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EULER CASE FOR A CLASS OF THIRD-ORDER DIFFERENTIAL EQUATION

Abstract. We deal with an Euler-Case for a class of third-order differential equation. A theorem on asymptotic behaviour at the infinity of three linearly independent solutions is proved. This theorem coveres different class of coefficients.

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1. Introduction

In this paper we investigate the form of three linearly independent solutions for a class of the third-order differential equation

$$(q(qy')')' - (py')' - ry = 0 \tag{1}$$

as $x \to \infty$, where x is the independent variable and the prime denotes d/dx. The functions q, p and r are defined on the interval $[a, \infty)$, are not necessarily real-valued and continuously differentiable, and all are non-zero everywhere in this interval. In this situation where p is sufficiently small compared to q and r as $x \to \infty$, (1) can be considered as a perturbation of the equation investigated by Eastham. In this paper,we consider the opposite situation where p is large compared to q and r. In this situation, we identify the Euler case:

$$\begin{aligned} \frac{(pr)'}{pr} &\sim const. \times \frac{p}{q^2}, \\ \frac{(pq^{-1})'}{pq^{-1}} &\sim const. \times \frac{p}{q^2} \end{aligned} \tag{2}$$

as $x \to \infty$. The various conditions imposed on the coefficients will be introduced when they are required in the development of the method. Al-Hammadi [1] considers (1) in the case where the solutions all have a similar exponential factor. A third-order equation similar to (1) has been considered previously by Unsworth [11] and Pfeiffer[10]. Eastham [6] considered the Euler case for a fourth-order differential equation and showed that this case represents a border line between situations where all solutions have a certain exponential character as $x \to \infty$ and where only two solutions have this character. The case (2) will appear in the method in Sections 4–6, where we use the recent asymptotic theorem of Eastham [4, Section 2] to obtain the solutions of (1). Two examples are considered in Section 6.

2. The General Method

We write (1) in the standard way [8] as a first order system

$$Y' = AY, \tag{3}$$

where the first component of Y is y and

$$A = \begin{pmatrix} 0 & q^{-1} & 0 \\ 0 & pq^{-2} & q^{-1} \\ r & 0 & 0 \end{pmatrix}. \tag{4}$$

As in [2], we express A in its diagonal form

$$T^{-1}AT = \Lambda \tag{5}$$

and we therefore require the eigenvalues λ_j and eigenvectors ν_j $(1 \le j \le 3)$ of A, with the eigenvalues λ_j are chosen as continuously differentiable function.

Writing

$$q^2 = s, \tag{6}$$

we obtain the characteristic equation of A as

$$s\lambda^3 - p\lambda^2 - r = 0. \tag{7}$$

An eigenvector ν_j of A corresponding to λ_j is

$$\nu_j = \left(1, s^{\frac{1}{2}}\lambda_j, r\lambda_j^{-1}\right)^t, \tag{8}$$

where the superscript denotes the transpose. We assume at this stage that the λ_j are distinct, and we define the matrix T in (5) by

$$T = \begin{pmatrix} m_1^{-1}v_1 & m_2^{-1}v_2 & m_3^{-1}v_3 \end{pmatrix}, \tag{9}$$

where the m_j $(1 \leq j \leq 3)$ are scalar factors to be specified according to the following procedure. Now from (4), we note that EA is symmetric, where

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

Hence, by [7, Section 2(i)], the v_j have the orthogonality property

$$(Ev_k)^t v_j = 0 \quad (k \neq j). \tag{11}$$

We then define the scalars

$$m_j = (Ev_j)^t v_j \tag{12}$$

and the row vectors

$$r_j = (Ev_j)^t. \tag{13}$$

Hence by [7, Section 2]

$$T^{-1} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \tag{14}$$

$$m_j = 3s\lambda_j^2 - 2p\lambda_j = s\lambda_j^2 + 2r\lambda_j^{-1}. \tag{15}$$

By (5), the transformation

$$Y = TZ \tag{16}$$

takes (3) into

$$Z' = (\Lambda - T^{-1}T')Z, \tag{17}$$

where

$$\Lambda = dg(\lambda_1, \lambda_2, \lambda_3). \tag{18}$$

From (8)–(12), we obtain $T^{-1}T' = (t_{jk})$, where

$$t_{jj} = -\frac{1}{2}\frac{m_j'}{m_j} \tag{19}$$

and, for $j \neq k$,

$$t_{jk} = \frac{1}{2}\frac{m_k'}{m_k} + \frac{\lambda_j - \lambda_k}{m_k}\left(s\lambda_k' + \frac{1}{2}\lambda_k s'\right) - \frac{m_k'}{m_k^2}\left(r\lambda_j^{-1} + s\lambda_j\lambda_k + r\lambda_k^{-1}\right). \tag{20}$$

Now we need to work out (19) and (20) in some detail in terms of s, p and r in order to determine the form of (17).

3. The Matrices Λ and $T^{-1}T'$

In our analysis, we impose a basic condition on the coefficients as follows: (I) p, r and s are all nowhere zero in some interval $[a,\infty)$, and

$$\left(\frac{r}{p}\right)^{\frac{1}{2}} = o\left(\frac{p}{s}\right) \quad (x \to \infty), \tag{21}$$

If we write

$$\delta = \frac{sr^{\frac{1}{2}}}{p^{\frac{3}{2}}}\,, \tag{22}$$

then by (21)

$$\delta = o(1) \quad (x \to \infty). \tag{23}$$

Now as in [1,2], we can solve the characteristic equation (7) asymptotically as $x \to \infty$. Using (21) and (23), we obtain the distinct eigenvalues λ_j as

$$\lambda_1 = i\left(\frac{r}{p}\right)^{\frac{1}{2}}(1+\delta_1), \tag{24}$$

$$\lambda_2 = -i\left(\frac{r}{p}\right)^{\frac{1}{2}}(1+\delta_2), \tag{25}$$

$$\lambda_3 = \left(\frac{p}{s}\right)(1+\delta_3), \tag{26}$$

where

$$\delta_1 = O(\delta), \quad \delta_2 = O(\delta), \quad \delta_3 = O(\delta^2). \tag{27}$$

By(21), the ordering of λ_j is such that

$$\lambda_j = o(\lambda_3) \quad (x \to \infty, \ \ j = 1, 2). \tag{28}$$

Now substituting (24)–(26) into (7) and differentiating, we obtain

$$\lambda_1' = \frac{1}{2}\, i\left(\frac{r}{p}\right)^{\frac{1}{2}}\left\{\frac{r'}{r} - \frac{p'}{p} + O(\varepsilon)\right\}, \tag{29}$$

$$\lambda_2' = -\frac{1}{2}\, i\left(\frac{r}{p}\right)^{\frac{1}{2}}\left\{\frac{r'}{r} - \frac{p'}{p} + O(\varepsilon)\right\}, \tag{30}$$

$$\lambda_3' = \left(\frac{p}{s}\right)\left\{\frac{p'}{p} - \frac{s'}{s} + O(\delta\varepsilon)\right\}. \tag{31}$$

Now we work out m_j $(1 \le j \le 3)$ asymptotically as $x \to \infty$; hence by (24)–(27), (15) gives,

$$m_1 = -2i(pr)^{\frac{1}{2}}\{1 + O(\delta)\}, \tag{32}$$

$$m_2 = 2i(pr)^{\frac{1}{2}}\{1 + O(\delta)\}, \tag{33}$$

$$m_3 = \left(\frac{p^2}{s}\right)\{1 + O(\delta^2)\}. \tag{34}$$

Also by substituting λ_j $(j = 1, 2, 3)$ into (15) and using (24), (25) and (26) respectively, and differentiating, we obtain

$$m_1' = -i(rp)^{\frac{1}{2}}\Big\{\frac{r'}{r} + \frac{p'}{p} + O(\varepsilon)\Big\}, \tag{35}$$

$$m_2' = i(rp)^{\frac{1}{2}}\Big\{\frac{r'}{r} + \frac{p'}{p} + O(\varepsilon)\Big\}, \tag{36}$$

$$m_3' = \Big(\frac{p^2}{s}\Big)\Big\{2\frac{p'}{p} - \frac{s'}{s} + O(\delta\varepsilon)\Big\}, \tag{37}$$

where

$$\varepsilon = \Big|\frac{r'}{r}\,\delta\Big| + \Big|\frac{s'}{s}\delta\Big| + \Big|\frac{p'}{p}\delta\Big|. \tag{38}$$

At this stage we also require the following condition:

(II)

$$\delta\,\frac{r'}{r}\,,\ \delta\,\frac{s'}{s}\,,\ \delta\,\frac{p'}{p}\ \text{ are all }\ L(a,\infty). \tag{39}$$

Now by (22)

$$\delta' = O\Big(\frac{r'}{r}\,\delta\Big) + O\Big(\frac{s'}{s}\,\delta\Big) + O\Big(\frac{p'}{p}\,\delta\Big). \tag{40}$$

Also by substituting (24)–(25) into (7) and differentiating, we obtain

$$\delta_j' = O\Big(\frac{r'}{r}\,\delta\Big) + O\Big(\frac{s'}{s}\,\delta\Big) + O\Big(\frac{p'}{p}\delta\Big) \quad (j = 1, 2) \tag{41}$$

and

$$\delta_3' = O\Big(\frac{r'}{r}\,\delta^2\Big) + O\Big(\frac{s'}{s}\,\delta^2\Big) + O\Big(\frac{p'}{p}\delta^2\Big). \tag{42}$$

Hence by (38), (40), (41), (42) and (39)

$$\varepsilon,\ \delta',\ \delta_j' \in L(a,\infty). \tag{43}$$

We can now substitute the estimates (24)–(27), (32)–(37) and (29)–(31) into (19) and (20) as in [1], we obtain the following expressions for t_{jk},

$$\begin{gathered} t_{11} = -\rho + O(\varepsilon), \quad t_{22} = -\rho + O(\varepsilon), \\ t_{33} = -\eta + O(\delta\varepsilon), \quad t_{12} = \rho + O(\varepsilon), \\ t_{21} = \rho + O(\varepsilon), \quad t_{13} = O(\varepsilon), \quad t_{23} = O(\varepsilon) \\ t_{31} = \frac{1}{2}\,\eta + O(\varepsilon), \quad t_{32} = \frac{1}{2}\,\eta + O(\varepsilon) \end{gathered} \tag{44}$$

with

$$\rho = \frac{1}{4}\,\frac{(rp)'}{rp}\,, \quad \eta = \frac{(ps^{-1/2})'}{ps^{-1/2}}\,. \tag{45}$$

It follows from (43) the O-terms in (44) are $L(a,\infty)$, and we can therefore write (17)

$$Z' = (\Lambda + R + S)Z, \tag{46}$$

where

$$R = \begin{bmatrix} \rho & -\rho & 0 \\ -\rho & \rho & 0 \\ -\frac{1}{2}\eta & -\frac{1}{2}\eta & \eta \end{bmatrix} \tag{47}$$

and $S \in L(a, \infty)$ by (43).

4. The Euler Case

Now we deal with (2) more generally. So we write (2) as

$$\frac{(pr)'}{pr} = 4\sigma \frac{p}{s}(1+\phi), \tag{48}$$

$$\frac{(ps^{-1/2})'}{ps^{-1/2}} = w \frac{p}{s}(1+\psi), \tag{49}$$

where σ and w are non zero constants, and $\phi(x) \to 0$, $\psi(x) \to 0$ $(x \to \infty)$. At this stage we let

$$\phi',\, \psi' \in L(a, \infty). \tag{50}$$

We note that by (48) and (49), the matrix Λ no longer dominates the matrix R and so Eastham's theorem [4, Section 2] is not satisfied which means that we have to carry out a second diagnolization of the system(46). First we write

$$\Lambda + R = \lambda_3\{S_1 + S_2\} \tag{51}$$

and we need to work out the two matrices $S_1 = const.$ with the matrix $S_2(x) = o(1)$ as $x \to \infty$ using (24), (25), (26) and Euler case (48) and (49). Hence after some calculations, we obtain

$$S_1 = \begin{pmatrix} \sigma & -\sigma & 0 \\ -\sigma & \sigma & 0 \\ -\frac{1}{2}\omega & -\frac{1}{2}\omega & 1+\omega \end{pmatrix}, \tag{52}$$

$$S_2(x) = \begin{pmatrix} u_1 & u_2 & 0 \\ u_2 & u_3 & 0 \\ u_4 & u_4 & u_5 \end{pmatrix}, \tag{53}$$

where

$$\begin{gathered} u_1 = \lambda_1\lambda_3^{-1} - u_2, \quad u_2 = -\sigma(1+\delta_3)^{-1}(\phi - \delta_3), \\ u_3 = \lambda_2\lambda_3^{-1} - u_2, \quad u_4 = -\frac{1}{2}\omega(1+\delta_3)^{-1}(\psi - \delta_3), \quad u_5 = -2u_4. \end{gathered} \tag{54}$$

It is clear that by (28) and (27), $S_2(x) \to 0$ as $x \to \infty$. Hence we diagonalize the constant matrix S_1. Now the eigenvalues $\alpha_j (1 \le j \le 3)$ of the matrix S_1 are given by

$$\alpha_1 = 0, \quad \alpha_2 = 2\sigma, \quad \alpha_3 = 1+\omega. \tag{55}$$

Let

$$\omega \neq -1 \text{ and } 2\sigma - \omega \neq 1. \tag{56}$$

Hence by (56), the eigenvalues α_j are distinct. Thus we use the transformation

$$Z = T_1 W \tag{57}$$

in (46), where T_1 diagonalizes the constant matrix S_1. Then (46) transforms to

$$W' = (\Lambda_1 + M + T_1^{-1} S T_1) W, \tag{58}$$

where

$$\begin{gathered} \Lambda_1 = \lambda_3 T_1^{-1} S_1 T_1 = dg(v_1, v_2, v_3) = \lambda_3 dg(\alpha_1, \alpha_2, \alpha_3), \\ M = \lambda_3 T_1^{-1} S_2 T_1, \quad T_1^{-1} S T_1 \in L(a, \infty). \end{gathered} \tag{59}$$

Now we can apply the asymptotic theorem of Eastham [4, Section 2] to (58) provided only that Λ_1 and M satisfy the conditions in [4, Section 2]. We first require that the v_j $(1 \le j \le 3)$ are distinct, and this holds because α_j $(1 \le j \le 3)$ are distinct. Second, we need to show that

$$\frac{M}{v_i - v_j} \to 0 \ \ (x \to \infty) \tag{60}$$

for $i \ne j$ and $1 \le i, j \le 3$. Now

$$\frac{M}{v_i - v_j} = (\alpha_i - \alpha_j)^{-1} T_1^{-1} S_2 T_1 = o(1) \ \ (x \to \infty). \tag{61}$$

Thus (60) holds. Third, we need to show that

$$S_2' \in L(a, \infty). \tag{62}$$

Thus it suffices to show that

$$u_i'(x) \in L(a, \infty) \ \ (1 \le i \le 5). \tag{63}$$

Now by (24), (25), (26) and (54)

$$\begin{aligned} u_1' &= O(\delta') + O(\delta_1'\delta) + O(\delta_3') + O(\phi'), \\ u_2' &= O(\delta_3') + O(\phi'), \\ u_3' &= O(\delta') + O(\delta_2'\delta) + O(\delta_3') + O(\phi'), \\ u_4' &= O(\delta_3') + O(\psi'), \\ u_5' &= O(\delta_3') + O(\psi'). \end{aligned} \tag{64}$$

Thus, by (64), (43) and (50), we see that (63) holds and consequently (62) holds. Now we state our main theorem for (1).

5. The Main Result

Theorem 5.1. *Let the coefficients p, r and s are $C^{(2)}[a, \infty)$. Let* (21)*,* (38)*,* (48)*,* (49) *and* (55) *hold. Let*

$$Re\, I(x), \tag{65}$$

$$Re\Big[\lambda_3 + \eta - \frac{1}{2}\,(2\rho + \lambda_1 + \lambda_2 \pm I)\Big] \tag{66}$$

be of one sign in $[a,\infty)$, *where*

$$I(x) = \left[4\rho^2 + (\lambda_1 - \lambda_2)^2\right]^{\frac{1}{2}}. \tag{67}$$

Then (1) *has the solutions*

$$\begin{aligned} y_1(x) &= o\left\{(r(x)p(x))^{\frac{-1}{4}} \exp\left(\frac{1}{2}\int_a^x \left[\lambda_1(t)+\lambda_2(t)-I(t)\right]dt\right)\right\},\\ y_2(x) &= [-i+o(1)](r(x)p(x))^{\frac{-1}{4}} \times\\ &\quad\times \exp\left(\frac{1}{2}\int_a^x \left[\lambda_1(t)+\lambda_2(t)+I(t)\right]dt\right),\\ y_3(x) &= o\left\{(r(x)s(x))^{\frac{-1}{2}} p^{1/2}(x) \exp\left(\int_a^x \lambda_3(t)\,dt\right)\right\}. \end{aligned} \tag{68}$$

Proof. Before applying the theorem in [4, Section 2], we show that the eigenvalues μ_k $(1 \le k \le 3)$ of $\Lambda_1 + M$ satisfy the dichotomy condition [9]. As in [2], the dichotomy condition holds if

$$Re(\nu_j - \nu_k) = f + g \quad (j \ne k,\ \ 1 \le k \le 3), \tag{69}$$

where f has one sign in $[a,\infty)$ and g belongs to $L(a,\infty)$ [4, (1.5)]. Now since the eigenvalues of $\Lambda_1 + M$ are the same as the eigenvalues of $\Lambda + R$, by (18) and (47) we have

$$\begin{aligned} \mu_k &= \frac{1}{2}\left[2\rho + \lambda_1 + \lambda_2 + (-1)^k I\right] \quad (k = 1,2),\\ \mu_3 &= \lambda_3 + \eta. \end{aligned} \tag{70}$$

Thus by (70) and (66), we see that (69) holds. Since (58) satisfies all the conditions for the asymptotic result [4, Section 2], it follows that, as $x \to \infty$, (58) has three linearly independent solutions

$$W_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t)\,dt\right), \tag{71}$$

where μ_k are given by (70) and e_k are the coordinate vectors with kth component unity and other components zero. Now we transform back to Y by means of (16) and (57), where T_1 in (57) is given by

$$T_1 = \begin{pmatrix} 1 & -1 & 0\\ 1 & 1 & 0\\ \dfrac{\omega}{1+\omega} & 0 & 1 \end{pmatrix}. \tag{72}$$

We obtain

$$Y_k(x) = T(x)T_1W_k(x) \quad (1 \le k \le 3). \tag{73}$$

Now using (9), (32), (33), (34), (71), (72) and (45) in (73) and carrying out the integration of $\frac{(ps^{\frac{-1}{2}})'}{ps^{\frac{-1}{2}}}$ and $(\frac{1}{4})\frac{(rp)'}{rp}$, for $1 \le k \le 3$, we obtain (68). □

6. Discussion

(1) In a familiar case, the coefficients covered by Theorem 5.1 are

$$s(x) = Ax^{\alpha}, \quad p(x) = Bx^{\beta}, \quad r(x) = Cx^{\gamma}, \tag{74}$$

where α, β, γ, $A(\neq 0)$, $B(\neq 0)$ and $C(\neq 0)$ are real constants. Then the Euler case (48)–(49) is given by

$$\alpha - \beta = 1. \tag{75}$$

The values of σ and ω are given by

$$\sigma = \frac{1}{4}\,\frac{(B+\gamma)A}{B}, \quad \omega = \frac{(\beta - \frac{1}{2}\,\alpha)A}{B}\,. \tag{76}$$

Also in this example $\phi(x) = \psi(x) = 0$ in (48) and (49).

(2) Theorem 5.1 coveres also the following class of coefficients

$$s = Ax^{\alpha}e^{x^b}, \quad p = Bx^{\beta}e^{x^b}, \quad r = Cx^{\gamma}e^{\frac{1}{2}x^b}, \tag{77}$$

where α, β, γ, $A(\neq 0)$, $B(\neq 0)$, $C(\neq 0)$ and $b(> 0)$ are real constants. Then the Euler case (48)–(49) is given by

$$b - 1 = \beta - \alpha. \tag{78}$$

The values of σ and ω are given by

$$\sigma = \frac{3}{8}\,\frac{bA}{B}, \quad \omega = \frac{1}{2}\,\frac{bA}{B}\,. \tag{79}$$

Also

$$\phi(x) = \frac{2}{3}\,b^{-1}(\beta+\gamma)x^{-b}, \tag{80}$$

$$\psi(x) = 2b^{-1}\Big(\beta - \frac{1}{2}\,\alpha\Big)x^{-b}. \tag{81}$$

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