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TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY

Abstract. The purpose of this paper is to consider two-dimensional version of quasistatic Aifantis' equation of the theory of consolidation with double porosity and to study the uniqueness and existence of solutions of basic boundary value problems (BVPs). The fundamental and some other matrices of singular solutions are constructed in terms of elementary functions for the steady-state quasistatic equations of the theory of consolidation with double porosity. Using the fundamental matrix we construct the simple and double layer potentials and study their properties near the boundary. Using these potentials, for the solution of the first basic BVP we construct Fredholm type integral equation of the second kind and prove the existence theorem of solution for the finite and infinite domains.

რეზიუმე. ნაშრომის მიზანია განვიხილოთ აიფანტის კვაზისტატიკის გან-ტოლებები ორგვარი ფოროვნობის მქონე ორგანზომილებიანი სხეულებისათვის. დამტკიცებულია ძირითადი სასაზღვრო ამოცანების ამონახსნის ერთადერთობის თეორემები კვაზისტატიკის განტოლებებისათვის ორგვარი ფოროვნების გათვა-ლისწინებით. აგებულია ამონახსნთა ფუნდამენტური და სხვა მატრიცები ელე-მენტარული ფუნქციების საშუალებით. ამ მატრიცების საშუალებით შედგენილია მარტივი და ორმაგი ფენის პოტენციალები და შესწავლილია მათი თვისებები. ამ პოტენციალების გამოყენებით პირველი სასაზღვრო ამოცანისათვის აგებულია ფრედჰოლმის მეორე გვარის ინტეგრალური განტოლებები და დამტკიცებულია მისი ამონახსნის არსებობის თეორემა როგორც სასრული, ისე უსასრულო არი-სათვის.

Introduction

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example, in a fissured rock (i.e. a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity. When fluid flow and deformations processes occur simultaneously, three coupled partial differential equations can be derived [1], [2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid. Inertia effects are neglected as they are in Biot's theory.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]–[3]. In part I of a series of papers on the subject, R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of this series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosoty (see [1]–[3] and the references cited therein). The basic results and the historical information on the theory of porous media were summarized by Boer [4].

The purpose of this paper is to consider a two-dimensional version of quasistatic Aifantis' equation of the theory of consolidation with double porosity and to study the uniqueness and existence of solutions of basic boundary value problems (BVPs). The fundamental and some other matrices of singular solutions are constructed in terms of elementary functions for the steady-state quasistatic equations of the theory of consolidation with double porosity. Using the fundamental matrix, we construct the simple and double layer potentials and study their properties near the boundary. Using these potentials, for solving the first basic BVP we construct a Fredholm type integral equation of the second kind and prove the existence theorem of solution for the finite and infinite domains.

1. Basic Equations, Boundary Value Problems and Uniqueness Theorems

The basic steady-state quasistatic Aifantis' equations of the theory of consolidation with double porosity in the case of plane deformation are

given by partial differential equations of the form [1], [2]

$$\mu\Delta u + (\lambda + \mu)\,\text{grad div}\,u - \text{grad}\,(\beta_1 p_1 + \beta_2 p_2) = 0,$$

$$\frac{i\omega\beta_1}{m_1}\,\text{div}\,u + \left(\Delta + \frac{\alpha_3}{m_1}\right)p_1 + \frac{k}{m_1}\,p_2 = 0, \qquad (1.1)$$

$$\frac{i\omega\beta_2}{m_2}\,\text{div}\,u + \frac{k}{m_2}\,p_1 + \left(\Delta + \frac{\alpha_4}{m_2}\right)p_2 = 0,$$

where $u = (u_1, u_2)$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores. $\alpha_3 = i\omega\alpha_1 - k$, $\alpha_4 = i\omega\alpha_2 - k$, $m_j = \frac{k_j}{\mu^*}$, $j = 1, 2$. The constant λ is the Lame modulus, μ is the shear modulus and the constants β_1 and β_2 measure the change of porosities due to an applied volumetric strain. The constants α_1 and α_2 measure the compressibilities of primary and secondary pores filled with pore fluid. The constants k_1 and k_2 are the permeabilities of the primary and secondary systems of pores, the constant μ^* denotes the viscosity of the pore fluid and the constant k measures the transfer of fluid from the secondary pores to the primary pores. The quantities λ, μ, α_j, β_j, k_j $(j = 1, 2)$ and μ^* are all positive constants. $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the two-dimensional Laplace operator, ω is the oscilation frequency $(\omega > 0)$.

We also rewrite the equation (1.1) in the matrix form

$$B(\partial x)U = 0, \qquad (1.2)$$

where

$$B(\partial x) = \parallel B_{pq}(\partial x) \parallel_{4x4}, \;\; p, q = 1, 2, 3, 4,$$

$$B_{jj}(\partial x) = \mu\Delta + (\lambda + \mu)\,\frac{\partial^2}{\partial x_j^2}, \;\; j = 1, 2,$$

$$B_{12}(\partial x) = B_{21}(\partial x) = (\lambda + \mu)\,\frac{\partial^2}{\partial x_1 \partial x_2},$$

$$B_{j3}(\partial x) = -\beta_1\,\frac{\partial}{\partial x_j}, \;\; B_{j4}(\partial x) = -\beta_2\,\frac{\partial}{\partial x_j}, \;\; j = 1, 2,$$

$$B_{3j}(\partial x) = \frac{i\omega\beta_1}{m_1}\,\frac{\partial}{\partial x_j}, \;\; B_{4j}(\partial x) = \frac{i\omega\beta_2}{m_2}\,\frac{\partial}{\partial x_j}, \;\; j = 1, 2,$$

$$B_{33}(\partial x) = \Delta + \frac{\alpha_3}{m_1}, \;\; B_{34}(\partial x) = \frac{k}{m_1}, \;\; B_{43}(\partial x) = \frac{k}{m_2},$$

$$B_{44}(\partial x) = \Delta + \frac{\alpha_4}{m_2}, \;\; U(u_1, u_2, p_1, p_2).$$

The conjugate system of the equation (2) is

$$\widetilde{B}(\partial x)U = B^T(-\partial x)U = 0.$$

Throughout this paper "T" denotes transposition.

Now we write the expressions for the components of the stress vector, which acts on elements of the arc with the normal $n = (n_1, n_2)$. Denoting

the stress vector by $P(\partial x, n)u$, we have

$$P(\partial x, n)u = T(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{1.3}$$

where [9]

$$T(\partial x, n) = \| T_{kj}(\partial x, n) \|_{2x2},$$

$$T_{kj}(\partial x, n) = \mu \delta_{kj} \frac{\partial}{\partial n} + \lambda n_k \frac{\partial}{\partial x_j} + \mu n_j \frac{\partial}{\partial x_k}, \quad k, j = 1, 2. \tag{1.4}$$

Let $D^+(D^-)$ be a finite (an infinite) two-dimensional region bounded by the contour S. Suppose that $S \in C^{1,\beta}$, $0 < \beta \leq 1$, i.e., S is a Lyapunov curve.

Introduce the definition of a regular vector-function.

Definition 1. A vector-function $U(x) = (u_1, u_2, p_1, p_2)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second derivatives in D^+ (D^-), and U itself and its first order derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i.e., $U \in C^2(D^+) \cap C^1(\overline{D^+})$, $(U \in C^2(D^+) \cap C^1(\overline{D^+}))$. Note that for the infinite domain D^- the vector $U(x)$ additionally satisfies the following conditions at infinity:

$$U(x) = O(1), \quad \frac{\partial U_k}{\partial x_j} = O(|x|^{-2}), \quad |x|^2 = x_1^2 + x_2^2, \quad j = 1, 2, \tag{1.5}$$

where $O(1)$ denotes a bounded function.

For the equation (1.1) we pose the following boundary value problems:
Find a regular vector U satisfying in D^+ (D^-) the equation (1.1), and on the boundary S one of the following conditions:

Problem 1. The displacement vector and the fluid pressures are given in the form

$$u^\pm(z) = f(z)^\pm, \quad p_1^\pm(z) = f_3^\pm, \quad p_2^\pm(z) = f_4^\pm(z), \quad z \in S;$$

Problem 2. The stress vector and the normal derivatives of the preasure functions $\frac{\partial p_j}{\partial n}$ are given in the form

$$(Pu)^\pm = f(z)^\pm, \quad \left(\frac{\partial p_1(z)}{\partial n}\right)^\pm = f_3^\pm, \quad \left(\frac{\partial p_2(z)}{\partial n}\right)^\pm = f_4^\pm(z), \quad z \in S;$$

Problem 3.

$$u^\pm(z) = f(z)^\pm, \quad \left(\frac{\partial p_1(z)}{\partial n}\right)^\pm = f_3^\pm(z), \quad \left(\frac{\partial p_2(z)}{\partial n}\right)^\pm = f_4^\pm(z), \quad z \in S;$$

Problem 4.

$$(Pu(z))^\pm = f(z)^\pm, \quad p_1^\pm(z) = f_3^\pm(z), \quad p_2^\pm(z) = f_4^\pm(z), \quad z \in S,$$

where $(\cdot)^\pm$ denotes the limiting values on S from $D^\pm$ and $f = (f_1, f_2)$, f_3, f_4 are given functions.

Generalized Green's Formulas. Let u and $\overline{u}$ be two regular solutions of the equation (1.1) in D^+. Multiply the first equation of (1.1) by $\overline{u}$, the second one by $\overline{p_1}$ and the third one by $\overline{p_2}$, where $\overline{u}$, $\overline{p_1}$ and $\overline{p_2}$ are the complex conjugate functions of u, p_1 and p_2 respectively, integrate over D^+ and sum to obtain

$$\int\limits_{D^+} \left[E(u, \overline{u}) + \alpha_1 |p_1|^2 + \alpha_2 |p_2|^2 + \right.$$

$$\left. + \frac{k}{i\omega} |p_1 - p_2|^2 + \frac{m_1}{i\omega} |\operatorname{grad} p_1|^2 + \frac{m_2}{i\omega} |\operatorname{grad} p_2|^2 \right] dx =$$

$$= \int\limits_{S} \left[\overline{u} P(\partial x, n) u + \frac{m_1}{i\omega} p_1 \frac{\partial \overline{p_1}}{\partial n} + \frac{m_2}{i\omega} p_2 \frac{\partial \overline{p_2}}{\partial n} \right] ds, \quad (1.6)$$

where

$$E(u, u) = (\lambda + \mu)(\operatorname{div} u)^2 + \mu \left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right)^2 + \mu \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right)^2.$$

For positive definiteness of the potential energy the inequalities $\lambda + \mu > 0$, $\mu > 0$ are necessary and sufficient.

One can generalize the formula (1.6) to the infinite domain D^-, provided the condition

$$\lim_{R \to \infty} \int\limits_{S(0,R)} \left[\overline{u} P(\partial x, n) u + \frac{m_1}{i\omega} p_1 \frac{\partial \overline{p_1}}{\partial n} + \frac{m_2}{i\omega} p_2 \frac{\partial \overline{p_2}}{\partial n} \right] ds = 0 \quad (1.7)$$

is fulfilled, where $S(0, R)$ is a circumference of radius R with center at the point O lying inside D^+. The radius R is taken so large that the region D^+ lies entirely inside the circumference $S(0, R)$.

Obviously, the condition (1.7) is fulfilled if the vector u and $\overline{u}$ satisfy the conditions (1.5).

If (1.7) is fulfilled, then Green's formula for the domain D^- takes the form

$$\int\limits_{D^-} \left[E(u, u) + \alpha_1 |p_1|^2 + \alpha_2 |p_2|^2 + \right.$$

$$\left. + \frac{k}{i\omega} |p_1 - p_2|^2 + \frac{m_1}{i\omega} |\operatorname{grad} p_1|^2 + \frac{m_2}{i\omega} |\operatorname{grad} p_2|^2 \right] dx =$$

$$= - \int\limits_{S} \left[\overline{u} P(\partial x, n) u + \frac{m_1}{i\omega} p_1 \frac{\partial \overline{p_1}}{\partial n} + \frac{m_2}{i\omega} p_2 \frac{\partial \overline{p_2}}{\partial n} \right] ds. \quad (1.8)$$

The Uniqueness Theorems. In this subsection we investigate the question of uniqueness of solutions of the above-mentioned problems.

Now let us prove the following theorems.

Theorem 1. *The first boundary value problem has at most one regular solution in the finite domain D^+.*

Proof. Let the first BVP have in the domain D^+ two regular solutions $U^{(1)}$ and $U^{(2)}$. Denote $u = U^{(1)} - U^{(2)}$. Evidently, the vector u satisfies (1.1) and the boundary condition $u^+ = 0$ on S. Note that if u is a regular solution of the equation (1.1), we have Green's formula (1.6). Using (1.6) and taking into account the fact that the potential energy is positive definite, we conclude that $U = C, x \in D^+$, where $C = const$. Since $U^+ = 0$, we have $C = 0$ and $U(x) = 0$, $x \in D^+$. $\qquad\square$

Theorem 2. *The first boundary value problem has at most one regular solution in the infinite domain D^-.*

Proof. The vectors $U^{(1)}$ and $U^{(2)}$ in the domain D^- must satisfy the condition (1.5). In this case the formula (1.8) is valid and $U(x) = C$, $x \in D^-$, where C is again a constant vector. But U on the boundary satisfies the condition $U^- = 0$, which implies that $C = 0$ and $U(x) = 0$, $x \in D^-$. $\qquad\square$

Theorem 3. *A regular solution of the second boundary value problem is not unique in the domain D^+. Two regular solutions may differ by the vector (u, p_1, p_2), where u is a rigid displacement vector and $p_j = 0$, $j = 1, 2$.*

Proof. Let

$$(P(\partial x, n)u)^+ = 0, \quad \left(\frac{\partial p_1}{\partial n}\right)^+ = 0, \quad \left(\frac{\partial p_2}{\partial n}\right)^+ = 0, \quad x \in S.$$

The positive definiteness of the potential energy implies

$$u_1 = c_1 - \varepsilon x_2, \quad u_2 = c_2 + \varepsilon x_1, \quad p_1 = 0, \quad p_2 = 0, \quad x \in D^+. \qquad\square$$

Theorem 4. *Two regular solutions of the second boundary value problem in the domain D^- may differ by the vector (u, p_1, p_2), where u is a constant vector and $p_j = 0$, $j = 1, 2$.*

Proof. For the exterior second homogeneous boundary value problem the vector u must satisfy the condition at infinity (1.5). In this case, the formula (1.8) is valid for a regular u. Using this formula, we obtain

$$u_1 = c_1 - \varepsilon x_2, \quad u_2 = c_2 + \varepsilon x_1, \quad p_1 = 0, \quad p_2 = 0, \quad x \in D^-.$$

Bearing in mind (1.5), we have $\varepsilon = 0$ and

$$u_1 = c_1, \quad u_2 = c_2, \quad p_1 = 0, \quad p_2 = 0, \quad x \in D^-. \qquad\square$$

Analogously, the following theorems are valid:

Theorem 5. *The boundary value problems $(III)^\pm$ have in the domains $D^\pm$ at most one regular solution.*

Theorem 6. *Two regular solutions of the boundary value problem $(IV)^+$ may differ by the vector $U(u, p_1, p_2)$, where u is a rigid displacement and $p_j = 0$, $j = 1, 2$. Two regular solutions of the boundary value problem $(IV)^-$ may differ by the vector (u, p_1, p_2), where u is a constant vector and $p_j = 0$, $j = 1, 2$.*

2. Matrix of Fundamental Solutions

Here we construct the matrix of fundamental solutions for the system (1.1).

Let

$$B^* = \frac{1}{a\mu} \begin{pmatrix} B_{11}^* - B_{12}^* \xi_1^2 & -B_{12}^* \xi_1 \xi_2 & \mu B_{13}^* \xi_1 & \mu B_{14}^* \xi_1 \\ -B_{12}^* \xi_1 \xi_2 & B_{11}^* - B_{12}^* \xi_2^2 & \mu B_{13}^* \xi_2 & \mu B_{14}^* \xi_2 \\ -\iota\omega\mu B_{31}^* \xi_1 & -\iota\omega\mu B_{31}^* \xi_2 & \mu B_{33}^* \Delta\Delta & -\mu B_{34}^* \Delta\Delta \\ -\iota\omega\mu B_{41}^* \xi_1 & -\iota\omega\mu B_{41}^* \xi_2 & -\mu B_{43}^* \Delta\Delta & \mu B_{44}^* \Delta\Delta \end{pmatrix},$$

where

$$B_{11}^* = a\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2),$$

$$B_{12}^* = a(\Delta + \lambda_1^2)(\Delta + \lambda_2^2) - \mu\left[\Delta\Delta + \left(\frac{\alpha_4}{m_2} + \frac{\alpha_3}{m_1}\right)\Delta + \frac{\alpha_3\alpha_4 - k^2}{m_1 m_2}\right],$$

$$B_{13}^* = \beta_1 \Delta\Delta + \Delta\frac{\alpha_4\beta_1 - k\beta_2}{m_2}, \quad B_{14}^* = \beta_2 \Delta\Delta + \Delta\frac{\alpha_3\beta_2 - k\beta_1}{m_1},$$

$$B_{31}^* = \frac{\beta_1}{m_1}\Delta\Delta + \Delta\frac{\alpha_4\beta_1 - k\beta_2}{m_1 m_2}, \quad B_{41}^* = \frac{\beta_2}{m_2}\Delta\Delta + \Delta\frac{\alpha_3\beta_2 - k\beta_1}{m_1 m_2},$$

$$B_{33}^* = a\left(\Delta + \frac{\alpha_4}{m_2}\right) + \frac{i\omega\beta_2^2}{m_2}, \quad B_{34}^* = \frac{ka + i\omega\beta_2\beta_1}{m_1},$$

$$B_{43}^* = \frac{ka + i\omega\beta_2\beta_1}{m_2}, \quad B_{44}^* = a\left(\Delta + \frac{\alpha_3}{m_1}\right) + \frac{i\omega\beta_1^2}{m_1}.$$

Supposing

$$U(x) = B^*(\partial x)\Psi, \tag{2.1}$$

where $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ is a four-dimensional vector function, we can write the equation (1.1) as

$$\mu a \Delta\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\Psi = 0; \tag{2.2}$$

here λ_j^2, $j = 1, 2$ are the roots of the characteristic equation

$$x^2 - \left[\frac{\alpha_4}{m_2} + \frac{\alpha_3}{m_1} + \frac{i\omega}{a}\left(\frac{\beta_2^2}{m_2} + \frac{\beta_1^2}{m_1}\right)\right]x + \frac{\alpha_3\alpha_4 - k^2}{m_1 m_2} +$$

$$+ \frac{i\omega}{am_1 m_2}\left(\alpha_4\beta_1^2 + \alpha_3\beta_2^2 - 2k\beta_1\beta_2\right) = 0, \quad a = \lambda + 2\mu. \tag{2.3}$$

We assume that $\lambda_1^2 \neq \lambda_2^2$. Without loss of generality we assume that $Im\lambda_j > 0$, $j = 1, 2$.

From (2.2) it follows that

$$\Psi(x) = -\frac{2i}{\pi} \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^4 \lambda_2^4} \ln r + \frac{2i}{\pi} \frac{r^2(\ln r - 1)}{4\lambda_1^2 \lambda_2^2} - \frac{H_0^{(1)}(\lambda_1 r)}{\lambda_1^4(\lambda_1^2 - \lambda_2^2)} + \frac{H_0^{(1)}(\lambda_2 r)}{\lambda_2^4(\lambda_1^2 - \lambda_2^2)}, \quad (2.4)$$

$H_0^{(1)}(\lambda r)$ is the first kind Hankel function of zero order [5]

$$H_0^{(1)}(\lambda r) = \frac{2i}{\pi} \ln r + \frac{2i}{\pi} \big[J_0(\lambda r) - 1 \big] \ln r +$$
$$+ \frac{2i}{\pi} J_0(\lambda r) \Big(\ln \frac{\lambda}{2} + C - \frac{i\pi}{2} \Big) -$$
$$- \frac{2i}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \Big(\frac{\lambda r}{2} \Big)^{2k} \Big(\frac{1}{k} + \frac{1}{k-1} + \cdots + 1 \Big), \quad (2.5)$$

$$J_0(\lambda r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \Big(\frac{\lambda r}{2} \Big)^{2k}.$$

Substituting $\Psi(x)$ in (2.1), after some calculations we obtain the fundamental matrix of solutions for the equation (1.1) which is denoted by $\Gamma(x-y)$

$$\Gamma(x-y) = \begin{pmatrix} \dfrac{2i}{\pi\mu} \ln r + \dfrac{\partial^2 \Psi_{11}}{\partial x_1^2} & \dfrac{\partial^2 \Psi_{11}}{\partial x_1 \partial x_2} & \dfrac{\partial \Psi_{13}}{\partial x_1} & \dfrac{\partial \Psi_{14}}{\partial x_1} \\[2mm] \dfrac{\partial^2 \Psi_{11}}{\partial x_1 \partial x_2} & \dfrac{2i}{\pi\mu} \ln r + \dfrac{\partial^2 \Psi_{11}}{\partial x_2^2} & \dfrac{\partial \Psi_{13}}{\partial x_2} & \dfrac{\partial \Psi_{14}}{\partial x_2} \\[2mm] -\dfrac{i\omega}{m_1} \dfrac{\partial \Psi_{13}}{\partial x_1} & -\dfrac{i\omega}{m_1} \dfrac{\partial \Psi_{13}}{\partial x_2} & \Psi_{33} & \Psi_{34} \\[2mm] -\dfrac{i\omega}{m_2} \dfrac{\partial \Psi_{14}}{\partial x_1} & -\dfrac{i\omega}{m_2} \dfrac{\partial \Psi_{14}}{\partial x_2} & \dfrac{m_1}{m_2}\Psi_{34} & \Psi_{44} \end{pmatrix}, \quad (2.6)$$

where

$$\Psi_{11} = \alpha_{11} \ln r + \alpha_{12} \frac{r^2(\ln r - 1)}{4} + \alpha_{21} H_0^{(1)}(\lambda_1 r) + \alpha_{22} H_0^{(1)}(\lambda_2 r),$$

$$\Psi_{13} = \beta_{11} \ln r + \beta_{12} H_0^{(1)}(\lambda_1 r) + \beta_{13} H_0^{(1)}(\lambda_2 r),$$

$$\Psi_{14} = \gamma_{11} \ln r + \gamma_{12} H_0^{(1)}(\lambda_1 r) + \gamma_{13} H_0^{(1)}(\lambda_2 r),$$

$$\Psi_{31} = \frac{1}{m_1} \Psi_{13}, \quad \Psi_{33} = \delta_{11} H_0^{(1)}(\lambda_1 r) + \delta_{12} H_0^{(1)}(\lambda_2 r),$$

$$\Psi_{41} = \frac{1}{m_2} \Psi_{14}, \quad \Psi_{34} = \delta_{34} \big[H_0^{(1)}(\lambda_2 r) - H_0^{(1)}(\lambda_1 r) \big],$$

$$\Psi_{43} = \frac{m_1}{m_2} \Psi_{34}, \quad \Psi_{44} = \delta_{41} H_0^{(1)}(\lambda_1 r) + \delta_{42} H_0^{(1)}(\lambda_2 r),$$

$$\alpha_{11} = \frac{2i}{\pi a \lambda_1^2 \lambda_2^2} \Big[\frac{\alpha_3}{m_1} + \frac{\alpha_4}{m_2} - \frac{(\lambda_1^2 + \lambda_2^2)(\alpha_3 \alpha_4 - k^2)}{m_1 m_2 \lambda_1^2 \lambda_2^2} \Big], \quad (2.7)$$

$$\alpha_{12} = \frac{2i}{\pi}\left[\frac{\alpha_3\alpha_4 - k^2}{am_1m_2\lambda_1^2\lambda_2^2} - \frac{1}{\mu}\right], \quad \delta_{34} = -\frac{ka + i\omega\beta_1\beta_2}{m_1a(\lambda_1^2 - \lambda_2^2)},$$

$$\alpha_{2k} = \frac{(-1)^k}{a(\lambda_1^2 - \lambda_2^2)}\left[1 - \frac{1}{\lambda_k^2}\left(\frac{\alpha_3}{m_1} + \frac{\alpha_4}{m_2}\right) + \frac{\alpha_3\alpha_4 - k^2}{m_1m_2\lambda_k^4}\right], \quad k = 1, 2,$$

$$\beta_{11} = \frac{2i(\alpha_4\beta_1 - k\beta_2)}{\pi m_2a\lambda_1^2\lambda_2^2}, \quad \gamma_{11} = \frac{2i(\alpha_3\beta_2 - k\beta_1)}{\pi m_1a\lambda_1^2\lambda_2^2},$$

$$\beta_{1k} = \frac{(-1)^k}{a(\lambda_1^2 - \lambda_2^2)}\left[-\beta_1 + \frac{\alpha_4\beta_1 - k\beta_2}{m_2\lambda_{k-1}^2}\right], \quad k = 2, 3,$$

$$\gamma_{1k} = \frac{(-1)^k}{a(\lambda_1^2 - \lambda_2^2)}\left[-\beta_2 + \frac{\alpha_3\beta_2 - k\beta_1}{m_1\lambda_{k-1}^2}\right], \quad k = 2, 3,$$

$$\delta_{1k} = \frac{1}{\lambda_1^2 - \lambda_2^2}\left[-\lambda_k^2 + \frac{\alpha_4a + i\omega\beta_2^2}{m_2a}\right], \quad k = 1, 2,$$

$$\delta_{4k} = \frac{1}{\lambda_1^2 - \lambda_2^2}\left[-\lambda_k^2 + \frac{\alpha_3a + i\omega\beta_1^2}{m_2a}\right], \quad k = 1, 2,$$

$$\alpha_{11} + \frac{2i}{\pi}\left[\alpha_{21} + \alpha_{22}\right] = 0, \quad \beta_{11} + \frac{2i}{\pi}\left[\beta_{12} + \beta_{13}\right] = 0,$$

$$\gamma_{11} + \frac{2i}{\pi}\left[\gamma_{12} + \gamma_{13}\right] = 0,$$

$$\delta_{11} + \delta_{33} = 1, \quad \delta_{22} + \delta_{44} = 1, \quad r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

Moreover, on the basis of the identity

$$H_0^{(1)}(\lambda r) = \frac{2i}{\pi}\ln r - \frac{2i}{4\pi}r^2\ln r + const + O(r^2)$$

we easily conclude that $\Gamma(x - y)$ has a logarithmic singularity. It can be shown that the columns of the matrix $\Gamma(x - y)$ are solutions to the equation (1.1) with respect to x for any $x \neq y$.

Denote $\widetilde{\Gamma}(x) = \Gamma^T(-x)$. Hence we have proved the following

Theorem. *The matrix* $\Gamma(x)$ *is a solution of the system* (1.1) *and the matrix* $\widetilde{\Gamma}(x)$ *is a solution of the adjoint system* $\widetilde{B}(\partial x)U = 0$.

3. Matrix of Singular Solutions

In solving boundary value problems of the theory of consolidation with double porosity by the method of potential theory, the fundamental matrix and some other matrices of singular solutions to the equation (1.1) are of great importance. These matrices will be constructed explicitly in the present section with the help of elementary functions. Using the basic fundamental matrix, we will construct the so-called singular matrices of solutions. For simplicity, we will introduce the special generalized stress vector.

Write now the expressions for the components of the generalized stress vector, which acts on elements of the arc with the normal $n = (n_1, n_2)$.

Denoting the generalized stress vector by $\overset{\kappa}{\mathrm{P}}(\partial x, n)u$, where κ is an arbitrary constant, we have

$$\overset{\kappa}{\mathrm{P}}(\partial x, n)u = \overset{\kappa}{\mathrm{T}}(\partial x, n)u - n(\beta_1 p_1 + \beta_2 p_2), \tag{3.1}$$

where

$$\overset{\kappa}{\mathrm{T}}(\partial x, n)u = \begin{pmatrix} \mu \dfrac{\partial}{\partial n} + (\lambda + \mu)n_1 \dfrac{\partial}{\partial x_1} & (\lambda + \mu)n_1 \dfrac{\partial}{\partial x_2} - \kappa \dfrac{\partial}{\partial s} \\[2mm] (\lambda + \mu)n_2 \dfrac{\partial}{\partial x_1} + \kappa \dfrac{\partial}{\partial s} & \mu \dfrac{\partial}{\partial n} + (\lambda + \mu)n_2 \dfrac{\partial}{\partial x_2} \end{pmatrix} u,$$

$$\frac{\partial}{\partial s} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}.$$

If $\kappa = \mu$, then we have the stress vector $P(\partial x, n)u$. The operator which will be obtained from $\overset{\kappa}{\mathrm{P}}(\partial x, n)$ for $\kappa = \kappa_n = \frac{\mu(\lambda+\mu)}{\lambda+3\mu}$ will be called the operator $N(\partial x, n)$, and the vector $N(\partial x, n)u$ will be called the pseudo-stress vector. The pseudo-stress operator succeeded in obtaining the Fredholm integral equation of the second kind for the first boundary value problem.

We introduce the following notation $\overset{\kappa}{\mathrm{R}}(\partial x, n)$, $\widetilde{\mathrm{R}}^{\kappa}(\partial x, n)$

$$\overset{\kappa}{\mathrm{R}}(\partial x, n) = \begin{pmatrix} \overset{\kappa}{\mathrm{T}}(\partial x, n)_{11} & \overset{\kappa}{\mathrm{T}}(\partial x, n)_{12} & -\beta_1 n_1 & -\beta_1 n_1 \\[2mm] \overset{\kappa}{\mathrm{T}}(\partial x, n)_{21} & \overset{\kappa}{\mathrm{T}}(\partial x, n)_{22} & -\beta_1 n_2 & -\beta_2 n_2 \\[2mm] 0 & 0 & \dfrac{\partial}{\partial n} & 0 \\[2mm] 0 & 0 & 0 & \dfrac{\partial}{\partial n} \end{pmatrix},$$

$$\widetilde{\mathrm{R}}^{\kappa}(\partial x, n) = \begin{pmatrix} \overset{\kappa}{\mathrm{T}}(\partial x, n)_{11} & \overset{\kappa}{\mathrm{T}}(\partial x, n)_{12} & -i\omega n_1 \dfrac{\beta_1}{m_1} & -i\omega n_1 \dfrac{\beta_2}{m_2} \\[2mm] \overset{\kappa}{\mathrm{T}}(\partial x, n)_{21} & \overset{\kappa}{\mathrm{T}}(\partial x, n)_{22} & -i\omega n_2 \dfrac{\beta_1}{m_1} & -i\omega n_2 \dfrac{\beta_2}{m_2} \\[2mm] 0 & 0 & \dfrac{\partial}{\partial n} & 0 \\[2mm] 0 & 0 & 0 & \dfrac{\partial}{\partial n} \end{pmatrix}.$$

By Applying the operator $\overset{\kappa}{\mathrm{R}}(\partial x, n)$ to the matrix $\Gamma(x)$, we will construct the so-called singular matrix of solutions. Let us consider the matrix $[\overset{\kappa}{\mathrm{R}}(\partial y, n)\Gamma(y - x)]^*$ which is obtained from $\overset{\kappa}{\mathrm{R}}(\partial x, n)\Gamma(x - y) = (\overset{\kappa}{\mathrm{R}}_{pq})_{4x4}$ by transposition of the columns and rows and the variables x and y. We can easily prove that every column of the matrix $[\overset{\kappa}{\mathrm{R}}(\partial y, n)\Gamma(y - x)]^*$ is a solution of the system $\widetilde{B}(\partial x)U = 0$ with respect to the point x, if $x \neq y$.

The elements $\overset{\kappa}{\mathrm{R}}_{pq}$ are as follows:

$$\overset{\kappa}{\mathrm{R}}_{pp} = \frac{2i}{\pi} \frac{\partial}{\partial n} \ln r + (-1)^p(\kappa + \mu) \frac{\partial}{\partial s} \frac{\partial^2 \Psi_{11}}{\partial x_1 \partial x_2}, \quad p = 1, 2,$$

$$\overset{\kappa}{\mathrm{R}}_{12} = -\frac{\partial}{\partial s}\left[\frac{2i}{\pi}\frac{\kappa}{\mu}\ln r + (\kappa+\mu)\frac{\partial^2\Psi_{11}}{\partial x_2^2}\right],$$

$$\overset{\kappa}{\mathrm{R}}_{21} = \frac{\partial}{\partial s}\left[\frac{2i}{\pi}\frac{\kappa}{\mu}\ln r + (\kappa+\mu)\frac{\partial^2\Psi_{11}}{\partial x_1^2}\right],$$

$$\overset{\kappa}{\mathrm{R}}_{13} = -(\kappa+\mu)\frac{\partial}{\partial s}\frac{\partial\Psi_{13}}{\partial x_2}, \quad \overset{\kappa}{\mathrm{R}}_{14} = -(\kappa+\mu)\frac{\partial}{\partial s}\frac{\partial\Psi_{14}}{\partial x_2}, \qquad (3.2)$$

$$\overset{\kappa}{\mathrm{R}}_{23} = (\kappa+\mu)\frac{\partial}{\partial s}\frac{\partial\Psi_{13}}{\partial x_1}, \quad \overset{\kappa}{\mathrm{R}}_{24} = (\kappa+\mu)\frac{\partial}{\partial s}\frac{\partial\Psi_{14}}{\partial x_1},$$

$$\overset{\kappa}{\mathrm{R}}_{3j} = -\frac{i\omega}{m_1}\frac{\partial}{\partial n}\frac{\partial\Psi_{13}}{\partial x_j}, \quad \overset{\kappa}{\mathrm{R}}_{4j} = -\frac{i\omega}{m_2}\frac{\partial}{\partial n}\frac{\partial\Psi_{14}}{\partial x_j}, \quad j=1,2,$$

$$\overset{\kappa}{\mathrm{R}}_{33} = \frac{\partial\Psi_{33}}{\partial n}, \quad \overset{\kappa}{\mathrm{R}}_{34} = \frac{\partial\Psi_{34}}{\partial n}, \quad \overset{\kappa}{\mathrm{R}}_{43} = \frac{m_1}{m_2}\frac{\partial\Psi_{34}}{\partial n}, \quad \overset{\kappa}{\mathrm{R}}_{44} = \frac{\partial\Psi_{44}}{\partial n},$$

Analogously, we obtain the matrix

$$\widetilde{\mathrm{R}}^{\kappa}(\partial y, n)\widetilde{\Gamma}(y-x) = \left([\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}]_{pq}\right)_{4x4},$$

where

$$\left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{pq} = \overset{\kappa}{\mathrm{R}}_{pp}, \quad p=1,2, \quad \left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{12} = \overset{\kappa}{\mathrm{R}}_{12}, \quad \left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{21} = \overset{\kappa}{\mathrm{R}}_{21},$$

$$\left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{13} = \frac{i\omega}{m_1}\overset{\kappa}{\mathrm{R}}_{13}, \quad \left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{14} = \frac{i\omega}{m_2}\overset{\kappa}{\mathrm{R}}_{14}, \quad \left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{23} = \frac{i\omega}{m_1}\overset{\kappa}{\mathrm{R}}_{23},$$

$$\left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{24} = \frac{i\omega}{m_2}\overset{\kappa}{\mathrm{R}}_{24}, \quad \left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{3j} = -\frac{\partial}{\partial n}\frac{\partial\Psi_{13}}{\partial x_j},$$

$$\left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{4j} = -\frac{\partial}{\partial n}\frac{\partial\Psi_{14}}{\partial x_j}, \quad j=1,2,$$

$$\left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{33} = \frac{\partial}{\partial n}\Psi_{33}, \quad \left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{34} = \frac{m_1}{m_2}\frac{\partial}{\partial n}\Psi_{34},$$

$$\left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{43} = \frac{\partial}{\partial n}\Psi_{34}, \quad \left[\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}\right]_{44} = \frac{\partial}{\partial n}\Psi_{44},$$

The matrix $\left[\widetilde{\mathrm{R}}^{\kappa}(\partial y, n)\widetilde{\Gamma}(y-x)\right]^*$ is a solution of the system (1.1). It shows, that the matrices $\left[\widetilde{\mathrm{R}}^{\kappa}(\partial x, n)\widetilde{\Gamma}\right]^*$ and $\left[\overset{\kappa}{\mathrm{R}}(\partial x, n)\Gamma\right]^*$ contain a singular part, which is integrable in the sense of the principal Cauchy value.

4. Potentials and Their Properties

Introduce the following definitions:

Definition 2. The vector-functions defined by the equalities

$$V^{(1)}(x) = \frac{1}{4i}\int\limits_{S}\Gamma(y-x)h(y)\,dy,$$

$$V^{(2)}(x) = \frac{1}{4i}\int\limits_{S}\widehat{\Gamma}(x-y)h(y)\,ds, \qquad (4.1)$$

where $\Gamma(x, y)$ is the fundamental matrix, $\widetilde{\Gamma}(x) = \Gamma^T(-x)$, h is a continuous (or Holder continuous) vector and S is a closed Lyapunov curve, will be called simple layer potentials.

Definition 3. The vector-function defined by the equalities

$$U^{(1)}(x) = \frac{1}{4i} \int\limits_S [\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y - x)]^* h(y)\, dy,$$

$$U^{(2)}(x) = \frac{1}{4i} \int\limits_S [N(\partial y, n)\Gamma(y - x)]^* h(y)\, dy,$$

(4.2)

will be called double layer potentials.

The potentials $V^{(1)}$, $U^{(1)}$ are solutions of the system (1.1) and the potentials $V^{(2)}$, $U^{(2)}$ are solutions of the system $\widetilde{B}(\partial x)U = 0$ both in the domains D^+ and D^-. When the point x tends to a point $z \in S$, the potential (4.2) has the discontinuity as the harmonic double layer potential

$$U^{(1)\pm} = \pm h(z) + \frac{1}{4i} \int\limits_S [\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y - z)]^* h(y)\, dy,$$

$$U^{(2)\pm} = \pm h(z) + \frac{1}{4i} \int\limits_S [N(\partial y, n)\Gamma(y - z)]^* h(y)\, dy.$$

(4.3)

Now let us investigate properties of the operation $\overset{\kappa}{R}(\partial x, n)$ acting on a simple layer potential. We obtain

$$\overset{\kappa}{R}(\partial x, n)V(x) = \frac{1}{4i} \int\limits_S \overset{\kappa}{R}(\partial x, n)\Gamma(y - x)h(y)\, dy.$$

(4.4)

When $\kappa = \kappa_n$ we obtain

$$\big[N(\partial y, n)V^{(1)}(z)\big]^{\mp} = \mp h(z) + \frac{1}{4i} \int\limits_S N(\partial y, n)\Gamma(z - y)h(y)\, dy,$$

$$\big[\widetilde{N}(\partial y, n)V^{(2)}(z)\big]^{\mp} = \mp h(z) + \frac{1}{4i} \int\limits_S \widetilde{N}(\partial y, n)\widehat{\Gamma}(z - y)h(y)\, dy.$$

(4.5)

It is well-known ([8]) that in the case of a Lyapunov curve $S \in C^{1,\alpha}$ the function $\frac{\partial \ln r}{\partial n}$ for $x, y \in S$ has a week singularity and $\frac{\partial \ln r}{\partial n}$ is integrable in the sense of the principal Cauchy value. Consequently, $\frac{\partial \ln r}{\partial n}$ is a singular kernel on S.

It is obvious that $\big[\overset{\kappa}{R}(\partial y, n)\Gamma(y - x)\big]^*$ is a singular kernel (in the sense of Cauchy). Note that if $\kappa = \kappa_n = \frac{\mu(\lambda+\mu)}{\lambda+3\mu}$, then $\big[\overset{\kappa}{R}(\partial x, n)\Gamma(x - y)\big]^*$ is a weakly singular kernel.

5. Solution of the First Boundary Value Problem

Problem $(I)^+$**.** Let us first prove the existence of solution of the first boundary value problem in the domain D^+. A solution is sought in the form of the double layer potential

$$U(x) = \frac{1}{4i} \int\limits_S \left[\widetilde{N}(\partial y, n) \widetilde{\Gamma}(y - x) \right]^* h(y) \, dy. \tag{5.1}$$

Then for determining the unknown real vector function h we obtain the following Fredholm integral equation of the second kind

$$-h(z) + \frac{1}{4i} \int\limits_S \left[\widetilde{N}(\partial y, n) \widetilde{\Gamma}(y - z) \right]^* h(y) \, dy = f^+. \tag{5.2}$$

Let us prove that the equation (5.2) is solvable for any continuous right-hand side. Consider the associated to (5.2) homogeneous equation

$$-h(z) + \frac{1}{4i} \int\limits_S N(\partial y, n) \Gamma(y - z) h(y) \, dy = 0 \tag{5.3}$$

and prove that it has only the trivial solution.Assume the contrary and denote by $\varphi(z)$ a nonzero solution of (5.3). Compose the simple layer potential

$$V(x) = \frac{1}{4i} \int\limits_S \Gamma(y - x) \varphi(y) \, dy. \tag{5.4}$$

It is obvious from (5.3), that

$$[N(\partial z, n) V(z)]^- = 0, \quad \int\limits_S \varphi(y) \, ds = 0.$$

Using the formula (1.8) for $\kappa = \kappa_n$ in D^-, we obtain $V(x) = 0$, $x \in D^-$.

Now taking into account the continuity of the simple layer potential and using the uniqueness theorem for the solution of the first boundary value problem, we have $V(x) = 0$, $x \in D^+$.

Note that $[NV]^+ - [NV]^- = 2\varphi(x) = 0$ and hence the equation (5.3) has only the trivial solution. This implies that the associated to (5.3) homogeneous equation also has only the trivial solution, and the equation(5.2) is solvable for any continuous right-hand side (according to the first Fredholm theorem).

For the regularity of the double layer potential in the domain D^+ it is sufficient to assume that $S \in C^{2,\beta}$, $(0 < \beta < 1)$ and $\frac{\partial f}{\partial s}$ is Holder continuous $f \in C^{1,\alpha}(S)$ $(0 < \alpha < \beta)$.

Problem $(I)^-$**.** Consider now the first boundary value problem in the domain D^-. Its solution is sought in the form

$$U(x) = \frac{1}{4i} \int\limits_S \left(\left[\widetilde{N}(\partial y, n) \widetilde{\Gamma}(y - x) \right]^* - \left[\widetilde{N}(\partial y, n) \widetilde{\Gamma}(y) \right]^* \right) \psi(y) \, dy. \tag{5.5}$$

Then for determining the unknown real valued vector function ψ we obtain the following Fredholm integral equation of the second kind

$$\psi(z) + \frac{1}{4i} \int\limits_S \left(\left[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y-z) \right]^* - \left[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y) \right]^* \right)\psi(y)\, dy = f^-. \quad (5.6)$$

Prove that the equation (5.6) is solvable for any continuous right-hand side. We consider the associated to (5.6) homogeneous equation

$$h(z) + \frac{1}{4i} \int\limits_S \left[N(\partial y, n)\Gamma(z-y) + N(\partial y, n)\Gamma(y) \right] h(y)\, dy = 0. \quad (5.7)$$

Let us prove that (5.7) has only the trivial solution. Suppose that it has a nonzero solution $h(z)$. From (5.7) by integration we obtain

$$\int\limits_S h\, ds = 0.$$

In this case the equation (5.7) corresponds to the boundary condition $[N(\partial x, n)V]^+ = 0$, where

$$V(x) = \frac{1}{4i} \int\limits_S \Gamma(y-x) h(y)\, dy. \quad (5.8)$$

We find that $V = C$, $x \in D^+$, where C is a constant vector.

Taking into account the equation $\int\limits_S h\, ds = 0$ and the fact that the single layer potential is continuous while passing through the boundary, and using Green's formula for $\kappa = \kappa_n$, we obtain $V = 0$, $x \in D^-$. Since $[NV]^+ - [NV]^- = 2h(x) = 0$, and $[NV]^+ = 0$, $[NV]^- = 0$, we get $h(x) = 0$.

Thus we conclude that the associated to (5.7) homogeneous equation has only the trivial solution, and the equation (5.6) is solvable for any continuous right-hand side.

To prove the regularity of the potential (5.5) in the domain D^-, it is sufficient to assume that $S \in C^{2,\beta}$ $(0 < \beta < 1)$ and $f \in C^{1,\alpha}(S)$ $(0 < \alpha < \beta)$.

References

1. R. K. Wilson and E. C. Aifantis, On the theory of consolidation with double porosity. *Internat. J. Engrg. Sci.* **20** (1982), 1009–1035.
2. D. E. Beskos and E. C. Aifantis, On the theory of consolidation with double porosity. II. *Internat. J. Engrg. Sci.* **24** (1986), No. 11, 1697–1716.

3. M. Y. Khaled, D. E. Beskos, and E. C. Aifantis, On the theory of consolidation with double porosity. III. A finite element formulation. *Int. J. Numer. Anal. Methods Geomech.* **8** (1984), 101–123.

4. R. de Boer, Theory of porous media. Highlights in historical development and current state. *Springer-Verlag, Berlin*, 2000.

5. M. Svanadze, Fundamental solution in the theory of consolidation with double porosity. *J. Mech. Behavior of Materials* **16** (2005), 123–130.

6. V. I. Smirnov, A course of higher mathematics. Vol. III. Part two. Complex variables. Special functions. Translated by D. E. Brown. Translation edited by I. N. Sneddon *Pergamon Press, Oxford-Edinburgh-New York-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass.-London,* 1964.

7. M. Basheleishvili, Two-dimensional problems of elasticity of anisotropic bodies. *Mem. Differential Equations Math. Phys.* **16** (1999), 9–140.

8. N. I. Muskhelishvili, Singular integral equations. Boundary problems of function theory and their application to mathematical physics. 2nd revised edition. (Russian) Third, corrected and augmented edition. With an appendix by B. Bojarski. *Izdat. "Nauka", Moscow,* 1968; English transl.: *P. Noordhoff N. V., Groningen,* 1953.

9. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. Translated from the second Russian edition. Edited by V. D. Kupradze. *North-Holland Series in Applied Mathematics and Mechanics,* 25. *North-Holland Publishing Co., Amsterdam–New York,* 1979.

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