}\right)^{\frac{p}{p-1}}, \quad n = 0, 1, 2, \ldots . \tag{8}$$

Proof. For $n = 0$ this estimate is obvious. Let $f^{(n)}(x) \geq (\frac{1}{2})^{\frac{p}{p-1}}$ be true. Taking into account (4) and (5), from (7) we get

$$f^{(n+1)}(x) \geq \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \int_{-\infty}^{x} k_0(t)\,dt \geq \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \int_{-\infty}^{0} k(t)\,dt = \left(\frac{1}{2}\right)^{\frac{p}{p-1}}. \tag{9}$$

The statement is proved. $\square$

Statements 1 and 2 imply that almost everywhere the limit of the sequence of functions $\{f^{(n)}(x)\}_0^\infty$ exists:

$$\lim_{n\to\infty} f^{(n)}(x) = f(x). \tag{10}$$

Furthermore,

$$\left(\frac{1}{2}\right)^{\frac{p}{p-1}} \leq f(x) \leq 1. \tag{11}$$

Using Levi's limit theorems, we conclude that $f(x)$ is a solution of the equation (6).

Statement 3. *The solution $f(x)$ of the equation* (6) *is monotonously increasing as x increases.*

Proof. First, we prove that the sequence of functions $\{f^{(n)}(x)\}_{n=0}^{\infty}$ is increasing in x. Indeed, for $n=0$ this is obvious. Suppose that $f^{(n-1)}(x)\uparrow$ as x increases. Let $x_1, x_2 \in (0,+\infty)$, $x_1 > x_2$, are two arbitrary numbers. We have

$$f^{(n)}(x_1) - f^{(n)}(x_2) =$$

$$= \int\limits_{-\infty}^{x_1} k_0(t)[\sqrt[p]{f^{(n-1)}(x_1-t)}\,dt - \int\limits_{-\infty}^{x_2} k_0(t)\sqrt[p]{f^{(n-1)}(x_2-t)]}\,dt \geq$$

$$\geq \int\limits_{-\infty}^{x_2} k_0(t)\left[\sqrt[p]{f^{(n-1)}(x_1-t)} - \sqrt[p]{f^{(n-1)}(x_2-t)}\right]dt \geq 0.$$

Therefore $f^{(n)}(x_1) \geq f^{(n)}(x_2)$, which implies that $f(x_1) \geq f(x_2)$. □

Statement 4. *The limit of the function $f(x)$ exists:*

$$\lim_{x\to+\infty} f(x) = 1. \tag{12}$$

Proof. Denote $\lim\limits_{x\to+\infty} f(x) = \delta$.

It is easy to check that

$$\lim_{x\to+\infty} \sqrt[p]{f(x)} = \lim_{x\to+\infty} \psi(x) = \sqrt[p]{\delta}. \tag{13}$$

We show that

$$\lim_{x\to+\infty} \int\limits_0^{\infty} k_0(x-t)\sqrt[p]{f(t)}\,dt = \sqrt[p]{\delta}. \tag{14}$$

Indeed,

$$\left|\int\limits_0^{\infty} k_0(x-t)\sqrt[p]{f(t)}\,dt - \sqrt[p]{\delta}\int\limits_{-\infty}^{+\infty} k_0(t)\,dt\right| =$$

$$= \left|\int\limits_{-\infty}^{x} k_0(t)\sqrt[p]{f(x-t)}\,dt - \sqrt[p]{\delta}\int\limits_{-\infty}^{x} k_0(t)dt - \int\limits_x^{\infty} \sqrt[p]{\delta}k_0(t)dt\right| \leq$$

$$\leq \int\limits_{-\infty}^{x} k_0(t)\left|\sqrt[p]{f(x-t)} - \sqrt[p]{\delta}\right|dt + \sqrt[p]{\delta}\int\limits_x^{\infty} k_0(t)\,dt = J_1 + J_2.$$

It is obvious that $\lim\limits_{x\to+\infty} J_2 = 0$. We have

$$J_1 = \int\limits_{-\infty}^{x} k_0(t) \left| \sqrt[p]{f(x-t)} - \sqrt[p]{\delta} \right| dt \le$$

$$\le \int\limits_{-\infty}^{\frac{x}{2}} k_0(t) \left| \sqrt[p]{f(x-t)} - \sqrt[p]{\delta} \right| dt + \int\limits_{\frac{x}{2}}^{x} k_0(t) \left| \sqrt[p]{f(x-t)} - \sqrt[p]{\delta} \right| dt =$$

$$= J_3 + J_4,$$

$$J_3 = \int\limits_{\frac{x}{2}}^{\infty} k_0(x-t) \left| \sqrt[p]{f(t)} - \sqrt[p]{\delta} \right| dt \le \sup_{t\ge\frac{x}{2}} \left| \sqrt[p]{f(t)} - \sqrt[p]{\delta} \right| dt \int\limits_{-\infty}^{+\infty} k_0(t)\, dt \to 0$$

as $x \to +\infty$.

$$J_4 = (1 + \sqrt[p]{\delta}) \int\limits_{\frac{x}{2}}^{x} k_0(t)\, dt \to 0$$

as x tends to ∞. Thus the formula (13) holds. Passing in (6) to limit, we obtain $\delta = \sqrt[p]{\delta} \Rightarrow \delta = 1$. From (14) it follows that

$$\lim_{x\to+\infty} \psi(x) = 1. \tag{15}$$

The statement is proved. □

Statement 5. *Let $f_1(x)$ and $f_2(x)$ be the constructed solutions of the equation* (6) *for the integers p_1 and p_2, respectively. If $p_1 > p_2$, then $f_1(x) \ge f_2(x)$.*

Proof. We consider the iterations for $p = p_1$ and $p = p_2$ separately.

$$f_i^{(n+1)}(x) = \int\limits_0^{\infty} k_0(x-t) \sqrt[p_i]{f_i^{(n)}(t)}\, dt, \quad f_i^{(0)} \equiv 1, \quad i = 1, 2, \quad n = 0, 1, 2, \ldots. \tag{16}$$

We will prove that

$$f_1^{(n)}(x) \ge f_2^{(n)}(x). \tag{17}$$

Indeed, for $n = 0$ the inequality (17) is obvious. Assuming that (17) holds for n, we check it for $n+1$. Taking into account the estimates $0 < f^{(n)}(x) \le 1$, from (16) we get

$$f_1^{(n+1)}(x) \ge \int\limits_0^{\infty} k_0(x-t) \sqrt[p_1]{f_2^{(n)}(t)}\, dt \ge$$

$$\ge \int\limits_0^{\infty} k_0(x-t) \sqrt[p_2]{f_2^{(n)}(t)}\, dt = f_2^{(n+1)}(x), \tag{18}$$

which implies that

$$f_1(x) \geq f_2(x). \tag{19}$$

Thus we have proved the statement. □

Theorem 1. *Under the conditions* (4), (5) *the equation* (3) *has a positive and bounded solution* $\psi(x)$ *which possesses the following properties:*

a) $\psi(x) \uparrow$ *in* x;

b) *the estimates* $(\frac{1}{2})^{\frac{1}{p-1}} \leq \psi(x) \leq 1$ *are valid;*

c) *there exists the limit* $\lim\limits_{x\to+\infty} \psi(x) = 1$.

Remark 1. The linear equation (3)–(5) ($p = 1$) represents the well-known homogeneous conservative Wiener–Hopf equation. Many works are devoted to the investigation of the corresponding linear equation (3) (see [7]–[9] and the literature therein). It is known (see [7]) that the corresponding linear equation in the symmetric case $k_0(-x) = k_0(x)$ has a positive solution, possessing the asymptotic $O(x)$ at $x \to +\infty$. Thus we confirm that there is a quantitative difference between solutions of nonlinear ($p > 1$) and linear ($p = 1$) equations.

2.2. **Nonsymmetric kernel.** We will assume that

$$\nu(k_0) = \int\limits_{-\infty}^{+\infty} x k_0(x)\, dx < 0. \tag{20}$$

The convergence of the integral (20) is understood in the Cauchy v.p. sense. Together with the equation (3) we consider the corresponding linear equation

$$S(x) = \int\limits_0^\infty k_0(x-t) S(t)\, dt, \quad x > 0. \tag{21}$$

It is well–known that if the function $k_0(x)$ satisfies the conditions (4), (20), then the equation (21) has a positive monotonously increasing and bounded solution $S(x)$ (see [8,9]). We denote $C = \sup\limits_{x>0} S(x)$. Due to the linearity of (21), the function $S^* = \dfrac{1}{C} S(x)$ will also satisfy the equation (21). Furthermore, $S^*(x) \uparrow 1$ as $x \to +\infty$. We consider the equation (7) with the kernel (4), (20).

Analogously, it is easy to verify that $f^{(n)}(x) \downarrow$ as n increases. We prove $f^{(n)}(x) \geq S^*(x)$. For $n = 0$ this is obvious. Taking into account (21) and $0 < S^*(x) \leq 1$, from (7) we obtain

$$f^{(n+1)}(x) \geq \int\limits_0^\infty k_0(x-t) \sqrt[p]{S^*(t)} dt \geq \int\limits_0^\infty k_0(x-t) S^*(t)\, dt = S^*(x).$$

Thus, there exists $f(x) = \lim\limits_{x\to+\infty} f^{(n)}(x)$. Moreover,

$$S^*(x) \le f(x) \le 1. \tag{22}$$

From Levi's theorem it follows that the limit function $f(x)$ satisfies the equation (3).

Acting analogously as in Theorem 1, we obtain that $f(x) \uparrow$ as x increases. Since $S^*(x) \to 1$ as $x \to +\infty$, it follows from (22) that

$$\lim_{x\to\infty} f(x) = 1.$$

Thus the following theorem holds.

Theorem 2. *Under the conditions* (4), (20) *the equation* (3) *has a positive monotonically increasing and bounded solution* $\psi(x)$. *Moreover,*

$$\lim_{x\to\infty} \psi(x) = 1, \quad S^*(x) \le \psi(x) \le 1.$$

Acting analogously we will be able to prove the following general theorem.

Theorem 3. *Let there exist* $k_0(x)$, $k_0(x) \ge 0$, $\int\limits_{-\infty}^{+\infty} k_0(x)\,dx = 1$, *such that* $K(x,t) \ge k_0(x-t)\ \forall\, x,t \in R^+ \times R^+$.

1) *if* $k_0(-x) = k_0(x)$, *then the equation* (1) *has a positive and bounded solution* $\varphi(x)$*:*

$$\left(\frac{1}{2}\right)^{\frac{1}{p-1}} \le \psi(x) \le \varphi(x) \le 1; \quad \lim_{x\to+\infty} \varphi(x) = 1;$$

2) *if* $\nu(k_0) < 0$, *then the equation* (1) *has a positive and bounded solution* $\varphi(x)$*:*

$$S^*(x) \le \psi(x) \le \varphi(x) \le 1; \quad \lim_{x\to+\infty} \varphi(x) = 1.$$

2.3. **Examples.** We bring two particular examples of the equation (1) satisfying the conditions of Theorem 3:

$$\begin{aligned} &1) \quad \varphi^p(x) = \int_0^\infty k_0(x-t)\varphi(t)\,dt + \int_0^\infty k_1(x+t)\varphi(t)\,dt, \quad \text{where} \\ &\quad 0 \le k_1 \in L_1(0,+\infty), \quad \int_x^\infty k_1(t)\,dt \le \int_x^\infty k_0(t)\,dt, \quad \forall\, x > 0; \end{aligned} \tag{23}$$

$$2) \quad \varphi^p(x) = \mu(x) \int_0^\infty k_0(x-t)\varphi(t)\,dt, \tag{24}$$

where $\mu(x)$ is a measurable function on $(0,+\infty)$ satisfying the condition $1 \le \mu(x) \le \frac{1}{\int\limits_{-\infty}^{x} k_0(t)\,dt}$.

3. On a Special Case Arising in Applications

We consider the equation (1) in the case where

$$K(x,t)=k_0(x-t)-k_1(x+t)\geq 0. \tag{25}$$

It should be noted that the condition $K(x,t)\geq k_0(x-t)$ doesn't work for the kernel (25) and it is necessary to develop a new approach for studying the problem of solvability of the equation (1), (25). We should also note that the nonlinear equation (1) with the kernel

$$K(x,t)=\frac{1}{\sqrt{\pi}}\,(e^{-(x-t)^2}-e^{-(x+t)^2}) \tag{26}$$

describes the dynamics (rolling) of $P-$adic closed strings for a scalar tachyon field (see [2], [3]).

First we consider the corresponding linear equation ($p=1$)

$$\eta(x)=\int\limits_0^\infty k_0(x-t)\eta(t)\,dt-\int\limits_0^\infty k_1(x+t)\eta(t)\,dt,\quad x>0, \tag{27}$$

where $\eta(x)$ is the unknown function.

We rewrite the equation (27) in the operator form

$$(I-\widehat{K}_0+\widehat{K}_1)\eta=0, \tag{28}$$

where I is the unit operator, $\widehat{K}_0$ is a Wiener–Hopf integral operator, and $\widehat{K}_1$ is a Henkel operator. Let E be one of the following Banach spaces: $L_p(0,+\infty)$, $1\leq p\leq\infty$, $M(0,+\infty)$, $C_u(0,+\infty)$, $C_0(0,+\infty)$, where $C_u(0,+\infty)$ is the space of continuous functions having a finite limit at infinity.

It is known (see [10]) that if $\nu(k_0)\leq 0$ and $m_2(k_1)=\int\limits_0^\infty x^2k_1(x)\,dx<+\infty$, then the operator $I-\widehat{K}_0+\widehat{K}_1$ admits the following three factor decomposition

$$I-\widehat{K}_0+\widehat{K}_1=(I-\widehat{V_-})(I+\widehat{W})(I-\widehat{V_+}), \tag{29}$$

where $\widehat{V}_\pm$ are Volterra operators:

$$(\widehat{V}_-f)(x)=\int\limits_x^\infty v_-(t-x)f(t)\,dt,\quad f\in E, \tag{30}$$

$$(\widehat{V}_+f)(x)=\int\limits_0^x v_+(x-t)f(t)\,dt,\quad f\in E, \tag{31}$$

$0\leq v_\pm\in L_1(0,+\infty)$, $\gamma_\pm=\int\limits_0^\infty v_\pm(x)\,dx\leq 1$, and $\widehat{W}$ is a Henkel type integral operator

$$(\widehat{W}f)(x)=\int\limits_0^\infty W(x+t)f(t)dt,\quad f\in E, \tag{32}$$

$0 \le W \in L_1(0,+\infty)$. It should be noted that (see [8])

i) if $\nu(k_0) < 0$, then $\gamma_- = 1$, $\gamma_+ < 1$;

ii) if $\nu(k_0) = 0$, then $\gamma_\pm = 1$.

At the same time, if the functions k_0 and k_1 are bounded, then $W \in M(0,+\infty)$, $v_\pm \in M(0,+\infty)$.

It is well known that $\widehat{W}$ is a compact operator in the spaces $L_1(0,+\infty)$ and $C_u(0,+\infty)$ (and in other natural functional spaces).

Taking into account the factorization (29), we rewrite the equation (28) in the form

$$(I - \widehat{V_-})(I + \widehat{W})(I - \widehat{V_+})\eta = 0. \tag{33}$$

Solving the equation (33) is equivalent to solving the following three coupled equations

$$(I - \widehat{V_-})\eta_1 = 0, \tag{34}$$

$$(I + \widehat{W})\eta_2 = \eta_1, \tag{35}$$

$$(I - \widehat{V_+})\eta = \eta_2. \tag{36}$$

Statement 6. *Let $\nu(k_0) < 0$. Then the equation (27) has a nontrivial solution $\eta(x) \in C_u(0,+\infty)$.*

Proof. Let us consider the following possibilities:

a) $\varepsilon = -1$ is an eigenvalue for the operator $\widehat{W}$;

b) $\varepsilon = -1$ is not an eigenvalue for the operator $\widehat{W}$.

a) We choose the trivial solution of the equation (34). Inserting it in (35), we obtain

$$\eta_2(x) = -\int\limits_0^\infty W(x+t)\eta_2(t)\,dt. \tag{37}$$

Since $\varepsilon = -1$ is an eigenvalue for the operator $\widehat{W}$, the equation (37) has a nontrivial solution $\eta_2 \in C_u(0,+\infty)$. Furthermore, from the estimate

$$|\eta_2(x)| \le \sup_{t>0} |\eta_2(t)| \int\limits_x^\infty W(\tau)\,d\tau$$

it follows that $\eta_2 \in C_0(0,+\infty)$.

Now we consider the equation (36)

$$\eta(x) = \eta_2(x) + \int\limits_0^x v_+(x-t)\eta(t)\,dt. \tag{38}$$

Since $\gamma_+ < 1$, the equation (38) in the space $C_0(0,+\infty)$ has a unique solution (see [9]).

b) It is easy to check that $\eta_1(x) = const \ne 0$ satisfies the equation (34) because $\gamma_- = 1$.

We choose $\eta_1(x) \equiv 1$ as η_1. Substituting it in (35), using the fact that $\varepsilon = -1$ is not an eigenvalue for $\widehat{W}$ and taking into account that $\widehat{W}$ is completely continuous (in $C_u(0,+\infty)$), we conclude that the equation (35) has a bounded solution $\eta_2 \in C_u(0,+\infty)$. Since $\gamma_+ < 1$, the equation (38) has a solution belonging to $C_u(0,+\infty)$. □

Statement 7. *Let $\nu(k_0) = 0$, $k_0 \in L_1(-\infty,+\infty) \cap M(-\infty,+\infty)$, $k_1 \in L_1(0,+\infty) \cap M(0,+\infty)$. If $\varepsilon = -1$ is an eigenvalue for the operator $\widehat{W}$, then the equation* (27) *has a nontrivial bounded solution.*

Proof. First we note that under the above-mentioned conditions and from the results of [9], [10] it follows that $W \in M(0,+\infty) \cap L_1(0,+\infty)$, $v_\pm \in M(0,+\infty) \cap L_1(0,+\infty)$. Choosing the trivial solution of the equation (34) and taking into account that $\varepsilon = -1$ is an eigenvalue for the completely compact operator $\widehat{W}$ (in $L_1(0,+\infty)$), we conclude that the equation (35) in $L_1(0,+\infty)$ has a nontrivial solution. Since $W \in M(0,+\infty) \cap L_1(0,+\infty)$, from the inequality

$$|\eta_2(x)| \le \sup_{x>0} |W(x)| \int_0^\infty |\eta_2(t)|\, dt$$

it follows that $\eta_2 \in M(0,+\infty)$. Thus we have proved that $\eta_2 \in L_1(0,+\infty) \cap M(0,+\infty)$. Now we consider the equation (36) in the conservative case (when $\gamma_+ = 1$). Using the results of the work [11], we conclude that the equation (36) has a bounded solution $\eta(x)$. Below we assume that one of the conditions of Statements 6 or 7 is fulfilled. Denote $C = \sup_{x>0} |\eta(x)|$. Due to the linearity of the equation (27), the function $\widetilde{\eta} = \frac{1}{C}\eta$ will be a nontrivial solution of the equation (27). Furthermore,

$$\sup_{x>0} |\widetilde{\eta}(x)| = 1. \tag{39}$$

Let us consider the following iteration

$$f^{(n+1)}(x) = \int_0^\infty K(x,t) \sqrt[p]{f^{(n)}(t)}\, dt, \quad f^{(n)}(x) \equiv 1, \quad n = 0,1,2,\dots, \tag{40}$$

where $K(x,t)$ is given by the formula (25).

It is easy to check that for arbitrary $n = 0,1,2,\dots$ the inequality

$$f^{(n)}(x) \ge |\widetilde{\eta}(x)| \tag{41}$$

holds. Indeed, for $n = 0$ it is obvious (see (39)). Assuming that the inequality (41) holds for some n, we will prove that it is true for $n+1$. Since $|\widetilde{\eta}(x)| \le 1$, we have

$$f^{(n+1)}(x) \ge \int_0^\infty K(x,t) \sqrt[p]{|\widetilde{\eta}(t)|} dt \ge \int_0^\infty K(x,t) |\widetilde{\eta}(t)| dt \ge |\widetilde{\eta}(x)|.$$

Hence the sequence of functions $\{f^{(n)}(x)\}_0^\infty$ has a limit as $n \to +\infty$,

$$\lim_{n\to\infty} f^{(n)}(x) = f(x). \tag{42}$$

At the same time,

$$|\widetilde{\eta}(x)| \leq f(x) \leq 1. \tag{43}$$

Using Levi's theorem, we conclude that $f(x)$ is a solution of the equation

$$f(x) = \int_0^\infty K(x,t) \sqrt[p]{f(t)}\, dt. \qquad \square$$

Statement 8. *$f(x) \uparrow$ as x increases.*

Proof. Let x_1, $x_2 \in (0,+\infty)$, $x_1 < x_2$, be arbitrary numbers and consider the following iteration process

$$f^{(n+1)}(x) = \int_{-\infty}^{x} k_0(t) \sqrt[p]{f^{(n)}(x-t)}\, dt - \int_x^\infty k_1(t) \sqrt[p]{f^{(n)}(t-x)}\, dt.$$

We have

$$\begin{aligned} &f^{(n+1)}(x_1) - f^{(n+1)}(x_2) = \\ &\quad = \int_{-\infty}^{x_1} k_0(t) \sqrt[p]{f^{(n)}(x_1-t)}\, dt - \int_{x_1}^\infty k_1(t) \sqrt[p]{f^{(n)}(t-x_1)}\, dt - \\ &\quad - \int_{-\infty}^{x_2} k_0(t) \sqrt[p]{f^{(n)}(x_2-t)}\, dt + \int_{x_2}^\infty k_1(t) \sqrt[p]{f^{(n)}(t-x_2)}\, dt \geq \\ &\quad \geq \int_{-\infty}^{x_2} k_0(t) \left[\sqrt[p]{f^{(n)}(x_1-t)} - \sqrt[p]{f^{(n)}(x_2-t)}\right] dt + \\ &\quad + \int_{x_2}^\infty k_1(t) \left[\sqrt[p]{f^{(n)}(t-x_2)} - \sqrt[p]{f^{(n)}(t-x_1)}\right] dt \geq 0. \end{aligned}$$

Therefore $f(x) \uparrow$ as x increases. From (39) and (43) it follows that $\lim\limits_{x\to\infty} f(x) = 1$. $\square$

Thus the following theorems are valid.

Theorem 4. *Let*

1) $0 \leq k_0 \in L_1(-\infty;+\infty)$, $\int\limits_{-\infty}^{+\infty} k_0(t)\, dt = 1$, $K(x,t) = k_0(x-t) - k_1(x+t) \geq 0$, $0 \leq k_1 \in L_1(0,+\infty)$, $m_2(k_1) = \int\limits_0^\infty x^2 k_1(x)\, dx < +\infty$;
2) $\nu(k_0) < 0$.

Then the equation (1) *has a nontrivial nonnegative solution* $\varphi(x)$ *and* $\lim\limits_{x\to\infty}\varphi(x)=1$.

Theorem 5. *Let*

1) *the condition* 1) *of Theorem* 4 *be fulfilled;*
2) *if* $\nu(k_0) = 0$ *and* $\varepsilon = -1$ *is an eigenvalue for the operator* $\widehat{W}$, *and* $k_0 \in M(-\infty,+\infty)\cap L_1(-\infty,+\infty)$, *then the equation* (1) *has a nontrivial, nonnegative solution* $\varphi(x)$ *and* $\lim\limits_{x\to\infty}\varphi(x) = 1$.

Remark 2. We note that Theorems 4, 5 are true for the kernels $K(x,t)$ satisfying the condition $K(x,t) \geq k_0(x-t) - k_1(x+t)$.

4. General Equation

We consider the general nonlinear equation (1^*). Acting analogously as in Theorem 1 and leaving out the details, we will formulate the following theorem.

Theorem 6. *Let the following conditions be fulfilled:*

1) *there exists* $k_0(x):\ k_0(-x) = k_0(x),\ \int\limits_{-\infty}^{+\infty} k_0(x)\,dx = 1$, *such that*

$$K(x,t) \geq k_0(x-t) \quad \forall\, x,\, t \in R^+ \times R^+; \tag{44}$$

2) *there exist* η, ζ, $\eta > 2\zeta$, *such that* $Q(\eta) = \eta$, $Q(\zeta) = 2\zeta$, $Q(x)\uparrow$ *on* $[\zeta,\eta]$, $Q \in C[\zeta,\eta]$,

where η *is the first positive root of the equation* $Q(x) = x$. (45)

Then the equation (1^*) *has a nonnegative and bounded solution* $f(x)$:

$$\lim_{x\to\infty} f(x) = \eta.$$

Moreover, if $K(x,t) \equiv k_0(x-t)$, *then the solution possesses the following properties:*

i) $\zeta \leq f(x) \leq \eta$;
ii) $f(x)\uparrow$ *as* x *increases.*

Examples. We bring some particular examples of the function $Q(x)$ (see below) which arise in applications:

(1) $Q(x) = x^{\frac{1}{p}}$, $x > 0$, $\zeta = (\frac{1}{2})^{\frac{p}{p-1}}$, $\eta = 1$;
(2) $Q(x) = sinx + x + 1$, $x > 0$, $\zeta \in (0, \frac{3}{4}\pi)$, $\eta = \frac{3}{2}\pi$;
(3) $Q(x) = ae^{-(x-a)^2}$, $x > 0$, $\zeta \in (0, \frac{\eta}{2})$, where η is the first positive root of the equation $ae^{-(x-a)^2} = x$;
(4) $Q(x) = e^{x-1}$, $x > 0$, $\zeta \in (0, \frac{1}{4})$, $\eta = 1$.

Summarizing, let us demonstrate one sample example. So, let $K(x,t) = k_0(x-t)$, $k_0(x) = \frac{1}{2}e^{-|x|}$, $Q(x) = e^{x-1}$, $\eta = 1$, ζ be the solution of equation $e^{x-1} = 2x$.

From (1*) we obtain

$$f''(x) - f(x) + e^{f(x)-1} = 0. \tag{46}$$

In spite of the fact that it is impossible to solve the obtained nonlinear differential equation analytically, the equation (46) has a positive and bounded solution $f(x) \not\equiv 1$ which has the following properties:

i) $\zeta \leq f(x) \leq 1$;
ii) $\lim\limits_{x\to\infty} f(x) = 1$;
iii) $f(x) \uparrow$ as x increases.

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