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Short Communications

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ON TWO-POINT BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH SINGULAR COEFFICIENTS

Abstract. Two-point boundary value problems for two-dimensional systems of linear differential equations with singular coefficients are considered. The cases are optimally described when the above-mentioned problems have the Fredholm property, and unimprovable in a certain sense conditions are established guaranteeing the unique solvability of those problems.

რეზიუმე. განხილულია ორწერტილოვანი სასაზღვრო ამოცანები წრფივ დიფერენციალურ განტოლებათა ორგანზომილებიანი სისტემებისათვის სინგულარული კოეფიციენტებით. ოპტიმალურადაა აღწერილი შემთხვევები, როცა აღნიშნულ ამოცანებს გააჩნიათ ფრედჰოლმის თვისება, და დადგენილია გარკვეული აზრით არაგაუმჯობესებადი პირობები, რომლებიც უზრუნველყოფენ ამ ამოცანების ცალსახად ამოხსნადობას.

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Boundary value problems for second and higher order linear differential equations, whose coefficients have nonintegrable singularities at the points bearing the boundary data, are investigated in full detail (see, e.g., [1], [2], [5]–[7], [9]–[16] and the references therein).

From the theorems proven by R. P. Agarwal and I. Kiguradze [10] for the second order differential equation

$$u'' = p(t)u + q(t),$$

it follow unimprovable in a certain sense results on the unique solvability of the boundary value problems

$$u(a) = 0, \quad u(b) = 0, \quad \int\limits_a^b u'^2(t)\,dt < +\infty$$

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and

$$u(a) = 0, \quad u'(b) = 0, \quad \int_a^b u'^2(t)\, dt < +\infty.$$

These results cover the cases where the concerned differential equation is strongly singular, more precisely, when the order of singularity of the function $t \to (|p(t)| - p(t))/2$ at the points a and b is equal to 2. In the present paper, the above-mentioned results are generalized for two-dimensional linear differential systems.

By $L_{loc}(]a,b[)$ we denote the space of functions $p :]a,b[\to \mathbb{R}$ Lebesgue integrable in the interval $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$. Analogously, by $L_{loc}(]a,b])$ we denote the space of functions $p :]a,b] \to \mathbb{R}$ Lebesgue integrable in the interval $[a+\varepsilon, b]$ for arbitrarily small $\varepsilon > 0$.

It is clear that the functions from the space $L_{loc}(]a,b[)$ may have nonintegrable singularities at the points a and b. As for the functions from the space $L_{loc}(]a,b])$, they may have nonintegrable singularities only at the point a.

For an arbitrary number x we set

$$[x]_- = \frac{|x| - x}{2}\,.$$

We consider the two-dimensional linear differential system

$$u_i' = p_{i1}(t)u_1 + p_{i2}(t)u_2 + p_{i0}(t) \ \ (i = 1, 2) \tag{1}$$

with locally integrable coefficients $p_{ik} \in L_{loc}(]a,b[)$ $(i = 1, 2;\ k = 0, 1, 2)$.

We do not exclude from consideration the cases where some (or all) of the coefficients of that system are not integrable on $[a,b]$, having singularities at the points a and b. In that sense the system (1) is singular.

It is naturally admitted the possibility that the functions p_{12} and p_{21} be equal to zero on the sets of positive measure. This is the most interesting case since in that case the system (1) cannot be reduced to a second order linear differential equation.

Denote

$$a_0 = \frac{a+b}{2}, \quad r_i(t) = \exp\left(\int_{a_0}^t p_{ii}(s)\, ds\right) \ (i = 1, 2), \quad r(t) = \frac{|p_{12}(t)|}{r_1(t)r_2(t)}\,;$$

$$p_1(t) = \frac{p_{12}(t)r_2(t)}{r_1(t)}\,, \quad p_2(t) = \frac{p_{21}(t)r_1(t)}{r_2(t)}\,; \quad q_i(t) = \frac{p_{i0}(t)}{r_i(t)} \ \ (i = 1, 2).$$

For the system (1) we consider the boundary value problems

$$\lim_{t\to a} \frac{u_1(t)}{r_1(t)} = 0, \quad \lim_{t\to b} \frac{u_1(t)}{r_1(t)} = 0, \quad \int_a^b r(t)u_2^2(t)\, dt < +\infty \tag{2}$$

and

$$\lim_{t\to a}\frac{u_1(t)}{r_1(t)}=0,\quad \lim_{t\to b}\frac{u_2(t)}{r_2(t)}=0,\quad \int\limits_a^b r(t)u_2^2(t)\,dt<+\infty. \tag{3}$$

Note that if the functions p_{11} and p_{22} are integrable on $[a,b]$, then the conditions (2) and (3), respectively, are equivalent to the conditions

$$u_1(a)=0,\quad u_1(b)=0,\quad \int\limits_a^b |p_{12}(t)|u_2^2(t)\,dt<+\infty$$

and

$$u_1(a)=0,\quad u_2(b)=0,\quad \int\limits_a^b |p_{12}(t)|u_2^2(t)\,dt<+\infty,$$

where by $u_i(a)$ and $u_i(b)$ it is understood, respectively, the right and the left limits of the function u_i at the points a and b.

Both the problems (1), (2) and (1), (3) we investigate in the case where the condition

$$0\le\sigma p_1(t)\le\ell_0 \ \text{ for } \ a<t<b,\quad \int\limits_a^b |p_1(t)|\,dt>0 \tag{4}$$

is satisfied. Here $\sigma\in\{-1,1\}$ and ℓ_0 is a positive number.

Along with (1) we consider the corresponding homogeneous differential system

$$u_i'=p_{i1}(t)u_1+p_{i2}(t)u_2 \ \ (i=1,2), \tag{1_0}$$

and we introduce

Definition 1. We say that the problem (1), (2) has *the Fredholm property* if the unique solvability of the corresponding homogeneous problem (1_0), (2) guarantees the unique solvability of the problem (1), (2) for any $p_{i0}\in L_{loc}(]a,b[)$ $(i=1,2)$ satisfying the conditions

$$q_1\in L([a,b]),\quad \int\limits_a^b (t-a)(b-t)\bigg(p_2(t)\int\limits_a^t |q_1(s)|\,ds\int\limits_t^b |q_1(s)|\,ds\bigg)^2 dt<+\infty; \tag{5}$$

$$\int\limits_a^b |p_1(t)|\,\bigg|\int\limits_{a_0}^t q_2(s)\,ds\bigg|^2 dt<+\infty. \tag{6}$$

The following theorem is valid.

Theorem 1. *If along with* (4) *the inequalities*

$$
\begin{aligned}
&\limsup_{t\to a}\bigg((t-a)\int\limits_t^{a_0}[\sigma p_2(s)]_-\,ds\bigg)<\frac{1}{4\ell_0}\,,\\
&\limsup_{t\to b}\bigg((b-t)\int\limits_{a_0}^{t}[\sigma p_2(s)]_-\,ds\bigg)<\frac{1}{4\ell_0}
\end{aligned}
\tag{7}
$$

are fulfilled, then the problem (1), (2) *has the Fredholm property.*

From this theorem it follows

Corollary 1. *If along with* (4) *the inequalities*

$$
\liminf_{t\to a}\Big(\sigma(t-a)^2p_2(t)\Big)>-\frac{1}{4\ell_0}\,,\quad \liminf_{t\to b}\Big(\sigma(b-t)^2p_2(t)\Big)>-\frac{1}{4\ell_0}
\tag{8}
$$

are fulfilled, then the problem (1), (2) *has the Fredholm property.*

On the basis of Theorem 1 the following theorem can be proved.

Theorem 2. *Let along with* (4) *the inequality*

$$
\bigg|\int\limits_{a_0}^{t}[\sigma p_2(s)]_-\,ds\bigg|\leq\frac{\ell(b-a)}{(t-a)(b-t)}\quad \textit{for}\ \ a<t<b
$$

be fulfilled, where ℓ is a non-negative constant such that

$$
\ell<\frac{1}{4\ell_0}\,.
\tag{9}
$$

If, moreover, the conditions (5) *and* (6) *are satisfied, then the problem* (1), (2) *has a unique solution.*

Theorem 2 yields

Corollary 2. *Let along with* (4) *the inequality*

$$
\sigma p_2(t)\geq-\ell\Big(\frac{1}{(t-a)^2}+\frac{1}{(b-t)^2}\Big)\quad \textit{for}\ \ a<t<b
$$

be fulfilled, where ℓ is a non-negative constant, satisfying the inequality (9). *If, moreover, the conditions* (5) *and* (6) *are satisfied, then the problem* (1), (2) *has a unique solution.*

Note that the conditions of Theorems 1 and 2 as well as the conditions of Corollary 1 and 2 are unimprovable. More precisely, none of the strict inequalities (7) and (8) can be replaced by the non-strict ones, and the inequality (9) cannot be replaced by the equality

$$
\ell=\frac{1}{4\ell_0}\,.
$$

As an example, we consider the differential system

$$u_1' = g_1(t)u_2 + (t-a)^\alpha (b-t)^\alpha g_{10}(t),$$
$$u_2' = \Big(\frac{g_2(t)}{(t-a)^\beta (b-t)^\beta} - \frac{\ell}{(t-a)^2} - \frac{\ell}{(b-t)^2}\Big)u_1 + \frac{g_{20}(t)}{(t-a)^\gamma (b-t)^\gamma}\,, \tag{10}$$

where $g_i : [a,b] \to [0,+\infty[$ and $g_{i0} : [a,b] \to \mathbb{R}$ $(i = 1,2)$ are continuous functions, and α, β, γ, and ℓ are positive constants. Moreover, $g_1(t) \not\equiv 0$ and

$$0 \le g_1(t) \le \Big(\frac{t-a}{b-a}\Big)^\lambda \Big(\frac{b-t}{b-a}\Big)^\lambda \quad \text{for } \ a < t < b,$$

where $\lambda > 0$.

The system (10), generally speaking, cannot be reduced to a second order linear differential equation since the restrictions, imposed on the functions g_1 and g_2, do not exclude, for example, the cases where

$$g_1(t) = g_2(t) = 0 \quad \text{for } \ t \in I =$$
$$= \bigcup_{k=1}^{\infty} \Big[a + \frac{b-a}{4k+1}, a + \frac{b-a}{4k}\Big] \bigcup \Big[b - \frac{b-a}{4k}, b - \frac{b-a}{4k+1}\Big],$$
$$\text{and } \ g_1(t) > 0, \ \ g_2(t) > 0 \ \text{ for } \ t \in [a,b] \setminus I.$$

From Corollary 2 it follows

Corollary 3. *If*

$$\ell < \frac{1}{4}\,, \ \ \alpha > 0, \ \ \beta < 2 + \alpha, \ \ \textit{and } \ \gamma < \frac{3+\lambda}{2}\,, \tag{11}$$

then the system (10) *has a unique solution satisfying the conditions*

$$u_1(a) = 0, \ \ u_1(b) = 0, \ \ \int_a^b g_1(t) u_2^2(t)\, dt < +\infty.$$

According to Corollary 3, the second equation in the system (10) may have the singularity of an arbitrary order. More precisely, β and γ may be arbitrarily large numbers if α and λ are also large.

Note that Corollary 3 does not follow from the previous well-known results on the unique solvability of two-point boundary value problems for linear differential systems (see [3], [4], [8], [17]).

Now we consider the problem (1), (3). First of all we introduce

Definition 2. We say that the problem (1), (3) has *the Fredholm property* if the unique solvability of the corresponding homogeneous problem (1_0), (3) guarantees the unique solvability of the problem (1), (3) for any

$p_{i0} \in L_{loc}(]a,b[)$ $(i=1,2)$ satisfying the conditions

$$q_1 \in L([a,b]), \quad \int\limits_a^b (t-a)\bigg(p_2(t)\int\limits_a^t |q_1(s)|\,ds\bigg)^2 dt < +\infty, \tag{12}$$

$$q_2 \in L_{loc}(]a,b]), \quad \int\limits_a^b |p_1(t)|\,\bigg|\int\limits_t^b q_2(s)\,ds\bigg|^2 dt < +\infty. \tag{13}$$

The following theorem is valid.

Theorem 3. *Let $p_2 \in L_{loc}(]a,b])$, and let along with* (4) *the inequality*

$$\limsup_{t\to a}\bigg(\sigma(t-a)\int\limits_t^b [\sigma p_2(s)]_-\,ds\bigg) < \frac{1}{4\ell_0} \tag{14}$$

be fulfilled. Then the problem (1), (3) *has the Fredholm property.*

Corollary 4. *Let $p_2 \in L_{loc}(]a,b])$, and let along with* (4) *the inequality*

$$\liminf_{t\to a}\big(\sigma(t-a)^2 p_2(t)\big) > -\frac{1}{4\ell_0} \tag{15}$$

be fulfilled. Then the problem (1), (3) *has the Fredholm property.*

Theorem 4. *Let $p_2 \in L_{loc}(]a,b])$, and let along with* (4) *the inequality*

$$\int\limits_t^b [\sigma p_2(s)]_-\,ds \le \frac{\ell}{t-a} \;\; \text{for } a<t<b, \;\; \text{where } \ell < \frac{1}{4\ell_0}\,, \tag{16}$$

be fulfilled. If, moreover, the conditions (12) *and* (13) *are satisfied, then the problem* (1), (3) *has a unique solution.*

Corollary 5. *Let $p_2 \in L_{loc}(]a,b])$, and let along with* (4) *the inequality*

$$\sigma p_2(t) \ge -\frac{\ell}{(t-a)^2} \;\; \text{for } a<t<b, \;\; \text{where } \ell < \frac{1}{4\ell_0}\,, \tag{17}$$

be fulfilled. If, moreover, the conditions (12) *and* (13) *are satisfied, then the problem* (1), (3) *has a unique solution.*

Note that the conditions (14)–(17) in Theorems 3, 4 and Corollaries 4, 5 are unimprovable.

As an example, we consider the differential system

$$\begin{aligned} u_1' &= g_1(t)u_2 + (t-a)^\alpha g_{10}(t),\\ u_2' &= \Big(\frac{g_2(t)}{(t-a)^\beta} - \frac{\ell}{(t-a)^2}\Big)u_1 + \frac{g_{20}(t)}{(t-a)^\gamma}\,, \end{aligned} \tag{18}$$

where $g_i : [a,b] \to [0,+\infty[$ and $g_{i0} : [a,b] \to \mathbb{R}$ $(i=1,2)$ are continuous functions, α, β, γ, and ℓ are positive constants. Moreover, $g_1(t) \not\equiv 0$ and

$$0 \le g_1(t) \le \Big(\frac{t-a}{b-a}\Big)^\lambda \;\; \text{for } a<t<b,$$

where $\lambda > 0$.

From Corollary 5 it follows

Corollary 6. *If the condition* (11) *is fulfilled, then the system* (18) *has a unique solution satisfying the conditions*

$$u_1(a) = 0, \quad u_2(b) = 0, \quad \int\limits_a^b g_1(t) u_2^2(t)\, dt < +\infty.$$

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