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SOLUTION OF THE BASIC CONTACT PROBLEM OF STATICS OF ELASTIC MIXTURES

Abstract. A finite domain D_1 and an infinite domain D_0 are considered with the common boundary S having Hölder continuous curvature. D_1 and D_0 are filled with isotropic elastic mixtures. In D_1 and D_0 $u^{(1)}$ and $u^{(0)}$ are displacement vectors while $T^{(1)}u^{(1)}$ and $T^{(2)}u^{(2)}$ are stress vectors. The main contact problem considered in the paper may be formulated as follows: in the domains D_1 and D_0, find regular vectors $u^{(1)}$ and $u^{(0)}$ satisfying on the boundary S the conditions

$$(u^{(1)})^+ - (u^{(0)})^- = f,$$
$$(T^{(1)}u^{(1)})^+ - (T^{(2)}u^{(2)})^- = F,$$

where f and F are given vectors. A uniqueness theorem is proved for this problem. A Fredholm system of integral equations is derived for the problem. An existence theorem is proved for the main contact problem via investigation of the latter system.

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Key words and phrases. Simple and double layer potentials, Fredholm system of integral equations of second kind.

რეზიუმე. განიხილება სასრული D_1 და უსასრულო D_0 არეები, რომელთა საერთო საზღვარს, S წირს, აქვს ჰელდერის აზრით უწყვეტი სიმრუდე. D_1 და D_0 შევსებული არიან იზოტროპული დრეკადი ნარევებით. D_1 და D_0 არეებში გადაადგილების ვექტორები არიან $u^{(1)}$ და $u^{(0)}$, ხოლო ძაბვის ვექტორები – $T^{(1)}u^{(1)}$ და $T^{(2)}u^{(2)}$. ძირითადი საკონტაქტო ამოცანა, რომელსაც ამ ნაშრომში ვიხილავთ შეიძლება ჩამოყალიბდეს შემდეგნაირად: მოიძებნოს D_1 და D_0 არეებში რეგულარული ვექტორები $u^{(1)}$ და $u^{(0)}$ ისე, რომ მათ S საზღვარზე დააკმაყოფილოს პირობები

$$(u^{(1)})^+ - (u^{(0)})^- = f,$$
$$(T^{(1)}u^{(1)})^+ - (T^{(2)}u^{(2)})^- = F,$$

სადაც f და F მოცემული ვექტორებია. მტკიცდება ამ ამოცანის ამონახსნის ერთადერთობის თეორემა. ამ ამოცანისათვის შედგენილია ფრედჰოლმის ინტეგრალურ განტოლებათა სისტემა. უკანასკნელი სისტემის გამოკვლევით დამტკიცებულია ძირითადი საკონტაქტო ამოცანის ამონახსნის არსებობის თეორემა.

1. Main Equations. Basic Contact Problem. The Uniqueness of a Solution

The main homogeneous equations of statics of an elastic mixture are of the form [1]:

$$Cu = 0, \tag{1.1}$$

where

$$C = \begin{bmatrix} C^{(1)} & C^{(2)} \\ C^{(3)} & C^{(4)} \end{bmatrix}, \quad C^{(1)} = [c^{(1)}_{kj}], \quad j = 1,4,$$

$$c^{(1)}_{kj} = a_1 \Delta \delta_{kj} + b_1 \frac{\partial^2}{\partial x_k \partial x_j}, \quad c^{(2)}_{kj} = c^{(3)}_{kj} = c\Delta\delta_{kj} + d \frac{\partial^2}{\partial x_k \partial x_j}, \tag{1.2}$$

$$c^{(4)}_{kj} = a_2 \Delta \delta_{kj} + b_2 \frac{\partial^2}{\partial x_k \partial x_j},$$

The constants appearing in (2.1) have the following values:

$$\begin{gathered} a_1 = \mu_1 - \lambda_5, \;\; a_2 = \mu_2 - \lambda_5, \;\; c = \mu_3 + \lambda_5, \\ b_1 = \mu_1 + \lambda_1 + \lambda_5 - \rho^{-1}\alpha_2\rho_2, \;\; b_2 = \mu_2 + \lambda_2 + \lambda_5 + \rho^{-1}\alpha_2\rho_1, \\ d = \mu_3 + \lambda_3 - \lambda_5 - \rho^{-1}\alpha_2\rho_1 \equiv \mu_3 + \lambda_4 - \lambda_5 + \rho^{-1}\alpha_2\rho_2, \\ \rho = \rho_1 + \rho_2, \;\; \alpha_2 = \lambda_3 - \lambda_4, \end{gathered} \tag{1.3}$$

where μ_1, μ_2, μ_3, λ_1, λ_2, λ_3, λ_4, λ_5 are constants characterizing physical properties of the elastic mixture and satisfying certain inequalities. The vector u is four-dimensional: $u = (u_1, u_2, u_3, u_4)$. The stress vector is defined as follows [1]:

$$\begin{aligned} (Tu)_1 &= \tau'_{11} n_1 + \tau'_{21} n_2, \quad (Tu)_2 = \tau'_{12} n_1 + \tau'_{22} n_2, \\ (Tu)_3 &= \tau''_{11} n_1 + \tau''_{21} n_2, \quad (Tu)_4 = \tau''_{12} n_1 + \tau''_{22} n_2, \end{aligned} \tag{1.4}$$

and the generalized stress vector has the form [1]

$$\begin{aligned} (\overset{\varkappa}{T}u)_1 &= \sigma'_{11} n_1 + \sigma'_{21} n_2, \quad (\overset{\varkappa}{T}u)_2 = \sigma'_{12} n_1 + \sigma'_{22} n_2, \\ (\overset{\varkappa}{T}u)_3 &= \sigma''_{11} n_1 + \sigma''_{21} n_2, \quad (\overset{\varkappa}{T}u)_4 = \sigma''_{12} n_1 + \sigma''_{22} n_2, \end{aligned} \tag{1.5}$$

where

$$\begin{gathered} \sigma'_{11} = L_1 + \frac{\partial M_2}{\partial x_2}, \quad \sigma'_{21} = -L_1 - \frac{\partial M_2}{\partial x_1}, \\ \sigma'_{12} = L_2 - \frac{\partial M_1}{\partial x_2}, \quad \sigma'_{22} = L_1 + \frac{\partial M_1}{\partial x_1}, \\ \sigma''_{11} = L_3 + \frac{\partial M_4}{\partial x_2}, \quad \sigma''_{21} = -L_4 - \frac{\partial M_4}{\partial x_1}, \\ \sigma''_{12} = L_4 - \frac{\partial M_3}{\partial x_2}, \quad \sigma''_{22} = L_3 + \frac{\partial M_3}{\partial x_1}, \end{gathered} \tag{1.6}$$

$$L_1 = (a_1 + b_1)\theta' + (c + d)\theta'', \quad L_2 = a_1\omega' + c\omega'',$$

$$L_3 = (c + d)\theta' + (a_2 + b_2)\theta'', \quad L_4 = c\omega' + a_2\omega'',$$

$$\begin{aligned} M_1 &= (\varkappa_1 - 2\mu_1)u_1 + (\varkappa_3 - 2\mu_3)u_3, \\ M_2 &= (\varkappa_1 - 2\mu_1)u_2 + (\varkappa_3 - 2\mu_3)u_4, \\ M_3 &= (\varkappa_3 - 2\mu_3)u_1 + (\varkappa_2 - 2\mu_2)u_3, \\ M_4 &= (\varkappa_3 - 2\mu_3)u_2 + (\varkappa_2 - 2\mu_2)u_4, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \theta' &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\,, \quad \theta'' = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_4}{\partial x_2}\,, \\ \omega' &= \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\,, \quad \omega'' = \frac{\partial u_4}{\partial x_1} - \frac{\partial u_3}{\partial x_2}\,, \end{aligned} \tag{1.8}$$

$\varkappa$ is a real constant matrix

$$\varkappa = \begin{bmatrix} 0 & \varkappa_1 & 0 & \varkappa_3 \\ -\varkappa_1 & 0 & -\varkappa_3 & 0 \\ 0 & \varkappa_3 & 0 & \varkappa_2 \\ -\varkappa_3 & 0 & -\varkappa_2 & 0 \end{bmatrix} \tag{1.9}$$

where $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ can take arbitrary real values. We point out some of them. If $\varkappa_1 = 2\mu_1$, $\varkappa_2 = 2\mu_2$, $\varkappa_3 = 2\mu_3$, then $\varkappa = \varkappa_L$. Taking into account (1.5) and (1.4), we have

$$\overset{\varkappa}{T}u = Tu + \varkappa\,\frac{\partial u}{\partial s(x)}\,, \tag{1.10}$$

where

$$\frac{\partial}{\partial s(x)} = n_1\,\frac{\partial}{\partial x_2} - n_2\,\frac{\partial}{\partial x_1}\,. \tag{1.11}$$

If $\varkappa$ assumes the above mentioned value, then $\overset{\varkappa}{T} \equiv L$, and from (1.10) we have

$$Lu = Tu + \varkappa_L\,\frac{\partial u}{\partial s(x)}\,. \tag{1.12}$$

If $\varkappa_N = \varkappa_L - m^{-1}E_1$, where

$$\begin{gathered} m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3 & 0 & -m_2 & 0 \\ 0 & m_3 & 0 & -m_2 \\ -m_2 & 0 & m_1 & 0 \\ 0 & -m_2 & 0 & m_1 \end{bmatrix}, \\ E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \Delta_0 = m_1m_3 - m_2^2 > 0, \end{gathered} \tag{1.13}$$

then we obtain

$$Nu = Tu + \varkappa_N\,\frac{\partial u}{\partial s(x)}\,. \tag{1.14}$$

Let D_1^+ be a finite domain in the plane E_2. Then $\overline{D}_1^+ = D_1 \cup S$, where S is a closed curve of continuous Hölder curvature. Denote $D_0 = E_2 - \overline{D}_1^+$. Obviously, D_0^- is an infinite domain bounded by the curve S. Find now a regular solution of the equation (1.1) [2].

Definition. The vector u is a regular solution of the equation (1.1) in the domain D_1^+ if it and its partial first order derivatives are continuous vectors up to the boundary S, and the second derivatives exist in D_1^+ and satisfy the equation (1.1).

In the domain D_0^-, to the above-mentioned conditions we add one which is fulfilled at infinity:

$$U = O(1), \quad \frac{\partial U}{\partial x_k} = O(\rho^{-2}), \quad \rho^2 = x_1^2 + x_2^2, \ \ k = 1, 2. \tag{1.15}$$

The basic contact problem can be formulated as follows: find regular solutions in D_1^+ and D_0^- of the equation (1.1) under the conditions

$$\left.\begin{aligned} (U^{(1)}(t))^+ - (U^{(0)}(t)) &= f(t), \\ (T^{(1)}U^{(1)}(t))^+ - (T^{(0)}U^{(0)}(t))^- &= F(t) \end{aligned}\right\} \quad t \in S; \tag{1.16}$$

here f and F are with a definite smoothness vectors given on the boundary. The signs $+$ and $-$ refer, respectively, to interior and exterior boundary values. Obviously, the constants take in D_1^+ and D_0^- different values. Therefore in the domain D_j^+, $j = 0, 1$, we supply the constants with an appropriate index j, $j = 0, 1$. The same refers to the vectors $u^{(1)}$ and $u^{(0)}$.

Let us prove a theorem of uniqueness of solution of the basic contact problem. Towards this end, we will need the following Green's formulas [1]:

$$\begin{aligned} \int\limits_{D_1^+} E^{(1)}(u^{(1)}, u^{(1)})\, d\sigma &= \int\limits_S u^{(1)} T^{(1)} u^{(1)}\, ds, \\ \int\limits_{D_0^+} E^{(0)}(u^{(0)}, u^{(0)})\, d\sigma &= -\int\limits_S u^{(0)} T^{(0)} u^{(0)}\, ds, \end{aligned} \tag{1.17}$$

where $E(u, u)$ is the doubled potential energy:

$$\begin{aligned} E^{(k)}(u^{(k)}, u^{(k)}) &= (b_1^{(k)} - \lambda_5^{(k)})\Big(\frac{\partial u_1^{(k)}}{\partial x_1} + \frac{\partial u_2^{(k)}}{\partial x_2}\Big)^2 + \\ &+ 2(d^{(k)} + \lambda_5^{(k)})\Big(\frac{\partial u_1^{(k)}}{\partial x_1} + \frac{\partial u_2^{(k)}}{\partial x_2}\Big)\Big(\frac{\partial u_3^{(k)}}{\partial x_1} + \frac{\partial u_4^{(k)}}{\partial x_2}\Big) + \\ &+ (b_2^{(k)} - \lambda_5^{(k)})\Big(\frac{\partial u_2^{(k)}}{\partial x_1} + \frac{\partial u_4^{(k)}}{\partial x_2}\Big)^2 + \\ &+ \mu_1^{(k)}\Big[\Big(\frac{\partial u_1^{(k)}}{\partial x_1} - \frac{\partial u_2^{(k)}}{\partial x_2}\Big)^2 + \Big(\frac{\partial u_2^{(k)}}{\partial x_1} + \frac{\partial u_1^{(k)}}{\partial x_2}\Big)^2\Big] + \\ &+ 2\mu_3^{(k)}\Big[\Big(\frac{\partial u_1^{(k)}}{\partial x_1} - \frac{\partial u_2^{(k)}}{\partial x_2}\Big)\Big(\frac{\partial u_3^{(k)}}{\partial x_1} - \frac{\partial u_4^{(k)}}{\partial x_2}\Big) + \Big(\frac{\partial u_2^{(k)}}{\partial x_1} + \frac{\partial u_2^{(k)}}{\partial x_2}\Big)\Big(\frac{\partial u_4^{(k)}}{\partial x_1} + \frac{\partial u_3^{(k)}}{\partial x_2}\Big)\Big] + \\ &+ \mu_2^{(k)}\Big[\Big(\frac{\partial u_3^{(k)}}{\partial x_1} - \frac{\partial u_4^{(k)}}{\partial x_2}\Big)^2 + \Big(\frac{\partial u_4^{(k)}}{\partial x_1} + \frac{\partial u_3^{(k)}}{\partial x_2}\Big)^2\Big] - \end{aligned}$$

$$- \lambda_5^{(k)} \Big[\Big(\frac{\partial u_3^{(k)}}{\partial x_1} - \frac{\partial u_1^{(k)}}{\partial x_2} \Big)^2 - \Big(\frac{\partial u_4^{(k)}}{\partial x_1} - \frac{\partial u_3^{(k)}}{\partial x_2} \Big)^2 \Big]. \quad (1.18)$$

Theorem. *A regular solution of the basic contact problem of the equation* (1.1) *satisfying the boundary conditions on the boundary S is zero.*

Proof. In (1.16), since $f = F = 0$, we have $(u^{(1)})^+ = (u^{(0)})^-$, $(T^{(1)}u^{(1)})^+ = (T^{(0)}u^{(0)})^-$, and taking into account (1.17), we find that

$$\int\limits_{D_1^+} E^{(1)}(u^{(1)}, u^{(1)})\, d\sigma + \int\limits_{D_0^-} E^{(0)}(u^{(0)}, u^{(0)})\, d\sigma = 0,$$

whence

$$U^{(k)} = C^{(k)} + \mathcal{E}^{(k)} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad k = 0, 1,$$

where $C^{(k)} = \begin{pmatrix} c_1^{(k)} \\ c_2^{(k)} \end{pmatrix}$, and $c_1^{(k)}$, $c_2^{(k)}$ and $\mathcal{E}^{(k)}$ are arbitrary real constants.

In our case, $\omega^{(1)} = \omega^{(0)}$, and therefore $\mathcal{E}^{(k)} = \mathcal{E}^{(0)}$. Moreover, $u^{(1)}(t) = u^{(0)}(t)$. It is required that $u^{(1)}(0) = 0$. (Note that the origin is in the domain D_1^+). Taking into account the fact that $u^{(1)}(x)$ and $u^{(0)}(x)$ are vectors continuous up to the boundary S, we obtain $c_1^{(1)} = 0$, $c_0^{(0)} = 0$, $\mathcal{E}^{(1)} = \mathcal{E}^{(0)} = 0$. Thus we have found that

$$u^{(1)}(x) = 0, \ \ x \in D_1^+, \quad u^{(0)}(x) = 0, \ \ x \in D_0^+. \quad (1.19)$$

Hence the proof of the theorem is complete. □

2. Integral Equations of the Basic Contact Problem

In this section, for the basic contact problem we will write the integral Fredholm equations of the second kind.

The solutions $u^{(1)}(x)$ and $u^{(0)}(x)$ are sought in the form

$$\begin{aligned} u^{(1)}(x) &= \frac{1}{\pi} \int\limits_S \operatorname{Re} \Big\{ \Big[(N^{(1)}\Gamma^{(1)})' X^{(1)} + \frac{\partial \Gamma^{(1)}}{\partial s(y)} Y^{(1)} \Big] g + \Gamma^{(1)} Z^{(1)} h \Big\}\, ds, \\ & \qquad x \in D_1^+, \\ u^{(0)}(x) &= \frac{1}{\pi} \int\limits_S \operatorname{Re} \Big\{ \Big[(N^{(0)}\Gamma^{(0)})' X^{(0)} + \frac{\partial \Gamma^{(0)}}{\partial s(y)} Y^{(0)} \Big] g + \Gamma^{(0)} Z^{(0)} h \Big\}\, ds, \\ & \qquad x \in D_0^-, \end{aligned} \quad (2.1)$$

where g and h are unknown real vectors which are determined from the boundary condition, and

$$\Gamma(y - x) = m \ln \sigma + \frac{n}{4} \frac{\overline{\sigma}}{\sigma}, \quad \sigma = (x_1 - y_1) + i(x_2 - y_2). \quad (2.2)$$

In this formula instead of $\ln\sigma$ we write $\ln\frac{\zeta-z}{\zeta}$ in the domain D_1^+, while in the domain D_0^- we write $\ln\frac{z-\zeta}{z}$, where $z=x_1+ix_2$, $\zeta=y_1+iy_2$. This remark is made due to the fact that the vectors $u^{(1)}(x)$ and $u^{(0)}(x)$ are continuous up to the boundary S,

$$m=\begin{bmatrix} m_1 & 0 & m_2 & 0\\ 0 & m_1 & 0 & m_2\\ m_2 & 0 & m_3 & 0\\ 0 & m_2 & 0 & m_3\end{bmatrix},\quad n=\begin{bmatrix} l_4 & il_4 & l_5 & il_5\\ il_4 & -l_4 & -il_5 & -l_5\\ l_5 & il_5 & l_6 & il_6\\ il_5 & -l_5 & il_6 & -l_6\end{bmatrix}, \tag{2.3}$$

$$\begin{gathered} l_1+l_4=\frac{a_2+b_2}{d_1}\,,\quad l_2+l_5=-\frac{e+d}{d_1}\,,\quad l_3+l_6=\frac{a_1+b_1}{d_1}\,,\\ e_2=-\frac{c}{d_2},\quad e_1=\frac{a_2}{d_2}\,,\quad e_3=\frac{a_1}{d_2}\\ d_1=(a_1+b_1)(a_2+b_2)-(c+d)^2>0,\quad d_2=a_1a_2-c^2>0. \end{gathered} \tag{2.4}$$

Thus we obtain

$$(N_y\,\mathrm{Re}\,\Gamma)'=E\,\frac{\partial\theta}{\partial s(y)}\,,\quad (N_y\,\mathrm{Im}\,\Gamma)'=-E\,\frac{\partial\ln r}{\partial s(y)}\,. \tag{2.5}$$

Taking into account (2.3) and (2.5), we rewrite (2.1) in the form

$$\begin{gathered} u^{(1)}(x)=\frac{1}{\pi}\int\limits_S\bigg\{\Big[\frac{\partial\theta}{\partial s(y)}\,X^{(1)}+m^{(1)}Y^{(1)}\,\frac{\partial\ln r}{\partial s(y)}\Big]g+\Gamma^{(1)}Z^{(1)}h\bigg\}\,ds,\\ x\in D_1^+,\\ u^{(0)}(x)=\frac{1}{\pi}\int\limits_S\bigg\{\Big[\frac{\partial\theta}{\partial s(y)}\,X^{(0)}+m^{(0)}Y^{(0)}\,\frac{\partial\ln r}{\partial s(y)}\Big]g+\Gamma^{(0)}Z^{(0)}h\bigg\}\,ds,\\ u\in D_0^+. \end{gathered} \tag{2.6}$$

Passing to limit as $x\to t\in S$, for finding unknown matrices X, Y, Z we obtain the equations (which will be written below):

$$\begin{aligned} (u^{(1)}(t))^+ &= X^{(1)}g+\\ &+\frac{1}{\pi}\int\limits_S\bigg\{\Big[\frac{\partial\theta}{\partial s(y)}\,X^{(1)}+m^{(1)}Y^{(1)}\,\frac{\partial\ln r}{\partial s(y)}\Big]g+\Gamma^{(1)}Z^{(1)}h\bigg\}\,ds,\\ (u^{(0)}(t))^- &= -X^{(0)}g+\\ &+\frac{1}{\pi}\int\limits_S\bigg\{\Big[\frac{\partial\theta}{\partial s(y)}\,X^{(0)}+m^{(0)}Y^{(0)}\,\frac{\partial\ln r}{\partial s(y)}\Big]g+\Gamma^{(0)}Z^{(0)}h\bigg\}\,ds, \end{aligned}$$

whence

$$\begin{aligned} &(u^{(1)}(t))^+-(u^{(0)}(t))^-=(X^{(1)}+X^{(0)})g+\\ &\quad+\frac{1}{\pi}\int\limits_S\bigg\{\Big[\frac{\partial\theta}{\partial s(y)}\,(X^{(1)}-X^{(0)})+\big(m^{(1)}Y^{(1)}-m^{(0)}Y^{(0)}\big)\,\frac{\partial\ln r}{\partial s(y)}\Big]g+ \end{aligned}$$

$$+\big(\Gamma^{(1)}Z^{(1)}-\Gamma^{(0)}Z^{(0)}\big)h\Big\}\,ds=f(t). \quad (2.7)$$

From (2.7), we adopt the following restrictions:

$$X^{(1)}+X^{(0)}=E, \quad m^{(1)}Y^{(1)}-m^{(0)}Y^{(0)}=0. \quad (2.8)$$

Under these conditions, (2.7) is a Fredholm equation of second kind of the form

$$(u^{(1)}(t))^{+}-(u^{(0)}(t))^{-}=$$
$$=g+\frac{1}{\pi}\int\limits_{S}\Big[\frac{\partial\theta}{\partial s(y)}\,(X^{(1)}-X^{(0)})g+\big(\Gamma^{(1)}Z^{(1)}-\Gamma^{(0)}Z^{(0)}\big)h\Big]\,ds=f(t). \quad (2.9)$$

Thus we have obtained one equation for finding the unknown vectors g and h. The second equation will be written below.

We now calculate $T^{(1)}u^{(1)}$ and $T^{(0)}u^{(0)}$ from (2.1). It should be noted that

$$T_x^{(1)}(N_y\Gamma)'=-i(E-i\varkappa_N m)\,\frac{\partial^2\ln\sigma}{\partial s(x)\partial s(y)}\,,$$
$$T_x^{(1)}\operatorname{Re}\frac{\partial\Gamma}{\partial s(y)}=-E\,\frac{\partial^2\theta}{\partial s(x)\partial s(y)}-\varkappa_N m\,\frac{\partial^2\ln r}{\partial s(x)\partial s(y)}\,,$$
$$T_x^{(1)}\operatorname{Re}\Gamma=-E\,\frac{\partial\theta}{\partial s(x)}+\varkappa_N m\,\frac{\partial\ln r}{\partial s(x)}\,.$$

Using the above formulas and performing partial integration with respect to the vector g, we obtain

$$T^{(1)}u^{(1)}(x)=\frac{1}{\pi}\int\limits_{S}\Big\{\Big[E\,\frac{\partial\theta}{\partial s(x)}\,X^{(1)}+\varkappa_N^{(1)}m^{(1)}X^{(1)}\,\frac{\partial\ln r}{\partial s(x)}+$$
$$+E\,\frac{\partial\theta}{\partial s(x)}\,Y^{(1)}+\varkappa_N^{(1)}m^{(1)}Y^{(1)}\Big]\,\frac{\partial\ln r}{\partial s(x)}\,\frac{\partial g}{\partial s(y)}+$$
$$+\Big[Z^{(1)}\,\frac{\partial\theta}{\partial s(x)}+\varkappa_N^{(1)}m^{(1)}Z^{(1)}\,\frac{\partial\ln r}{\partial s(x)}\Big]h\Big\}\,ds, \quad x\in D_1^{+},$$
$$T^{(0)}u^{(0)}(x)=\frac{1}{\pi}\int\limits_{S}\Big\{\Big[E\,\frac{\partial\theta}{\partial s(x)}\,X^{(0)}+\varkappa_N^{(0)}m^{(0)}X^{(0)}\,\frac{\partial\ln r}{\partial s(x)}+$$
$$+E\,\frac{\partial\theta}{\partial s(x)}\,Y^{(0)}+\varkappa_N^{(0)}m^{(0)}Y^{(0)}\,\frac{\partial\ln r}{\partial s(x)}\Big]\,\frac{\partial g}{\partial s(y)}+$$
$$+\Big[Z^{(0)}\,\frac{\partial\theta}{\partial s(x)}+\varkappa_N^{(0)}m^{(0)}Z^{(0)}\,\frac{\partial\ln r}{\partial s(x)}\Big]h\Big\}\,ds, \quad x\in D_0^{+}.$$

Passing to limit as $x\to t\in S$, we find that

$$\big(T^{(1)}u^{(1)}(t)\big)^{+}=-(X^{(1)}+Y^{(1)})\,\frac{\partial g}{\partial s(t)}+Z^{(1)}h+$$

$$+\frac{1}{\pi}\int\limits_S\bigg\{\Big[(X^{(1)}+Y^{(1)})\frac{\partial\theta}{\partial s(x)}+\big(\varkappa_N^{(1)}m^{(1)}X^{(1)}+\varkappa_N^{(1)}m^{(1)}Y^{(1)}\big)\,\frac{\partial\ln r}{\partial s(x)}\Big]\,\frac{\partial g}{\partial s(x)}+$$

$$+\Big[Z^{(1)}\,\frac{\partial\theta}{\partial s(x)}+\varkappa_N^{(1)}m^{(1)}\,\frac{\partial\ln r}{\partial s(x)}\,Z^{(1)}\Big]h\bigg\}\,ds,$$

$$\big(T^{(0)}u^{(0)}(t)\big)^-=(X^{(0)}+Y^{(0)})\,\frac{\partial g}{\partial s(t)}+Z^{(0)}h+$$

$$+\frac{1}{\pi}\int\limits_S\bigg\{\Big[(X^{(0)}+Y^{(0)})\frac{\partial\theta}{\partial s(x)}+\big(\varkappa_N^{(0)}m^{(0)}X^{(0)}+\varkappa_N^{(0)}m^{(0)}Y^{(0)}\big)\,\frac{\partial\ln r}{\partial s(x)}\Big]\,\frac{\partial g}{\partial s(x)}+$$

$$+\Big[Z^{(0)}\,\frac{\partial\theta}{\partial s(x)}+\varkappa_N^{(0)}m^{(0)}\,\frac{\partial\ln r}{\partial s(x)}\,Z^{(0)}\Big]h\bigg\}\,ds.$$

Thus we have

$$\big(T^{(1)}u^{(1)}(t)\big)^+-\big(T^{(0)}u^{(0)}(t)\big)^-=$$

$$=-\big(X^{(1)}+X^{(0)}+Y^{(0)}+Y^{(1)}\big)\,\frac{\partial g}{\partial s(x)}+\big(Z^{(1)}+Z^{(0)}\big)h+$$

$$+\frac{1}{\pi}\int\limits_S\Big[2(X^{(1)}+Y^{(1)})\,\frac{\partial\theta}{\partial s(x)}\,\frac{\partial g}{\partial s(y)}+(Z^{(1)}-Z^{(0)})\,\frac{\partial\theta}{\partial s(x)}\,h\Big]\,ds. \tag{2.10}$$

It is assumed here that

$$\begin{gathered}X^{(1)}+X^{(0)}+Y^{(0)}+Y^{(1)}=O,\quad Z^{(1)}+Z^{(0)}=E,\\ \varkappa_N^{(1)}m^{(1)}X^{(1)}-\varkappa_N^{(0)}m^{(0)}X^{(0)}+(\varkappa_N^{(1)}-\varkappa_N^{(0)})m^{(0)}Y^{(0)}=0,\\ \varkappa_N^{(1)}m^{(1)}z^{(1)}+\varkappa_N^{(0)}m^{(0)}z^{(0)}=0.\end{gathered} \tag{2.11}$$

Hence (2.10) takes the form

$$\big(T^{(1)}u^{(1)}(t)\big)^+-\big(T^{(0)}u^{(0)}(t)\big)^-=$$

$$=-h+\frac{1}{\pi}\int\limits_S\bigg\{2(X^{(1)}+Y^{(1)})\,\frac{\partial\theta}{\partial s(x)}\,\frac{\partial g}{\partial s(y)}+(Z^{(1)}-Z^{(0)})\,\frac{\partial\theta}{\partial s(x)}\,h\bigg\}\,ds=$$

$$=F(t). \tag{2.12}$$

(2.8) and (2.11) form a complete system allowing one to determine the unknown matrices $X^{(1)}$, $X^{(0)}$, $Y^{(1)}$, $Y^{(0)}$, $Z^{(1)}$, $Z^{(0)}$. Solving this system, we finally get

$$\begin{gathered}X^{(1)}=(m^{(1)}+m^{(0)})^{-1}m^{(0)},\;\;Y^{(1)}=-(m^{(1)}+m^{(0)})^{-1}m^{(0)},\\ Z^{(1)}=(A^{(1)}+A^{(0)})^{-1}A^{(0)},\\ X^{(0)}=(m^{(1)}+m^{(0)})^{-1}m^{(1)},\;\;Y^{(0)}=-(m^{(1)}+m^{(0)})^{-1}m^{(1)},\\ Z^{(0)}=(A^{(1)}+A^{(0)})^{-1}A^{(1)},\end{gathered} \tag{2.13}$$

where

$$A=E_1-\varkappa_L m. \tag{2.14}$$

In what follows, the unknown matrices will be meant to be defined by formula (2.13).

Thus for g and h we have obtained integral Fredholm equations of second kind. For $\frac{\partial g}{\partial s(y)}$ in (2.12) we perform partial integration and finally obtain

$$\big(T^{(1)}u^{(1)}(t)\big)^{+}-\big(T^{(0)}u^{(0)}(t)\big)^{-}=$$
$$=-h+\frac{1}{\pi}\int\limits_S\Big[-2(X^{(1)}+Y^{(1)})\,\frac{\partial^2\theta}{\partial s(x)\partial s(y)}\,g+(Z^{(1)}-Z^{(0)})\,\frac{\partial\theta}{\partial s(t)}\,h\Big\}\,ds=$$
$$=F(t). \tag{2.15}$$

Recall some formulas [1]. If $W=u+iv$, then u and v are conjugate if they satisfy the following conditions:

$$Nu=m^{-1}\,\frac{\partial v}{\partial s(x)}\,,\quad Nv=-m^{-1}\,\frac{\partial u}{\partial s(x)}\,, \tag{2.16}$$

where N is the pseudo-stress operator which is of importance when solving the first boundary value problem.

In constructing the equations (2.9) and (2.12), we paid no attention to the terms

$$\pi(x)=\int\limits_S\frac{\partial}{\partial s(x)}\,\frac{\overline{\sigma}}{\sigma}\,g\,ds\ \text{ and }\ \frac{\partial\pi(x)}{\partial s(x)}=\int\limits_S\frac{\partial^2}{\partial s(x)\partial s(y)}\,\frac{\overline{\sigma}}{\sigma}\,g\,ds. \tag{2.17}$$

Let us consider this question. Let $\sigma=re^{i\theta}$, $\overline{\sigma}=re^{-i\theta}$. Then $\pi(x)=-2i\int\limits_S e^{-2i\theta}\,\frac{\partial\theta}{\partial s(y)}\,g\,ds$. This expression represents a function continuous on the whole plane E_2. This happens due to the fact that $\frac{\partial\theta}{\partial s(y)}=\frac{1}{2\rho(y)}$, where $\frac{1}{\rho(y)}$ is the curvature of the curve S at the point y, and by our assumption, this function is Hölder continuous. To study the properties of $\frac{\partial\pi(x)}{\partial s(x)}$, we note that

$$\frac{\partial\pi(x)}{\partial s(x)}=-4\int\limits_S e^{-2i\theta}\,\frac{\partial\theta}{\partial s(x)}\,\frac{\partial\theta}{\partial s(y)}\,g\,ds-2i\int\limits_S e^{-2i\theta}\,\frac{\partial^2\theta}{\partial s(x)\partial s(y)}\,g\,ds.$$

It should also be noted that the identity

$$\frac{\partial^2\ln\sigma}{\partial s(x)\partial s(y)}=-\frac{\partial\ln\sigma}{\partial s(x)}\,\frac{\partial\ln\sigma}{\partial s(y)}=-\Big(\frac{\partial\ln r}{\partial s(x)}+i\,\frac{\partial\theta}{\partial s(x)}\Big)\Big(\frac{\partial\ln r}{\partial s(y)}+i\,\frac{\partial\theta}{\partial s(y)}\Big)$$

holds from which, calculating the imaginary parts of the both sides, we obtain

$$\frac{\partial^2\theta}{\partial s(x)\partial s(y)}=-\frac{\partial\ln r}{\partial s(x)}\,\frac{\partial\theta}{\partial s(y)}-\frac{\partial\ln r}{\partial s(y)}\,\frac{\partial\theta}{\partial s(x)}\,.$$

Since

$$\frac{\partial\theta}{\partial s(x)}=\frac{1}{2\rho(x)}\,,\quad\frac{\partial\theta}{\partial s(y)}=\frac{1}{2\rho(y)},$$

we have

$$\frac{\partial\theta}{\partial s(x)\partial s(y)} = -\frac{\partial \ln r}{\partial s(x)}\,\frac{1}{2\rho(y)} - \frac{\partial \ln r}{\partial s(y)}\,\frac{1}{2\rho(x)} =$$
$$= -\frac{\partial \ln r}{\partial s(x)}\,\frac{1}{2}\Big(\frac{1}{\rho(y)} - \frac{1}{\rho(x)}\Big) - \Big(\frac{\partial \ln r}{\partial s(x)} + \frac{\partial \ln r}{\partial s(y)}\Big)\,\frac{1}{\rho(x)}\,.$$

The above expression represents a Hölder continuous function on the whole plane.

Thus we have proved that when we pay no attention to the terms in (2.16), the validity of the above calculations becomes obvious.

From the equation (2.12), integrating over S, we obtain

$$\int\limits_S h\,ds + (Z^{(0)} - Z^{(1)})\int\limits_S h\,ds = \int\limits_S F\,ds.$$

that is, taking into account (2.11),

$$2Z^{(0)}\int\limits_S h\,ds = \int\limits_S F\,ds.$$

If $\int\limits_S F\,ds = 0$, then taking into account that $\det Z^{(0)} \neq 0$, we find that

$$\int\limits_S h\,ds = 0. \tag{2.18}$$

This condition ensures that $u^{(0)}(x)$ is equal to zero at infinity.

3. Investigation of the Integral Equations of the Basic Contact Problem

Let us prove that the integral Fredholm equations (2.9) and (2.12) for $f = F = 0$ have only trivial solutions. Assume the contrary that (2.9) and (2.12) have nontrivial solutions which we denote again by g and h. Using the uniqueness theorems, we obtain

$$u^{(1)}(x) = u^{(0)}(x) = c_1.$$

But $u^{(1)}(0) = 0$ (without restriction of generality, we assume that the origin is in the domain D_1^+), and we find that $c_1 = 0$ and hence

$$\begin{aligned} u^{(1)}(x) = 0, \;\; x \in D_1^+,\\ u^{(0)}(x) = 0, \;\; x \in D_0^+. \end{aligned} \tag{3.1}$$

Taking into account (3.1) and (2.15), we have

$$v^{(1)}(x) = c_1,$$
$$v^{(0)}(x) = c_0,$$

where c_1 and c_0 are constants. It is already known that $v^{(0)}(\infty)=0$ and $c_0=0$. Since $v^{(1)}$ is defined to within a constant, we can choose it such that $v^{(1)}(x)=0$, $x\in D_1^+$. Finally, we obtain

$$\begin{aligned} v^{(1)}(x)&=\frac{1}{\pi}\int\limits_S \operatorname{Im}\Big\{\Big[(N_y^{(1)}\Gamma^{(1)})'X^{(1)}+\frac{\partial\Gamma^{(1)}}{\partial s(y)}\,Y^{(1)}\Big]g+\Gamma^{(1)}Z^{(1)}h\Big\}\,ds=0,\\ &\qquad\qquad x\in D_1^+,\\ v^{(0)}(x)&=\frac{1}{\pi}\int\limits_S \operatorname{Im}\Big\{\Big[(N_y^{(0)}\Gamma^{(0)})'X^{(0)}+\frac{\partial\Gamma^{(0)}}{\partial s(y)}\,Y^{(0)}\Big]g+\Gamma^{(0)}Z^{(0)}h\Big\}\,ds=0,\\ &\qquad\qquad x\in D_0^+. \end{aligned}\tag{3.2}$$

Taking into account the formulas

$$(N_y\operatorname{Im}\Gamma^{(1)})'=-\frac{\partial\ln r}{\partial s(y)}\,,\quad \frac{\partial\operatorname{Im}\Gamma^{(1)}}{\partial s(y)}=m^{(1)}Y^{(1)}\,\frac{\partial\theta}{\partial s(y)}\,,$$

we can rewrite (3.2) in the form

$$\begin{aligned} v^{(1)}(x)&=\frac{1}{\pi}\int\limits_S\Big\{\Big[-\frac{\partial\ln r}{\partial s(y)}\,X^{(1)}+m^{(1)}Y^{(1)}\,\frac{\partial\theta}{\partial s(y)}\Big]g-\Gamma^{(1)}Z^{(1)}h\Big\}\,ds=0,\\ &\qquad\qquad x\in D_1^+,\\ v^{(0)}(x)&=\frac{1}{\pi}\int\limits_S\Big\{\Big[-\frac{\partial\ln r}{\partial s(y)}\,X^{(0)}+m^{(0)}Y^{(0)}\,\frac{\partial\theta}{\partial s(y)}\Big]g-\Gamma^{(0)}Z^{(0)}h\Big\}\,ds=0,\\ &\qquad\qquad x\in D_0^+. \end{aligned}\tag{3.3}$$

whence, passing to limit as $x\to t\in S$, we obtain

$$\begin{aligned} (v^{(1)}(t))^+-(v^{(1)}(t))^-&=2m^{(1)}Y^{(1)}g,\\ (v^{(0)}(t))^+-(v^{(0)}(t))^-&=2m^{(0)}Y^{(0)}g, \end{aligned}$$

As is known, $(v^{(1)}(t))^+=(v^{(0)}(t))^-=0$, and thus we have

$$(v^{(1)}(t))^-=-2m^{(1)}Y^{(1)}g,\quad (v^{(0)}(t))^+=2m^{(0)}Y^{(0)}g.\tag{3.4}$$

With regard for (2.8), we find that

$$(v^{(1)}(t))^-=-(v^{(0)}(t))^+.\tag{3.5}$$

From (3.2), we can now calculate $T^{(1)}v^{(1)}$ and $T^{(0)}v^{(0)}$.

Taking into account the formulas

$$\begin{aligned} T_x^{(1)}(N_y\Gamma^{(1)})'&=(-iE-\varkappa_N^{(1)}m^{(1)})\,\frac{\partial^2\ln\sigma}{\partial s(x)\partial s(y)}\,,\\ T_x^{(1)}\Big(\frac{\partial\Gamma^{(1)}}{\partial s(y)}\Big)&=(-iE-\varkappa_N^{(1)}m^{(1)})\,\frac{\partial^2\ln\sigma}{\partial s(x)\partial s(y)}\,,\\ T_x^{(1)}\Gamma^{(1)}&=(-iE-\varkappa_N^{(1)}m^{(1)})Z^{(1)}\,\frac{\partial\ln\sigma}{\partial s(x)} \end{aligned}$$

and performing partial integration with respect to the vector g, we obtain

$$\begin{aligned}
T^{(1)}v^{(1)}(x) = \frac{1}{\pi}\int\limits_S \bigg\{ \Big[X^{(1)}\frac{\partial \ln r}{\partial s(x)} + \varkappa_N^{(1)} m^{(1)} X^{(1)}\frac{\partial\theta}{\partial s(x)}\Big]\frac{\partial g}{\partial s(x)} + \\
+ \Big[Y^{(1)}\frac{\partial \ln r}{\partial s(x)} - \varkappa_N^{(1)} m^{(1)} Y^{(1)}\frac{\partial\theta}{\partial s(x)}\Big]\frac{\partial g}{\partial s(y)} + \\
+ \Big[\varkappa_N^{(1)} m^{(1)} Z^{(1)}\frac{\partial\theta}{\partial s(x)} - Z^{(1)}\frac{\partial \ln r}{\partial s(x)}\Big] h\bigg\}\, ds, \quad x\in D_1^+, \\
T^{(0)}v^{(0)}(x) = \frac{1}{\pi}\int\limits_S \bigg\{ \Big[X^{(0)}\frac{\partial \ln r}{\partial s(x)} + \varkappa_N^{(0)} m^{(0)} X^{(0)}\frac{\partial\theta}{\partial s(x)}\Big]\frac{\partial g}{\partial s(x)} + \\
+ \Big[Y^{(0)}\frac{\partial \ln r}{\partial s(x)} - \varkappa_N^{(0)} m^{(0)} Y^{(0)}\frac{\partial\theta}{\partial s(x)}\Big]\frac{\partial g}{\partial s(y)} + \\
+ \Big[\varkappa_N^{(0)} m^{(0)} Z^{(0)}\frac{\partial\theta}{\partial s(x)} - Z^{(0)}\frac{\partial \ln r}{\partial s(x)}\Big] h\bigg\}\, ds, \quad x\in D_0^+,
\end{aligned} \tag{3.6}$$

whence

$$\begin{gathered}
(T^{(1)}v^{(1)}(t))^+ - (T^{(1)}v^{(1)}(t))^- = \\
= -2\big(\varkappa_N^{(1)} m^{(1)} X^{(1)} - \varkappa_N^{(1)} m^{(1)} Y^{(1)}\big)\frac{\partial g}{\partial s(t)} - 2(\varkappa_N^{(1)} m^{(1)} Z^{(1)})h, \\
(T^{(1)}v^{(1)}(t))^+ = 0, \\
(T^{(0)}v^{(0)}(t))^+ - (T^{(0)}v^{(0)}(t))^- = \\
= -2\big(\varkappa_N^{(0)} m^{(0)} X^{(0)} - \varkappa_N^{(0)} m^{(0)} Y^{(0)}\big)\frac{\partial g}{\partial s(t)} - 2(\varkappa_N^{(0)} m^{(0)} Z^{(0)})h, \\
(T^{(0)}v^{(0)}(t))^- = 0.
\end{gathered}$$

Thus we have

$$\begin{gathered}
(T^{(1)}v^{(1)}(t))^- = \\
= 2\big(\varkappa_N^{(1)} m^{(1)} X^{(1)} - \varkappa_N^{(1)} m^{(1)} Y^{(1)}\big)\frac{\partial g}{\partial s(t)} + 2(\varkappa_N^{(1)} m^{(1)} Z^{(1)})h, \\
(T^{(0)}v^{(0)}(t))^+ = \\
= 2\big(\varkappa_N^{(0)} m^{(0)} X^{(0)} - \varkappa_N^{(0)} m^{(0)} Y^{(0)}\big)\frac{\partial g}{\partial s(t)} + 2(\varkappa_N^{(0)} m^{(0)} Z^{(0)})h,
\end{gathered} \tag{3.7}$$

whence, bearing in mind (2.11), we find that

$$(T^{(1)}v^{(1)}(t))^- + (T^{(0)}v^{(0)}(t))^+ \equiv 0. \tag{3.8}$$

Using Green's formulas as well as (3.6) and (3.7), we obtain

$$\int\limits_{D_0^-} E(v^{(1)}, v^{(1)})\, d\sigma = -\int\limits_S (v^{(1)}(t))^- (T^{(1)}v^{(1)})^-\, ds,$$

$$\int\limits_{D_1^+} E(v^{(0)}, v^{(0)})\, d\sigma = \int\limits_S (v^{(0)}(t))^+ (T^{(0)} v^{(0)})^+\, ds,$$

whence

$$\int\limits_{D_0^-} E(v^{(1)}, v^{(1)})\, d\sigma + \int\limits_{D_1^+} E(v^{(0)}, v^{(0)})\, d\sigma = 0.$$

Thus we obtain

$$v^{(1)}(x) = C^{(1)} + \mathcal{E}^{(1)} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad x \in D_0^-,$$

$$v^{(0)}(x) = C^{(0)} + \mathcal{E}^{(0)} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad x \in D_1^+.$$

$v^{(1)}(\infty) = 0$, that is, $c^{(1)} = 0$, $\mathcal{E}^{(1)} = 0$, $v^{(0)} = 0$, i.e. $c^{(0)} = 0$, $\mathcal{E}^{(0)} = 0$.

Taking now into account (3.4) and (3.7), we get

$$g = 0 \ \text{ and } \ h = 0.$$

Thus we have proved that the homogeneous integral equations corresponding to (2.9) and (2.12) have only trivial solutions if $\int\limits_s F\, ds = 0$ and the curvature of the curve S is Hölder continuous.

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