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MULTIDIMENSIONAL GRONWALL-BELLMAN-TYPE INTEGRAL INEQUALITIES WITH APPLICATIONS

Abstract. Several linear and nonlinear integral inequalities for multivariable functions developed in the literature are presented. These inequalities can be used as ready and powerful tools in the analysis of various classes of hyperbolic partial differential, integral, and integro-differential equations. In addition, some new nonlinear retarded integral inequalities for Gronwall–Bellman-type multi-variable functions of are established. Applications of some of the existing inequalities as well as the new ones are included.

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რეზიუმე. ნაშრომში წარმოდგენილია ლიტერატურაში განხილული მრავალი წრფივი და არაწრფივი ინტეგრალური უტოლობა მრავალი ცვლადის ფუნქციებისათვის. ეს უტოლობები შეიძლება გამოყენებულ იქნეს როგორც ხელმისაწვდომი და მძლავრი იარაღი ჰიპერბოლური კერძოწარმოებულებიანი, ინტეგრალური და ინტეგრო-დიფერენციალური განტოლებების ანალიზისათვის. გარდა ამისა, დამტკიცებულია ზოგიერთი ახალი გრონუოლ–ბელმანის ტიპის არაწრფივი დაგვიანებულარგუმენტიანი ინტეგრალური უტოლობა მრავალი ცვლადის ფუნქციებისათვის. მოყვანილია როგორც ზოგიერთი ცნობილი, ისე ახალი უტოლობების გამოყენებები.

1. Introduction

The importance of inequalities has long been recognized in the field of mathematics. The mathematical foundations of the theory of inequalities were established in part during the 18th and 19th centuries by mathematicians such as K. F. Gauss (1777–1855), A. L. Cauchy (1789–1857) and P. L. Chebyshev (1821–1894). In the years thereafter the influence of inequalities has been immense and the subject has attracted many distinguished mathematicians, including H. Poincaré (1854–1912), A. M. Lyapunov (1857–1918), O. Hölder (1859–1937) and J. Hadamard (1865–1963).

Analysis has been the dominant branch of mathematics for the last three centuries and inequalities are the heart of analysis. Although inequalities play a fundamental role in all branches of mathematics, the subject was developed as a branch of modern mathematics during the 20th century through the pioneering work *Inequalities* by G. H. Hardy, J. E. Littlewood and G. Pólya [39], which appeared in 1934. This theoretical foundation, which was further developed by other mathematicians, has in turn led to the discovery of many new inequalities and interesting applications in various fields of mathematics.

Since 1934, when the key work of Hardy et al. [39], *Inequalities,* was published, several papers devoted to inequalities were published. These dealt with new inequalities that are useful in many applications. It appears that in the theory of inequality, the three fundamental inequalities, namely, **AM-GM** inequality, the Hölder (in particular, Cauchy–Schwarz) inequality and the Minkowski inequality, have played dominant roles. A detailed discussion of these inequalities can be found in the book *Inequalities* by E. F. Beckenbach and R. Bellman [9], which appeared in 1961, and the two books *Analytic Inequalities Involving Functions and Their Integrals and Derivatives* and *Classical and New Inequalities in Analysis* by D. S. Mitrinović *et al.* [45], [46], which appeared in 1991 and 1992, respectively.

Many problems, arising in a wide variety of application areas, give rise to mathematical models involving boundary value problems for ordinary or partial differential equations. The foremost desire of an investigator is to solve the problem explicitly. If little theory is available and no explicit solution is readily obtainable, generally the ensuing line of attack is to identify circumstances under which the complexity of the problem may be reduced. In the past rew years the growth of the concerned theory has taken beautiful and unexpected paths and will continue with great vigor in the next few decades.

Integral inequalities that give explicit bounds on unknown functions provide a very useful and important device in the study of many qualitative as well as quantitative properties of solutions of nonlinear differential equations. One of the best known and widely used inequalities in the study of nonlinear differential equations can be stated as follows.

If u is a continuous function defined on the interval $J = [\alpha, \alpha + h]$ and

$$0 \le u(t) \le \int_\alpha^t [bu(s) + a]\, ds, \quad t \in J, \tag{1.1}$$

where a and b are nonnegative constants, then

$$0 \le u(t) \le ah \exp(bh), \quad t \in J. \tag{1.2}$$

This inequality was found by Gronwall [38] in 1919 while investigating the dependence of systems of differential equations with respect to a parameter. In fact the roots of such an inequality can be found in the work of Peano [69], which explicitly dealt with the special case of the above inequality having $a = 0$, and proved some quite general results concerning differential inequalities as well as maximal and minimal solutions of differential equations.

In a paper published in 1943, Bellman [15] proved the following inequality:

Let u and f be non-negative continuous functions on an interval $J = [\alpha, \beta]$, and let c be a nonnegative constant. Then the inequality

$$u(t) \le c + \int_\alpha^t f(s)u(s)\, ds, \quad t \in J, \tag{1.3}$$

implies that

$$u(t) \le c \exp\left(\int_\alpha^t f(s)\, ds \right), \quad t \in J. \tag{1.4}$$

It is clear that Bellman's result includes that of Gronwall, since $\int_\alpha^t a\, ds \le ah$ for $\alpha \le t \le \alpha + h$.

Bellman's inequality exerted tremendous influence in the subsequent years, and the study of such inequalities has grown into a subsequent field with many important applications in various branches of differential and integral equations.

After the discovery of the above inequality by Bellman, inequalities of this type are known in the literature as 'Bellman's lemma or inequality', the 'Gronwall–Bellman inequality', Gronwall's inequality', or the Bellman–Gronwall inequality'; For examples, Agarwal at al. [2], [3], [5], Akinyele [7], Bainov and Simeonov [8], Beesack [10], [12], [13], Chandra and Davis [21], Cho at al. [24], Conlan and Wang [25], [26], Dragomir and Kim [30], [31], Ghoshal and Masood [34], Ghoshal [35], Headley [40], Kasture and Deo [73], Pachpatte [48], [50], [54], [66], Snow [73], [74], Young [80], and Gao and Ding [82].

Bihari [16] in 1956 proved the following useful nonlinear generalization of the Gronwall–Bellman inequality:

Let u and f be nonnegative continuous functions defined on R_+. Let $w(u)$ be a continuous nondecreasing function defined on R_+ and $w(u) > 0$ on $(0, \infty)$. If

$$u(t) \le k + \int_0^t f(s)w(u(s))\,ds, \tag{1.5}$$

for $t \in R_+$, where k is a nonnegative constant, then for $0 \le t \le t_1$,

$$u(t) \le G^{-1}\biggl(G(k) + \int_0^t f(s)\,ds\biggr), \tag{1.6}$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{w(s)}, \quad r > 0, \ \ r_0 > 0, \tag{1.7}$$

and G^{-1} is the inverse function of G and $t_1 \in R_+$ is chosen so that

$$G(k) + \int_0^t f(s)\,ds \in \mathrm{Dom}(G^{-1}), \tag{1.8}$$

for all $t \in R_+$ in the interval $0 \le t \le t_1$.

Since 1975 an enormous amount of effort has been devoted to the discovery of new types of inequalities and their applications in various branches of ordinary and partial differential and integral equations. Owing to the tremendous success enjoyed during the past few years, a large number of papers have appeared in the literature, which are partly inspired by the challenge of research in various branches of differential and integral equations, where inequalities are often the bases of important lemmas for proving various theorems or approximating various functions. Several isolated researchers have developed valuable work in the field of integral inequalities. The concerned application in the theory of differential and integral equations is vast and is rapidly growing. Part of this growth is due to the fact that the subject is genuinely rich and lends itself to many different approaches and applications.

Beckenbach and Bellman [9] stated without proof a two-independent-variable generalization of the well-known Gronwall–Bellman inequality due to Wendroff, which has its origin in the field of partial differential equations. A new beginning in the theory of such inequalities due to Wendroff, given in Beckenbach and Bellman [9, p. 154], is embodied in the following statement.

Let $u(x, y)$ and $c(x, y)$ be nonnegative continuous functions defined on x, $y \in R_+$. If

$$u(x, y) \le a(x) + b(y) + \int_0^x \int_0^y c(s, t)u(s, t)\,ds\,dt,$$

for x, $y \in R_+$, *where* $a(x), b(y)$ *are positive continuous functions for* x, $y \in R_+$, *having derivatives such that* $a'(s) \geq 0, b'(y) \geq 0$ *for* x, $y \in R_+$, *then*

$$u(x,y) \leq E(x,y) \exp\left(\int_0^x \int_0^y c(s,t)\, ds\, dt \right), \tag{1.9}$$

for x, $y \in R_+$, *where*

$$E(x,y) = [a(x) + b(0)][a(0) + b(y)]/[a(0) + b(0)], \tag{1.10}$$

for x, $y \in R_+$.

While analyzing the dynamics of physical systems governed by various nonlinear partial differential equations, one often needs some new ideas and methods. It is well-known that the method of differential and integral inequalities plays an important role in the qualitative theory of partial differential, integral, and integro-differential equations. During the past few years, many papers have appeared in the literature which deal with integral inequalities in more than one independent variable, which are motivated by certain applications in the theory of hyperbolic partial differential and integral equations.

The main aim of this paper is to present a number of two- as well as n-dimensional linear and nonlinear integral inequalities developed in the literature. These inequalities can be used as ready and powerful tools in the analysis of various classes of hyperbolic partial differential, integral, and integro-differential equations. Applications of some of the inequalities are also presented and some new nonlinear retarded integral inequalities of Gronwall–Bellman-type are established. These inequalities can be used as basic tools in the study of certain classes of functional differential equations as well as integral equations.

2. Gronwall–Bellman-Type Linear Inequalities I

It is well known that the method of integral inequalities plays an important role in the qualitative theory of partial differential, integral, and integro-differential equations. During the past few years, many papers have appeared in the literature which deal with integral inequalities in two independent variables: For instance, refer Bondge and Pachpatte [17], [18], Bondge et al. [20], Ghoshal and Masood [34], Kasture and Deo [41], Pachpatte [51], [55]–[57], Shastri and Kasture [70], and Snow [74].

In this section we present a number of two-dimensional linear integral inequalities developed in the literature. These inequalities can be used as ready and powerful tools in the analysis of various classes of hyperbolic partial differential, integral, and integro-differential equations.

Beckenbach and Bellman [9] stated without proof a two-independent-variable generalization of the well-known Gronwall–Bellman inequality due to Wendroff, which has its origin in the field of partial differential equations.

The result due to Wendroff, given in Backenbach and Bellman [9, p. 154] is embodied in the following theorem.

Theorem 2.1 (Wendroff). *Let $u(x,y), c(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$. If*

$$u(x,y) \le a(x) + b(y) + \int_0^x \int_0^y c(s,t)u(s,t)\,ds\,dt, \tag{2.1}$$

for x, $y \in R_+$, where $a(x)$, $b(y)$ are positive continuous functions for x, $y \in R_+$, having derivative such that $a'(t) \ge 0$, $b'(y) \ge 0$ for x, $y \in R_+$, then

$$u(x,y) \le E(x,y) \exp\left(\int_0^x \int_0^y c(s,t)\,ds\,dt \right), \tag{2.2}$$

for x, $y \in R_+$, where

$$E(x,y) = [a(x) + b(0)][a(0) + b(y)]/[a(0) + b(0)], \tag{2.3}$$

for x, $y \in R_+$.

Snow [73], [74] gave one of the first Gronwall-type integral inequalities involving two independent variables for scalar and vector functions using the notation of a Riemann function. The inequalities given in Snow [73], [74] have significant applications in the study of various properties of the solutions of partial differential equations.

Snow [73] established a useful generalization of Theorem 2.1 in the following form.

Theorem 2.2 (Snow, 1971). *Suppose $u(x,y), a(x,y)$ and $b(x,y)$ are continuous on a domain D with $b \ge 0$. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two point in D such that $(x-x_0)(y-y_0) \ge 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t;x,y)$ be the solution of the characteristic initial value problem*

$$L[v] = v_{st} - b(s,t)v = 0, \quad v(s,y) = v(x,t) = 1,$$

and let D^+ be connected subdomain of D which contains P and on which $v \ge 0$. If $R \subset D^+$ and $u(x,y)$ satisfies

$$u(x,y) \le a(x,y) + \int_0^x \int_0^y b(s,t)u(s,t)\,ds\,dt,$$

then $u(x,y)$ also satisfies

$$u(x,y) \le a(x,y) + \int_0^x \int_0^y a(s,t)b(s,t)v(s,t;x,y)\,ds\,dt.$$

Wendroff's inequality given in Theorem 2.1 is very effective in the study of various properties of the solutions of certain hyperbolic partial differential equations. In the past few years many authors have obtained various interesting and useful generalizations and extensions of this inequality.

The following theorem deals with a useful two-independent-variable generalization of the inequality given by Greene [37].

Theorem 2.3 (Greene, 1977). *Let $u(x,y)$, $v(x,y)$, $h_i(x,y)$, $i=1,2,3,4$ be nonnegative continuous functions defined for x, $y \in R_+$ and c_1, c_2 and μ be nonnegative constants. If*

$$u(x,y) \le c_1 + \int_0^x \int_0^y h_1(s,t)u(s,t)\,ds\,dt + \int_0^x \int_0^y h_2(s,t)\overline{v}(s,t)\,ds\,dt, \tag{2.4}$$

$$v(x,y) \le c_2 + \int_0^x \int_0^y h_3(s,t)\overline{u}(s,t)\,ds\,dt + \int_0^x \int_0^y h_4(s,t)v(s,t)\,ds\,dt, \tag{2.5}$$

for x, $y \in R_+$, where $\overline{u}(x,y) = \exp(-\mu(x+y))u(x,y)$ and $\overline{v}(x,y) = \exp(-\mu(x+y))v(x,y)$ for x, $y \in R_+$, then

$$u(x,y) \le (c_1+c_2)\exp\left(\mu(x+y) + \int_0^x \int_0^y h(s,t)\,ds\,dt\right), \tag{2.6}$$

$$v(x,y) \le (c_1+c_2)\exp\left(\int_0^x \int_0^y h(s,t)\,ds\,dt\right), \tag{2.7}$$

for x, $y \in R_+$, where

$$h(x,y) = \max\Big\{[h_1(x,y)+h_3(x,y)], [h_2(x,y)+h_4(x,y)]\Big\},$$

for all x, $y \in R_+$.

Following are some inequalities of the Wendroff type given by Pachpatte [56] and analogous inequalities useful in the study of qualitative properties of the solutions of certain integro-differential and integral equations. An interesting and useful integro-differential inequality given by Pachpatte [56] reads as follows.

Theorem 2.4 (Pachpatte, 1980). *Let $u(x,y)$, $u_x(x,y)$, $u_y(x,y)$, $u_{xy}(x,y)$ and $c(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$ and $u(x,0) = u(0,y) = 0$. If*

$$u_{xy}(x,y) \le a(x)+b(y)+M\left[u(x,y) + \int_0^x \int_0^y c(s,t)(u(s,t)+u_{st}(s,t))\,ds\,dt\right] \tag{2.8}$$

for x, $y \in R_+$, where $a(x) > 0, b(y) > 0$ are continuous for x, $y \in R_+$, having derivatives such that $a'(x) \ge 0$, $b'(y) \ge 0$ for x, $y \in R_+$, and $M \ge 0$

is constant, then

$$u_{xy}(x,y) \le E(x,y)\exp\Big(\int_0^x\int_0^y \big[M+(1+M)c(s,t)\big]\,ds\,dt\Big), \tag{2.9}$$

for $x,\ y\in R_+$, *where* $E(x,y)$ *is defined by* (2.3) *in Theorem* 2.1.

Pachpatte [61] established the following inequality, which can be widely used in various applications.

Theorem 2.5 (Pachpatte, 1995). *Let* $u(x,y)$, $p(x,y)$, *and* $q(x,y)$ *be nonnegative continuous functions defined for* $x,\ y\in R_+$. *Let* $k(x,y,s,t)$ *and its partial derivatives* $k_x(x,y,s,t)$, $k_y(x,y,s,t)$, $k_{xy}(x,y,s,t)$ *be nonnegative continuous functions for* $0\le s\le x<\infty$, $0\le t\le y<\infty$. *If*

$$u(x,y)\le p(x,y)+q(x,y)\int_0^x\int_0^y k(x,y,s,t)u(s,t))\,ds\,dt, \tag{2.10}$$

for $x,\ y\in R_+$, *then*

$$u(x,y)\le p(x,y)+q(x,y)\Big(\int_0^x\int_0^y A(\sigma,\tau)\ d\sigma\,d\tau\Big)\exp\Big(\int_0^x\int_0^y B(\sigma,\tau)\ d\sigma\,d\tau\Big), \tag{2.11}$$

for $x,\ y\in R_+$, *where*

$$A(x,y)=k(x,y,x,y)p(x,y)+\int_0^x k_x(x,y,s,y)p(s,y)\,ds+$$

$$+\int_0^y k_y(x,y,x,t)p(x,t)\,dt+\int_0^x\int_0^y k_{xy}(x,y,s,t)p(s,t)\ ds\,dt, \tag{2.12}$$

$$B(x,y)=k(x,y,x,y)q(x,y)+\int_0^x k_x(x,y,s,y)q(s,y)\,ds+$$

$$+\int_0^y k_y(x,y,x,t)q(x,t)\,dt+\int_0^x\int_0^y k_{xy}(x,y,s,t)q(s,t)\ ds\,dt, \tag{2.13}$$

for all $x,\ y\in R_+$.

An interesting and useful generalization of Wendroff's inequality, given in Pachpatte [63, p. 325] is embodied in the following theorem.

Theorem 2.6 (Pachpatte, 1998). *Let* $u(x,y)$, $n(x,y)$ *and* $c(x,y)$ *be nonnegative continuous functions defined for* $x,\ y\in R_+$, *and let* $n(x,y)$ *be nondecreasing in each variable* $x,\ y\in R_+$. *If*

$$u(x,y)\le n(x,y)+\int_0^x\int_0^y c(s,t)u(s,t)\ ds\,dt, \tag{2.14}$$

for $x,\ y \in R_+$, *then*

$$u(x,y) \le n(x,y) \exp\Big(\int_0^x \int_0^y c(s,t)\ ds\, dt \Big), \tag{2.15}$$

for $x,\ y \in R_+$.

Given a continuous function $a : R_+ \times R_+ \to R_+$, we write

$$\widehat{a}(x,y) = \max\big\{ a(s,t) :\ 0 \le s \le x,\ 0 \le s \le y \big\} \tag{2.16}$$

for $x,\ y \in R_+$. Theorem 2.6 is a part of the following more general result.

Theorem 2.7. *Let* $u(x,y)$, $a(x,y)$, *and* $c(x,y)$ *be nonnegative continuous functions defined for* $x,\ y \in R_+$. *If*

$$u(x,y) \le a(x,y) + \int_0^x \int_0^y c(s,t)u(s,t)\ ds\, dt, \tag{2.17}$$

for $x,\ y \in R_+$, *then*

$$u(x,y) \le \widehat{a}(x,y) \exp\Big(\int_0^x \int_0^y c(s,t)\, ds\, dt \Big), \tag{2.18}$$

for $x,\ y \in R_+$, *where the function* $\widehat{a}$ *is defined in* (2.16).

Proof. Let $X > 0$ and $Y > 0$ be fixed. For $0 \le x \le X$, $0 \le y \le Y$, define a function $v(x,y)$ by

$$v(x,y) = \widehat{a}(X,Y) + \int_0^x \int_0^y c(s,t)u(s,t)\ ds\, dt. \tag{2.19}$$

Then $v(x,y)$ is nondecreasing in each variable x, y, and $v(0,y) = \widehat{a}(X,Y)$ and

$$\frac{\partial v}{\partial x}(x,y) = \int_0^y c(x,t)u(x,t)\, dt \le v(x,y) \int_0^y c(x,t)\, dt, \tag{2.20}$$

since $u(x,y) \le v(x,t) \le v(x,y)$. Using the fact that Lemma 1.1 [2, p. 2], the differential inequality (2.20) implies

$$v(x,y) \le \widehat{a}(X,Y) \exp\Big(\int_0^x \int_0^y c(s,t)\ ds\, dt \Big) \tag{2.21}$$

for $0 \le x \le X$, $0 \le y \le Y$. Setting $X = x$ and $Y = y$ and changing notation we get the required inequality in (2.18). □

Another interesting integro-differential inequality, useful in the qualitative analysis of hyperbolic partial differential equations with retarded arguments, given by Pachpatte [64] is as follows. R denotes the set of real numbers; $R_+ = [0,\infty)$, $R_1 = [1,\infty)$, and $J_1 = [x_0, X)$ and $J_2 = [y_0, Y)$ are the given subsets of R; $\triangle = J_1 \times J_2$.

Theorem 2.8 (Pachpatte, 2002)**.** *Let* a, $b \in C(\triangle, R_+)$ *and* $\alpha \in C^1(J_1, J_1)$, $\beta \in C^1(J_2, J_2)$ *be nondecreasing with* $\alpha(x) \le x$ *on* J_1, $\beta(y) \le y$ *on* J_2. *Let* $k \ge 0$ *be a constant. If* $u \in C(\triangle, R_+)$ *and*

$$u(x,y) \le k + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u(s,t)\,ds\,dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u(s,t)\,ds\,dt,$$

for $(x,y) \in \triangle$, *then*

$$u(x,y) \le k \exp\Bigg(\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,ds\,dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)\,ds\,dt,\Bigg),$$

for $(x,y) \in \triangle$.

By a reasoning similar to the proofs of Theorem 2.7 and Theorem 2.8 we can prove the following assertion.

Theorem 2.9. *Let* a, b, $c \in C(\triangle, R_+)$ *and* $\alpha \in C^1(J_1, J_1)$, $\beta \in C^1(J_2, J_2)$ *be nondecreasing with* $\alpha(x) \le x$ *on* J_1, $\beta(y) \le y$ *on* J_2. *If* $u \in C(\triangle, R_+)$ *and*

$$u(x,y) \le c(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u(s,t)\,ds\,dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u(s,t)\,ds\,dt,$$

for $(x,y) \in \triangle$, *then*

$$u(x,y) \le \widehat{c}(x,y) \exp\Bigg(\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,ds\,dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)\,ds\,dt\Bigg),$$

for $(x,y) \in \triangle$, *where the function* $\widehat{c}$ *is defined in* (2.16).

3. Gronwall–Bellman-Type Linear Inequalities II

During the past several years some new inequalities of the Wendroff type have been developed which provide a natural and effective means for further development of the theory of partial integro-differential and integral equations. Here we present some inequalities of the Wendroff type given by Pachpatte [56] and analogous inequalities which can be used in the study of qualitative properties of the solutions of certain integro-differential and integral equations. Pachpatte [56] established the following inequality.

Theorem 3.1 (Pachpatte, 1980)**.** *Let $u(x,y)$, $f(x,y)$ and $g(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$. If*

$$u(x,y) \le a(x)+b(y)+\int_0^x\int_0^y f(s,t)\left(u(s,t)+\int_0^s\int_0^t g(\sigma,\eta)u(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt, \quad (3.1)$$

for x, $y \in R_+$, where $a(x) > 0$, $b(y) > 0$ are continuous for x, $y \in R_+$, having derivatives such that $a'(x) \ge 0$, $b'(y) \ge 0$ for x, $y \in R_+$, then

$$u(x,y) \le a(x)+b(y)+ \\ + \int_0^x\int_0^y f(s,t)E(s,t)\exp\left(\int_0^s\int_0^t \big[f(\sigma,\eta)+g(\sigma,\eta)\big]\,d\sigma\,d\eta\right)ds\,dt, \quad (3.2)$$

for x, $y \in R_+$, where $E(x,y)$ is defined by (2.3) *in Theorem* 2.1*.*

An interesting and useful generalization of Theorem 3.1 given in Pachpatte [63, p. 336] is embodied in the following theorem.

Theorem 3.2 (Pachpatte, 1998)**.** *Let $u(x,y)$, $f(x,y)$, $g(x,y)$ and $c(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$, and let $c(x,y)$ be nondecreasing in each variable x, $y \in R_+$. If*

$$u(x,y) \le c(x,y)+\int_0^x\int_0^y f(s,t)\left(u(s,t)+\int_0^s\int_0^t g(\sigma,\eta)u(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt, \quad (3.3)$$

for x, $y \in R_+$, then

$$u(x,y) \le c(x,y)H(x,y), \quad (3.4)$$

for x, $y \in R_+$, where

$$H(x,y) = 1+\int_0^x\int_0^y f(s,t)\exp\left(\int_0^s\int_0^t \big[f(\sigma,\eta)+g(\sigma,\eta)\big]\,d\sigma\,d\eta\right)ds\,dt, \quad (3.5)$$

for x, $y \in R_+$.

An interesting two-independent-variable theorem given in Pachpatte [63, p. 337] is as follows.

Theorem 3.3 (Pachpatte, 1998)**.** *Let $u(x,y)$, $f(x,y)$, $g(x,y)$, $h(x,y)$ and $p(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$ and u_0 be a nonnegative constant.*

(a_1) *If*

$$u(x,y) \le u_0+\int_0^x\int_0^y \big[f(s,t)u(s,t)+p(x,t)\big]\,ds\,dt+ \\ +\int_0^x\int_0^y f(s,t)\left(\int_0^s\int_0^t \big[g(\sigma,\eta)u(\sigma,\eta)\big]\,d\sigma\,d\eta\right)ds\,dt, \quad (3.6)$$

for $x,\ y\in R_+$, *then*

$$u(x,y)\le\left(u_0+\int_0^x\int_0^y p(s,t)\,ds\,dt\right)H(x,y),\tag{3.7}$$

for $x,\ y\in R_+$, *where* $H(x,y)$ *is as defined by* (3.5) *in Theorem* 3.2.

(a_2) *If*

$$u(x,y)\le u_0+\int_0^x\int_0^y f(s,t)u(s,t)\,ds\,dt+$$
$$+\int_0^x\int_0^y f(s,t)\left(\int_0^s\int_0^t\big[g(\sigma,\eta)u(\sigma,\eta)+p(\sigma,\eta)\big]\,d\sigma\,d\eta\right)ds\,dt,\tag{3.8}$$

for $x,\ y\in R_+$, *then*

$$u(x,y)\le\left(u_0+\int_0^x\int_0^y f(s,t)\left(\int_0^s\int_0^t p(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt\right)H(x,y),\tag{3.9}$$

for $x,\ y\in R_+$, *where* $H(x,y)$ *is as defined by* (3.5) *in Theorem* 3.2.

(a_3) *If*

$$u(x,y)\le u_0+\int_0^x\int_0^y h(s,t)u(s,t)\,ds\,dt+$$
$$+\int_0^x\int_0^y f(s,t)\left(u(s,t)+\int_0^s\int_0^t g(\sigma,\eta)u(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt,\tag{3.10}$$

for $x,\ y\in R_+$, *then*

$$u(x,y)\le u_0\exp\left(\int_0^x\int_0^y h(s,t)\,ds\,dt\right)H(x,y),\tag{3.11}$$

for $x,\ y\in R_+$, *where* $H(x,y)$ *is as defined by* (3.5) *in Theorem* 3.2.

(a_4) *If*

$$u(x,y)\le h(x,y)+$$
$$+p(x,y)\int_0^x\int_0^y f(s,t)\left(u(s,t)+p(s,t)\int_0^s\int_0^t g(\sigma,\eta)u(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt,\tag{3.12}$$

for $x,\, y \in R_+$, *then*

$$u(x,y) \le h(x,y) + p(x,y)M(x,y)\Bigg[1+$$

$$+\int_0^x\int_0^y f(s,t)p(s,t)\left(\int_0^s\int_0^t \big[f(\sigma,\eta)+g(\sigma,\eta)\big]p(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt\Bigg], \quad (3.13)$$

for $x,\, y \in R_+$, *where*

$$M(x,y)=\int_0^x\int_0^y f(s,t)\left(h(s,t)+p(s,t)\int_0^s\int_0^t g(\sigma,\eta)h(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt \quad (3.14)$$

for $x,\, y \in R_+$.

The following assertion is related to work of Pachpatte [63, p. 340].

Theorem 3.4. *Let* $u(x,y)$, $a(x,y)$, $f(x,y)$, $g(x,y)$, $h(x,y)$, *and* $p(x,y)$ *be nonnegative continuous functions defined for* $x,\, y \in R_+$ *and let* $f(x,y)$ *and* $g(x,y)$ *be positive and sufficiently smooth functions for* $x,\, y \in R_+$. *Define the function* $\widehat{a}$ *by* $\widehat{a}(x,y) = \max\big\{a(s,t):\ 0 \le s \le x,\ 0 \le s \le y\big\}$ *for* $x,\, y \in R_+$.

(b$_1$) *If*

$$u(x,y)\le a(x,y)+\int_0^x\int_0^y f(s,t)\left(h(s,t)+\int_0^s\int_0^t p(\sigma,\eta)u(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt, \quad (3.15)$$

for $x,\, y \in R_+$, *then*

$$u(x,y) \le Q(x,y)\exp\left(\int_0^x\int_0^y f(s,t)\left(\int_0^s\int_0^t p(\sigma,\eta)\,d\sigma\,d\eta\right)ds\,dt\right), \quad (3.16)$$

for $x,\, y \in R_+$, *where*

$$Q(x,y) = \widehat{a}(x,y) + \int_0^x\int_0^y f(s,t)h(s,t)\,ds\,dt, \quad (3.17)$$

for $x,\, y \in R_+$.

(b$_2$) *If*

$$u(x,y) \le a(x,y)+$$

$$+\int_0^x\int_0^y f(s,t)\left(\int_0^s\int_0^t g(\sigma,\eta)\left(\int_0^\sigma\int_0^\eta p(s_1,t_1)u(s_1,t_1)\,ds_1\,dt_1\right)d\sigma\,d\eta\right)ds\,dt, \quad (3.18)$$

for $x,\ y \in R_+$, *then*

$$u(x,y) \leq Q(x,y)\times$$
$$\times \exp\Big(\int_0^x\int_0^y f(s,t)\Big(\int_0^s\int_0^t g(\sigma,\eta)\Big(\int_0^\sigma\int_0^\eta p(s_1,t_1)\,ds_1\,dt_1\Big)\,d\sigma\,d\eta\Big)\,ds\,dt\Big), \quad (3.19)$$

for $x,\ y \in R_+$, *where* $Q(x,y)$ *is as defined by* (3.17).

Proof. We provide here the proof of (b_1) only; the proof of (b_2) can be completed similarly.

(b_1) Let $X > 0$ and $Y > 0$ be fixed. For $0 \leq x \leq X$, $0 \leq y \leq Y$, we assume that $\widehat{a}(X,Y)$ is positive. From (3.15) we have

$$u(x,y) \leq \widehat{q}(x,y) + \int_0^x\int_0^y f(s,t)\Big(\int_0^s\int_0^t p(\sigma,\eta)u(\sigma,\eta)\,d\sigma\,d\eta\Big)\,ds\,dt, \quad (3.20)$$

where $\widehat{q}(x,y) = \widehat{a}(X,Y) + \int_0^x\int_0^y f(s,t)h(s,t)\,ds\,dt$. Clearly $\widehat{q}(x,y)$ is nonnegative and nondecreasing in each variable $x,\ y \in R_+$. From (3.20) we observe that

$$\frac{u(x,y)}{\widehat{q}(x,y)} \leq 1 + \int_0^x\int_0^y f(s,t)\Big(\int_0^s\int_0^t p(\sigma,\eta)\frac{u(\sigma,\eta)}{\widehat{q}(\sigma,\eta)}\,d\sigma\,d\eta\Big)\,ds\,dt. \quad (3.21)$$

Define a function $z(x,y)$ by the right-hand side of (3.21). Then $z(0,y) = z(x,0) = 1$ and

$$z_{xy}(x,y) = f(x,y)\int_0^x\int_0^y p(\sigma,\eta)\,\frac{u(\sigma,\eta)}{\widehat{q}(\sigma,\eta)}\,d\sigma\,d\eta. \quad (3.22)$$

From (3.22) it is easy to observe that

$$\frac{\partial^2}{\partial x\partial y}\Big(\frac{z_{xy}(x,y)}{f(x,y)}\Big) = p(x,y)\,\frac{u(x,y)}{\widehat{q}(x,y)}\,. \quad (3.23)$$

Now by using the fact that $u(x,y)/\widehat{q}(x,y) \leq z(x,y)$ in (3.23) we have

$$\frac{\frac{\partial}{\partial y}\big(\frac{\partial}{\partial x}\big(\frac{z_{xy}(x,y)}{f(x,y)}\big)\big)}{z(x,y)} \leq p(x,y). \quad (3.24)$$

Since

$$\frac{\partial}{\partial x}\Big(\frac{z_{xy}(x,y)}{f(x,y)}\Big) \geq 0, \quad \frac{\partial}{\partial y}z(x,y) \geq 0, \quad z(x,y) > 0,$$

from (3.24) we observe that

$$\frac{\frac{\partial}{\partial y}\big(\frac{\partial}{\partial x}\big(\frac{z_{xy}(x,y)}{f(x,y)}\big)\big)}{z(x,y)} \leq p(x,y) + \frac{\frac{\partial}{\partial x}\big(\frac{z_{xy}(x,y)}{f(x,y)}\big)\frac{\partial}{\partial y}z(x,y)}{z^2(x,y)}\,,$$

i.e.

$$\frac{\partial}{\partial y}\left(\frac{\frac{\partial}{\partial x}\left(\frac{z_{xy}(x,y)}{f(x,y)}\right)}{z(x,y)}\right) \leq p(x,y). \tag{3.25}$$

Keeping x fixed in (3.25), set $y = \eta$ and integrate with respect to η from 0 to y to obtain the estimate

$$\frac{\frac{\partial}{\partial x}\left(\frac{z_{xy}(x,y)}{f(x,y)}\right)}{z(x,y)} \leq \int_0^y p(x,\eta)\, d\eta. \tag{3.26}$$

Since

$$\frac{z_{xy}(x,y)}{f(x,y)} \geq 0, \quad z_x(x,y) \geq 0, \quad z(x,y) > 0,$$

again as above from (3.26) we observe that

$$\frac{\partial}{\partial x}\left(\frac{\left(\frac{z_{xy}(x,y)}{f(x,y)}\right)}{z(x,y)}\right) \leq \int_0^y p(x,\eta)\, d\eta. \tag{3.27}$$

Keeping y fixed in (3.27), set $x = \sigma$ and integrate with respect to σ from 0 to x to obtain the estimate

$$\frac{z_{xy}(x,y)}{z(x,y)} \leq f(x,y) \int_0^x \int_0^y p(\sigma,\eta)\, d\sigma\, d\eta. \tag{3.28}$$

Since $z_x(x,y) \geq 0$, $z_y(x,y) \geq 0$ and $z(x,y) > 0$, from (3.28) we observe that

$$\frac{\partial}{\partial y}\left(\frac{\frac{\partial}{\partial x} z(x,y)}{z(x,y)}\right) \leq f(x,y) \int_0^x \int_0^y p(\sigma,\eta)\, d\sigma\, d\eta. \tag{3.29}$$

Keeping x fixed in (3.29), set $y = t$ and integrate with respect to t from 0 to y to obtain the estimate

$$\frac{\frac{\partial}{\partial x} z(x,y)}{z(x,y)} \leq \int_0^y f(x,t)\left(\int_0^x \int_0^t p(\sigma,\eta)\, d\sigma\, d\eta\right) dt. \tag{3.30}$$

Keeping y fixed in (3.30), set $x = s$ and integrate with respect to s from 0 to x to obtain the estimate

$$z(x,y) \leq \exp\left(\int_0^x \int_0^y f(s,t)\left(\int_0^s \int_0^t p(\sigma,\eta)\, d\sigma\, d\eta\right) ds\, dt\right). \tag{3.31}$$

Using (3.31) in (3.21) where for $0 \leq x \leq X, 0 \leq y \leq Y$, we get the required inequality in (3.16). The proof of the case where $\widehat{a}(X,Y)$ is nonnegative can be carried out as above with $\widehat{a}(X,Y) + \epsilon$ instead of $\widehat{a}(X,Y)$, where $\epsilon > 0$ is an arbitrary small constant and subsequently allowing $\epsilon \to 0$ (in the limit) to obtain (3.16). $\square$

Remark 3.1. In special cases when $a(x,y) = u_0$, in (3.15) and (3.18) for x, $y \in R_+$, where $u_0 \geq 0$ is a constant, the bounds obtained in (3.16) and (3.19) reduce to

$$u(x,y) \leq \widetilde{Q}(x,y) \exp\Bigg(\int_0^x \int_0^y f(s,t)\Bigg(\int_0^s \int_0^t p(\sigma,\eta)\,d\sigma\,d\eta\Bigg) ds\,dt\Bigg), \qquad (3.32)$$

$$u(x,y) \leq \widetilde{Q}(x,y)\times$$
$$\times \exp\Bigg(\int_0^x\int_0^y f(s,t)\Bigg(\int_0^s\int_0^t g(\sigma,\eta)\Bigg(\int_0^\sigma\int_0^\eta p(s_1,t_1)\,ds_1\,dt_1\Bigg)\,d\sigma\,d\eta\Bigg)ds\,dt\Bigg), \quad (3.33)$$

for x, $y \in R_+$, respectively, where

$$\widetilde{Q}(x,y) = u_0 + \int_0^x \int_0^y f(s,t)h(s,t)\,ds\,dt.$$

The requirement $k > 0$ can be weakened to $k \geq 0$ as noted in the proof of Theorem 3.4. Note that the inequalities given in (3.32) and (3.33) are two-independent-variable inequalities established by Pachpatte [63, p. 340].

4. Gronwall–Bellman-Type Nonlinear Inequalities I

The fundamental role played by Wendroff's inequality and its generalizations and variants in the development of the theory of partial differential and integral equations is well known. In this section, we present some basic nonlinear generalizations of Wendroff's inequality established by Bondge and Pachpatte [18], [19] and some new variants, which can be used as tools in the study of certain partial differential and integral equations. Bondge and Pachpatte [18] proved the following useful nonlinear generalization of Wendroff's inequality.

Theorem 4.1 (Bondge and Pachpatte, 1979)**.** *Let $u(x,y)$ and $p(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$. Let $g(u)$ be continuously differentiable function defined for $u \geq 0$, $g(u) > 0$ for $u > 0$ and $g'(u) \geq 0$ for $u \geq 0$. If*

$$u(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y p(s,t)g(u(s,t))\,ds\,dt, \qquad (4.1)$$

for x, $y \in R_+$, where $a(x) > 0$, $b(y) > 0$, $a'(x) \geq 0$, $b'(y) \geq 0$ are continuous functions for x, $y \in R_+$, then for $0 \leq x \leq x_1$, $0 \leq y \leq y_1$

$$u(x,y) \leq \Omega^{-1}\Bigg(\Omega(a(0)+b(y)) + \int_0^x \frac{a'(s)}{g(a(s)+b(0))}\,ds + \int_0^x\int_0^y p(s,t)\,ds\,dt\Bigg), \quad (4.2)$$

where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \quad r > 0, \quad r_0 > 0, \tag{4.3}$$

Ω^{-1} *is the inverse function of* Ω *and* x_1, y_1 *are chosen so that*

$$\Omega(a(0) + b(y)) + \int_0^x \frac{a'(s)}{g(a(s) + b(0))}\, ds + \int_0^x \int_0^y p(s,t)\, ds\, dt \in \mathrm{Dom}(\Omega^{-1}).$$

for all x, y *lying in the subintervals* $0 \le x \le x_1$, $0 \le y \le y_1$ *of* R_+.

Bondge and Pachpatte [19] gave the following generalization of Wendroff's inequality.

Theorem 4.2 (Bondge and Pachpatte, 1980)**.** *Let* $u(x,y)$, $a(x,y)$, $b(x,y)$, *and* $c(x,y)$ *be nonnegative continuous functions defined for* x, $y \in R_+$. *Let* $g(u)$, $h(u)$ *be continuously differentiable functions defined for* $u \ge 0$, $g(u) > 0$, $h(u) > 0$ *for* $u > 0$ *and* $g'(u) \ge 0$, $h'(u) \ge 0$ *for* $u \ge 0$, *and let* $g(u)$ *be subadditive and submultiplicative for* $u \ge 0$. *If*

$$u(x,y) \le a(x,y) + b(x,y) h\Big(\int_0^x \int_0^y c(s,t) g(u(s,t))\, ds\, dt \Big), \tag{4.4}$$

for x, $y \in R_+$, *then for* $0 \le x \le x_2$, $0 \le y \le y_2$

$$u(x,y) \le a(x,y) + \\ + b(x,y) h\Big(G^{-1}\Big[G(A(x,y)) + \int_0^x \int_0^y c(s,t) g(b(s,t))\, ds\, dt \Big] \Big), \tag{4.5}$$

where

$$A(x,y) = \int_0^x \int_0^y c(s,t) g(a(s,t))\, ds\, dt, \tag{4.6}$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(h(s))}, \quad r > 0, \quad r_0 > 0, \tag{4.7}$$

G^{-1} *is the inverse function of* G *and* x_2, y_2 *are so chosen that*

$$G(A(x,y)) + \int_0^x \int_0^y c(s,t) g(b(s,t))\, ds\, dt \in \mathrm{Dom}(G^{-1}).$$

for all x, y *lying in the subintervals* $0 \le x \le x_2$, $0 \le y \le y_2$ *of* R_+.

Various branches of Gronwall-like inequalities can be found in the book, *Inequalities for Differential and Integral Equations* by Pachpatte [63]. The following theorem provides another useful generalization of Wendroff's inequality, which appeared in Pachpatte' book [63, p. 465].

Theorem 4.3 (Pachpatte, 1998). *Let $u(x,y)$, $c(x,y)$, and $p(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$. Let $g(u)$, $g'(u)$, $a(x)$, $a'(x)$, $b(y)$, and $b'(y)$ be as in Theorem* 4.1 *and let $g(u)$ be submultiplicative on R_+. If*

$$u(x,y) \le a(x)+b(y)+\int_0^x\int_0^y c(s,t)u(s,t)\,ds\,dt+\int_0^x\int_0^y p(s,t)g(u(s,t))\,ds\,dt, \quad (4.8)$$

for x, $y \in R_+$, then for $0 \le x \le x_3$, $0 \le y \le y_3$

$$u(x,y) \le q(x,y)\Bigg[\Omega^{-1}\Bigg(\Omega(a(0)+b(y))+ + \int_0^x \frac{a'(s)}{g(a(s)+b(0))}\,ds + \int_0^x\int_0^y p(s,t)g(q(s,t))\,ds\,dt\Bigg)\Bigg], \quad (4.9)$$

where

$$q(x,y) = \exp\Bigg(\int_0^x\int_0^y c(s,t)\,ds\,dt\Bigg), \quad (4.10)$$

Ω, Ω^{-1} *are as defined in Theorem 4.1, and x_3, y_3 are so chosen that*

$$\Omega(a(0)+b(y))+\int_0^x \frac{a'(s)}{g(a(s)+b(t))}\,ds+\int_0^x\int_0^y p(s,t)g(q(s,t))\,ds\,dt \in \mathrm{Dom}(\Omega^{-1}).$$

for all x, y lying in the subintervals $0 \le x \le x_3$, $0 \le y \le y_3$ of R_+.

A slight variant of Theorem 4.3 is given in the following theorem appearing in Pachpatte [63, p. 467].

Theorem 4.4 (Pachpatte, 1998). *Let $u(x,y)$, $c(x,y)$, $p(x,y)$, $g(u)$, $g'(u)$, $a(x)$, $a'(x)$, $b(y)$, and $b'(y)$ be as in Theorem* 4.1. *If*

$$u(x,y) \le a(x)+b(y)+\int_0^x\int_0^y c(s,y)u(s,y)\,ds\,dt+\int_0^x\int_0^y p(s,t)g(u(s,t))\,ds\,dt, \quad (4.11)$$

for x, $y \in R_+$, then for $0 \le x \le x_4$, $0 \le y \le y_4$

$$u(x,y) \le F(x,y)\Bigg[\Omega^{-1}\Bigg(\Omega(a(0)+b(y))+ + \int_0^x \frac{a'(s)}{g(a(s)+b(0))}\,ds + \int_0^x\int_0^y p(s,t)g(F(s,t))\,ds\,dt\Bigg)\Bigg], \quad (4.12)$$

where

$$F(x,y) = \exp\Bigg(\int_0^x c(s,y)\,ds\,dt\Bigg), \quad (4.13)$$

Ω, Ω^{-1} are as defined in Theorem 4.1, *and x_4, y_4 are chosen so that*

$$\Omega(a(0)+b(y))+\int_0^x \frac{a'(s)}{g(a(s)+b(t))}\,ds+\int_0^x\int_0^y p(s,t)g(F(s,t))\,ds\,dt\in \mathrm{Dom}(\Omega^{-1}).$$

for all x, y lying in the subintervals $0 \le x \le x_4$, $0 \le y \le y_4$ of R_+.

Another interesting and useful two-independent-variable inequality given by Pachpatte [64] reads as the following two theorems which can be used in the qualitative analysis of hyperbolic partial differential equations with retarded arguments. In what follows, R denotes the set of real numbers; $R_+ = [0, \infty)$, $R_1 = [1, \infty)$, and $J_1 = [x_0, X)$ and $J_2 = [y_0, Y)$ are the given subsets of R; $\triangle = J_1 \times J_2$.

Theorem 4.5 (Pachpatte, 2002)**.** *Let a, $b \in C(\triangle, R_+)$ and $\alpha \in C^1(J_1, J_1)$, $\beta \in C^1(J_2, J_2)$ be nondecreasing with $\alpha(x) \le x$ on J_1, $\beta(y) \le y$ on J_2. Let $c \ge 1$, $k \ge 0$, and $p > 1$ are constants.*

(a$_1$) *If $u \in C(\triangle, R_1)$ and*

$$u(x,y) \le k + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u(s,t)\log u(s,t)\,ds\,dt+ \\ + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u(s,t)\log u(s,t)\,ds\,dt,$$

for $(x,y) \in \triangle$, then

$$u(x,y) \le c^{\exp(A(x,y)+B(x,y))},$$

for $(x,y)\in\triangle$, where $A(x,y)=\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,ds\,dt$, $B(x,y)=\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)\,ds\,dt$.

(a$_2$) *If $u \in C(\triangle, R_+)$ and*

$$u^p(x,y) \le k + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u(s,t)\,ds\,dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u(s,t)\,ds\,dt,$$

for $(x,y) \in \triangle$, then

$$u(x,y) \le \left[k^{(p-1)/p} + \left(\frac{p-1}{p}\right)[A(x,y)+B(x,y)]\right]^{1/(p-1)} \tag{4.14}$$

for $(x,y)\in\triangle$, where $A(x,y)=\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)\,ds\,dt$, $B(x,y)=\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)\,ds\,dt$.

Theorem 4.6 (Pachpatte, 2002)**.** *Let a, $b \in C(\triangle, R_+)$ and $\alpha \in C^1(J_1, J_1)$, $\beta \in C^1(J_2, J_2)$ be nondecreasing with $\alpha(x) \le x$ on J_1, $\beta(y) \le y$ on J_2. Let*

$c \geq 1$, $k \geq 0$, and $p > 1$ are constants. For $i = 1, 2$, let $g_i \in C(R_+, R_+)$ be nondecreasing functions with $g_i(u) > 0$ for $u > 0$.

(a$_1$) *If $u \in C(\triangle, R_+)$ and for $(x, y) \in \triangle$,*

$$u(x,y) \leq k + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u(s,t)g_1(u(s,t))\,ds\,dt + \\ + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u(s,t)g_2(u(s,t))\,ds\,dt,$$

then for $x_0 \leq x \leq x_1$, $y_0 \leq y \leq y_1$,

(i) *in case $g_2(u) \leq g_1(u)$,*

$$u(x,y) \leq G_1^{-1}\big[G_1(k) + A(x,y) + B(x,y)\big],$$

(ii) *in case $g_1(u) \leq g_2(u)$,*

$$u(x,y) \leq G_2^{-1}\big[G_2(k) + A(x,y) + B(x,y)\big],$$

where $A(x,y)$, $B(x,y)$ are as defined in Theorem 4.5 and for $i = 1, 2:$ G_i^{-1} are the inverse functions of

$$G_i(r) = \int_{r_0}^{r} \frac{ds}{g_i(s)}, \quad r > 0, \quad r_0 > 0,$$

and $x_1 \in J_1$, $y_1 \in J_2$ are so chosen that for $i = 1, 2$,

$$G_i(k) + A(x,y) + B(x,y)) \in \mathrm{Dom}(G_i^{-1}),$$

for all x and y lying in $[x_0, x_1]$ and $[y_0, y_1]$, respectively.

(a$_2$) *If $u \in C(\triangle, R_1)$ and for $(x, y) \in \triangle$,*

$$u(x,y) \leq c + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u(s,t)g_1(\log u(s,t))\,ds\,dt + \\ + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u(s,t)g_2(\log u(s,t))\,ds\,dt,$$

then for $x_0 \leq x \leq x_2$, $y_0 \leq y \leq y_2$,

(i) *in case $g_2(u) \leq g_1(u)$,*

$$u(x,y) \leq \exp\Big(G_1^{-1}\big[G_1(\log c) + A(x,y) + B(x,y)\big]\Big),$$

(ii) *in case $g_1(u) \leq g_2(u)$,*

$$u(x,y) \leq \exp\Big(G_2^{-1}\big[G_2(\log c) + A(x,y) + B(x,y)\big]\Big),$$

where $A(x,y)$, $B(x,y)$, G_i, G_i^{-1} *are as in* (a_1) *and* $x_2 \in J_1$, $y_2 \in J_2$ *are so chosen that for* $i = 1,2$,

$$G_i(\log c) + A(x,y) + B(x,y) \in \mathrm{Dom}(G_i^{-1}),$$

for all x *and* y *lying in* $[x_0, x_2]$ *and* $[y_0, y_2]$, *respectively.*

(a_3) *If* $u \in C(\triangle, R_+)$ *and for* $(x,y) \in \triangle$,

$$u^p(x,y) \le k + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)u(s,t)g_1(u(s,t))\,ds\,dt + \\ + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)u(s,t)g_2(u(s,t))\,ds\,dt,$$

then for $x_0 \le x \le x_3$, $y_0 \le y \le y_3$,

(i) *in case* $g_2(u) \le g_1(u)$,

$$u(x,y) \le H_1^{-1}\big[H_1(k) + A(x,y) + B(x,y)\big],$$

(ii) *in case* $g_1(u) \le g_2(u)$,

$$u(x,y) \le H_2^{-1}\big[H_2(k) + A(x,y) + B(x,y)\big],$$

where $A(x,y)$, $B(x,y)$ *are as defined in Theorem* 4.5 *and for* $i = 1,2$, H_i^{-1} *are the inverse functions of*

$$H_i(r) = \int_{r_0}^{r} \frac{ds}{g_i(s^{1/p})}, \quad r > 0, \quad r_0 > 0,$$

and $x_3 \in J_1$, $y_3 \in J_2$ *are so chosen that for* $i = 1,2$,

$$H_i(k) + A(x,y) + B(x,y)) \in \mathrm{Dom}(H_i^{-1}),$$

for all x *and* y *lying in* $[x_0, x_3]$ *and* $[y_0, y_3]$ *respectively.*

Another interesting and useful two-independent-variable inequality given by Pachpatte [68] reads as in the following theorem. This can be used in the qualitative analysis of retarded partial differential equations with retarded arguments. In what follows, R denotes the set of real numbers; $R_+ = [0,\infty)$, $R_1 = [1,\infty)$, and $J_1 = [x_0, X)$ and $J_2 = [y_0, Y)$ are the given subsets of R; $\triangle = J_1 \times J_2$.

Theorem 4.7 (Pachpatte, 2004)**.** *Let* u, a_i, $b_i \in C(\triangle, R_+)$ *and* $\alpha_i \in C^1(J_1, J_1)$, $\beta_i \in C^1(J_2, J_2)$ *be nondecreasing with* $\alpha_i(x) \le x$ *on* J_1, $\beta_i(y) \le y$ *on* J_2 *for* $i = 1, \ldots, n$. *Let* $c \ge 0$ *and* $p > 1$ *be constants.*

(a_1) *If*

$$u^p(x,y) \le c + p\sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} \big[a_i(s,t)u^p(s,t) + b_i(s,t)u(s,t)\big]\,ds$$

for $(x, y) \in \triangle$, *then*

$$u(x,y) \le \left\{ B(x,y) \exp\left((p-1) \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a_i(s,t)\,ds\,dt \right) \right\}^{\frac{1}{p-1}},$$

for $(x, y) \in \triangle$, *where*

$$B(x,y) = \{c\}^{\frac{p-1}{p}} + (p-1) \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b_i(s,t)\,ds\,dt,$$

for $(x, y) \in \triangle$.

(a_2) *Let* $w \in C(R_+, R_+)$ *be nondecreasing with* $w(u) > 0$ *of* $(0, \infty)$. *If for* $(x, y) \in \triangle$,

$$u^p(x,y) \le c + p \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \big[a_i(s,t) w(u(s,t)) + b_i(s,t) u(s,t)\big]\,ds\,dt,$$

then for $x_0 \le x \le x_1$, $y_0 \le y \le y_1$,

$$u(x,y) \le \left\{ G^{-1}\left[G(B(x,y)) + (p-1) \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a_i(s,t)\,ds\,dt \right] \right\}^{\frac{1}{p-1}},$$

where $B(x, y)$ *is as defined* (a_1), G^{-1} *is the inverse functions of*

$$G(r) = \int_{r_0}^{r} \frac{ds}{w(s^{\frac{1}{p-1}})}, \quad r > 0, \quad r_0 > 0,$$

and $x_1 \in J_1$, $y_1 \in J_2$ *are chosen so that*

$$G(B(x,y)) + (p-1) \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a_i(s,t)\,ds\,dt \in \mathrm{Dom}(G_i^{-1}),$$

for all x *and* y *lying in* $[x_0, x_1]$ *and* $[y_0, y_1]$, *respectively.*

More interesting and useful two-independent-variable inequalities given by Cheung and Ma [23] read as in the following Theorems 4.8–4.11. These can be used in the qualitative analysis of partial differential equations with retarded arguments.

Theorem 4.8 (Cheung and Ma, 2005)**.** *Let* $u(x, y)$, $a(x, y)$, $c(x, y)$, *and* $d(x, y)$ *be nonnegative continuous functions defined for* x, $y \in R_+$ *and* $w(u)$ *be a nonnegative, nondecreasing continuous function for* $u \in R_+$ *with* $w(u) > 0$ *for* $u > 0$ *and* $e(x, y) \in C(R_+^2, R_+)$. *Let* $\varphi(u) \in C^1(R_+, R_+)$ *with*

$\varphi'(u) > 0$ for $u > 0$. Here φ' denotes the derivative of φ. Assume that $a(x,y)$ and $c(x,y)$ are nondecreasing in x and nonincreasing in y for $x,\, y \in R_+$. If

$$\varphi(u(x,y)) \le a(x,y) + c(x,y) \int_0^x \int_y^\infty \varphi'(u(s,t))\big[d(s,t)w(u(s,t)) + e(s,t)\big]\, ds\, dt,$$

for $x,\, y \in R_+$, then

$$u(x,y) \le G^{-1}\Big\{G\big[\varphi^{-1}(a(x,y)) + E(x,y)\big] + c(x,y) \int_0^x \int_y^\infty d(s,t)\, ds\, dt\Big\},$$

for all $0 \le x \le x_1, y_1 \le y < \infty$, where

$$E(x,y) = c(x,y) \int_0^x \int_y^\infty e(s,t)\, ds\, dt,$$

G^{-1} is the inverse function of

$$G(r) = \int_{r_0}^r \frac{ds}{w(s)}, \quad r \ge r_0 > 0,$$

and $x_1,\, y_1 \in R_+$ are so chosen that

$$G\big[\varphi^{-1}(a(x,y)) + E(x,y)\big] + c(x,y) \int_0^x \int_y^\infty d(s,t)\, ds\, dt \in \mathrm{Dom}(G^{-1}).$$

Theorem 4.9 (Cheung and Ma, 2005)**.** *Let $u(x,y)$, $a(x,y)$, $c(x,y)$, and $d(x,y)$ be nonnegative continuous functions defined for $x,\, y \in R_+$ and $w(u)$ be a nonnegative, nondecreasing continuous function for $u \in R_+$ with $w(u) > 0$ for $u > 0$ and $e(x,y) \in C(R_+^2, R_+)$. Let $\varphi(u) \in C^1(R_+, R_+)$ with $\varphi'(u) > 0$ for $u > 0$, here φ' denotes the derivative of φ. Assume that $a(x,y)$ and $c(x,y)$ are nonincreasing in y in each of the variables $x,\, y \in R_+$. If*

$$\varphi(u(x,y)) \le a(x,y) + c(x,y) \int_x^\infty \int_y^\infty \varphi'(u(s,t))\big[d(s,t)w(u(s,t)) + e(s,t)\big]\, ds\, dt,$$

for $x,\, y \in R_+$, then

$$u(x,y) \le G^{-1}\Big\{G\big[\varphi^{-1}(a(x,y)) + \overline{E}(x,y)\big] + c(x,y) \int_x^\infty \int_y^\infty d(s,t)\, ds\, dt\Big\},$$

for all $x_2 \le x < \infty$, $y_2 \le y < \infty$, where

$$\overline{E}(x,y) = c(x,y) \int_x^\infty \int_y^\infty e(s,t)\, ds\, dt,$$

G and G^{-1} are as defined in Theorem 4.8 and x_2, $y_2 \in R_+$ are so chosen that

$$G\big[\varphi^{-1}(a(x,y)) + \overline{E}(x,y)\big] + c(x,y)\int\limits_x^\infty\int\limits_y^\infty d(s,t)\,ds\,dt \in \mathrm{Dom}(G^{-1}).$$

Theorem 4.10 (Cheung and Ma, 2005)**.** *Let $u(x,y)$, $a(x,y)$, $c(x,y)$, and $d(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$ and $w(u)$ be a nonnegative, nondecreasing continuous function for $u \in R_+$ with $w(u) > 0$ for $u > 0$ and $e(x,y)$, $f(x,y) \in C(R_+^2, R_+)$. Let $\varphi(u) \in C^1(R_+, R_+)$ with $\varphi'(u) > 0$ for $u > 0$, where φ' denotes the derivative of φ. Assume that $b(x,y)$ and $d(x,y)$ be are nondecreasing in x and non-increasing in y. If*

$$\varphi(u(x,y)) \le a(x,y) + b(x,y)\int\limits_\alpha^x c(s,y)\varphi(u(s,y))\,ds + \\ + d(x,y)\int\limits_0^x\int\limits_y^\infty \varphi'(u(s,t))\big[f(s,t)w(u(s,t)) + e(s,t)\big]\,ds\,dt,$$

for x, y, $\alpha \in R_+$ with $\alpha \le x$, then

$$u(x,y) \le G^{-1}\bigg\{G\big[\varphi^{-1}(p(x,y)a(x,y)) + p(x,y)E_1(x,y)\big] + \\ + p(x,y)d(x,y)\int\limits_0^x\int\limits_y^\infty f(s,t)\,ds\,dt\bigg\},$$

for all $0 \le x \le x_3$, $y_3 \le y < \infty$, where

$$p(x,y) = 1 + b(x,y)\int\limits_\alpha^x c(s,y)\exp\bigg(\int\limits_s^x b(m,y)c(m,y)\,dm\bigg)\,ds,$$

$$E_1(x,y) = d(x,y)\int\limits_0^x\int\limits_y^\infty e(s,t)\,ds\,dt,$$

G and G^{-1} are as defined in Theorem 4.8 and x_3, $y_3 \in R_+$ are so chosen that

$$G\big[\varphi^{-1}(p(x,y)a(x,y)) + p(x,y)E_1(x,y)\big] + \\ + p(x,y)d(x,y)\int\limits_0^x\int\limits_y^\infty f(s,t)\,ds\,dt \in \mathrm{Dom}(G^{-1}).$$

Theorem 4.11 (Cheung and Ma, 2005)**.** *Let $u(x,y)$, $f(x,y)$, $e(x,y)$, $\varphi(u)$, and $w(u)$ be defined as in Theorem 4.10. Let $a(x,y)$, $b(x,y)$, $c(x,y)$,* *and $d(x,y)$ be nonnegative continuous and non-increasing in each variable x, $y \in R_+$. If*

$$\varphi(u(x,y)) \le a(x,y) + b(x,y)\int_x^\beta c(s,y)\varphi(u(s,y))\,ds + $$
$$+ d(x,y)\int_x^\infty\int_y^\infty \varphi'(u(s,t))\big[f(s,t)w(u(s,t)) + e(s,t)\big]\,ds\,dt,$$

for $x, y, \beta \in R_+$ with $x \le \beta$, then

$$u(x,y) \le G^{-1}\Big\{G\big[\varphi^{-1}(\overline{p}(x,y)a(x,y)) + \overline{p}(x,y)\overline{E}_1(x,y)\big] + $$
$$+ \overline{p}(x,y)d(x,y)\int_x^\infty\int_y^\infty f(s,t)\,ds\,dt\Big\},$$

for all $x_4 \le x < \infty$, $y_4 \le y < \infty$, where

$$\overline{p}(x,y) = 1 + b(x,y)\int_x^\beta c(s,y)\exp\bigg(\int_x^s b(m,y)c(m,y)\,dm\bigg)\,ds,$$
$$\overline{E}_1(x,y) = d(x,y)\int_x^\infty\int_y^\infty e(s,t)\,ds\,dt.$$

G and G^{-1} are as defined in Theorem 4.8 *and x_4, $y_4 \in R_+$ are so chosen that*

$$G\big[\varphi^{-1}(\overline{p}(x,y)a(x,y)) + \overline{p}(x,y)\overline{E}_1(x,y)\big] + $$
$$+ \overline{p}(x,y)d(x,y)\int_x^\infty\int_y^\infty f(s,t)\,ds\,dt \in \mathrm{Dom}(G^{-1}).$$

In [43], Ma and Pečarić established new explicit bounds on the solutions to a class of new nonlinear retarded Volterra–Fredholm type integral inequalities in two independent variables. These are embodied in Theorems 4.12 and 4.13, which can be used as effective tools in the study of certain integral equations. In what follows,

$$E = \Big\{(x,y,s,t) \in \triangle^2 :\ x_0 \le s \le x \le M,\ y_0 \le t \le y \le N\Big\}.$$

Theorem 4.12 (Ma and Pečarić, 2008)**.** *Let $u(x,y)$, $l(x,y) \in C(\triangle, R_+)$, $a(x,y,s,t)$, $b(x,y,s,t) \in C(E, R_+)$ with $a(x,y,s,t)$ and $b(x,y,s,t)$ be non-decreasing in x and y for each $s \in J_1$, and $t \in J_2$, and $\alpha \in C^1(J_1, J_1)$,*

$\beta \in C^1(J_2, J_2)$ be nondecreasing with $\alpha(x) \leq x$ on J_1, $\beta(y) \leq y$ on J_2. If

$$u^p(x,y) \leq l(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) u^q(s,t)\, ds\, dt + \\ + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) u^r(s,t)\, ds\, dt,$$

for $(x,y) \in \triangle$, where $p \geq q \geq 0$, $p \geq r \geq 0$, p, q, and r are constants and

$$\lambda_1 = \frac{r}{p} \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) K_2^{\frac{r-p}{p}}(s,t) \exp(A_1(s,t))\, ds\, dt < 1,$$

for $(x,y) \in \triangle$, then

$$u(x,y) \leq \left[l(x,y) + \frac{\overline{A}_1(x,y) + B_1(x,y)}{1 - \lambda_1(x,y)} \exp(A_1(x,y)) \right]^{\frac{1}{p}},$$

for $(x,y) \in \triangle$ and any $K_i(x,y) \in C(\triangle, R_0)$, $i = 1, 2$, where

$$A_1(x,y) = \frac{q}{p} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) K_1^{\frac{q-p}{p}}(s,t)\, ds\, dt,$$

$$\overline{A}_1(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) \left[\frac{q}{p} K_1^{\frac{q-p}{p}}(s,t) l(s,t) + \frac{p-q}{p} K_1^{\frac{q}{p}}(s,t) \right] ds\, dt,$$

$$\overline{B}_1(x,y) = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) \left[\frac{r}{p} K_2^{\frac{r-p}{p}}(s,t) l(s,t) + \frac{p-r}{p} K_2^{\frac{r}{p}}(s,t) \right] ds\, dt,$$

for $(x,y) \in \triangle$.

Theorem 4.13 (Ma and Pečarić, 2008)**.** *Let $u(x,y)$, $l(x,y) \in C(\triangle, R_+)$, $a_i(x,y,s,t)$, $b_i(x,y,s,t) \in C(E, R_+)$ with $a_i(x,y,s,t)$ and $b_i(x,y,s,t)$ be nondecreasing in x and y for each $s \in J_1$, and $t \in J_2$. Let α_i, $\gamma_j \in C^1(J_1, J_1)$, β_i, $\delta_j \in C^1(J_2, J_2)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ be nondecreasing with $\alpha_i(x) \leq x$, $\gamma_j(x) \leq x$ on J_1, $\beta_i(y) \leq y$, $\delta_j(y) \leq y$ on J_2. Let $p \geq q_i \geq 0$ and $p > r_j \geq 0$ be constants. If*

$$u^p(x,y) \leq l(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} a_i(x,y,s,t) u^{q_i}(s,t)\, ds\, dt +$$

$$+\sum_{j=1}^{m}\int\limits_{\gamma_j(x_0)}^{\gamma_j(M)}\int\limits_{\delta_j(y_0)}^{\delta_j(N)} b_j(x,y,s,t)u^{r_j}(s,t)\,ds\,dt,$$

for $(x,y)\in\triangle$, *and*

$$\lambda_2=\sum_{j=1}^{m}\frac{r_j}{p}\int\limits_{\gamma_j(x_0)}^{\gamma_j(M)}\int\limits_{\delta_j(y_0)}^{\delta_j(N)} b_j(x,y,s,t)K_{2j}^{\frac{r_j-p}{p}}(s,t)\exp(A_2(s,t))\,ds\,dt<1,$$

for $(x,y)\in\triangle$, *then*

$$u(x,y)\le\left[l(x,y)+\frac{\overline{A}_2(x,y)+B_2(x,y)}{1-\lambda_2(x,y)}\exp(A_2(x,y))\right]^{\frac{1}{p}},$$

for $(x,y)\in\triangle$ *and any* $K_{1i}(x,y),K_{2j}(x,y)\in C(\triangle,R_0)$, $i=1,2\dots,n$, $j=1,2\dots,m$, *where*

$$A_2(x,y)=\sum_{i=1}^{n}\frac{q_i}{p}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} a_i(x,y,s,t)K_{1i}^{\frac{q_i-p}{p}}(s,t)\,ds\,dt,$$

$$\overline{A}_2(x,y)=\sum_{i=1}^{n}\frac{q_i}{p}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} a_i(x,y,s,t)\times$$

$$\times\left[\frac{q_i}{p}K_{1i}^{\frac{q_i-p}{p}}(s,t)l(s,t)+\frac{p-q_i}{p}K_{1i}^{\frac{q_i}{p}}(s,t)\right]ds\,dt,$$

$$\overline{B}_2(x,y)=\sum_{j=1}^{m}\int\limits_{\gamma_j(x_0)}^{\gamma_j(M)}\int\limits_{\delta_j(y_0)}^{\delta_j(N)} b_j(x,y,s,t)\times$$

$$\times\left[\frac{r_j}{p}K_{2j}^{\frac{r_j-p}{p}}(s,t)l(s,t)+\frac{p-r_j}{p}K_{2j}^{\frac{r_j}{p}}(s,t)\right]ds\,dt,$$

for $(x,y)\in\triangle$.

In the next we present some new nonlinear retarded Gronwall–Bellman-type integral inequalities in two independent variables. These are stated as the following Theorems, which can be used as effective tools in the study of certain integral equations. In what follows, R denotes the set of real numbers; $R_+=[0,\infty)$, $R_1=[1,\infty)$, and $J_1=[x_0,X)$ and $J_2=[y_0,Y)$ are the given subsets of R; $\triangle=J_1\times J_2$. Given a continuous function $a:\ J_1\times J_2\to R_+$, we write

$$\widehat{a}(x,y)=\max\Big\{a(s,t):\ x_0\le s\le x,\ y_0\le s\le y\Big\}. \tag{4.15}$$

Theorem 4.14. *Let* u, a, c, f_i, $g_i\in C(\triangle,R_+)$, $i=1,\dots,n$ *and let* $\alpha_i\in C^1(J_1,J_1)$ *be nondecreasing with* $\alpha_i(t)\le t$, $i=1,\dots,n$, *and* $\beta_i\in C^1(J_2,J_2)$ *be nondecreasing with* $\beta_i(t)\le t$, $i=1,\dots,n$. *Suppose that* $q>0$

is a constant, $\varphi \in C(R_+, R_+)$ *is an increasing function with* $\varphi(\infty) = \infty$ *and* $\psi(u)$ *is a nondecreasing continuous function for* $u \in R_+$ *with* $\psi(u) > 0$ *for* $u > 0$. *If*

$$\varphi(u(x,y)) \leq a(x,y)+$$
$$+ c(x,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\big[f_i(s,t)\psi(u(s,t)) + g_i(s,t)\big]\, dt\, ds \quad (4.16)$$

for all $(x,y) \in \triangle$, *then*

$$u(x,y) \leq$$
$$\leq \varphi^{-1}\left\{G^{-1}\left[\Phi^{-1}\left(\Phi(k(x_0,y)) + \widehat{c}(x,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} [f_i(s,t)]\, dt\, ds\right)\right]\right\} \quad (4.17)$$

for all $(x,y) \in [x_0, x_1] \times [y_0, y_1]$, *where*

$$k(x_0,y) = G(\widehat{a}(x,y)) + \widehat{c}(x,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} [g_i(s,t)]\, dt\, ds,$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{[\varphi^{-1}(s)]^q}, \quad r \geq r_0 > 0, \quad (4.18)$$

$$\Phi(r) = \int_{r_0}^{r} \frac{ds}{\psi[\varphi^{-1}(G^{-1}(s))]}, \quad r \geq r_0 > 0, \quad (4.19)$$

G^{-1}, Φ^{-1} *denote the inverse function of* G, Φ *and* $(x_1, y_1) \in \triangle$ *is so chosen that*

$$\left[\Phi(k(x_0,y)) + \widehat{c}(x,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} [f_i(s,t)]\, dt\, ds\right] \in \mathrm{Dom}(\Phi^{-1}).$$

Proof. Fixing any numbers X and Y with $x_0 \leq x \leq X$ and $y_0 \leq y \leq Y$, we assume that $\widehat{a}(X,Y)$ is positive and define a positive function $z(x,y)$ by

$$z(x,y) = \widehat{a}(X,Y)+$$
$$+\widehat{c}(X,Y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)[f_i(s,t)\psi(u(s,t)) + g_i(s,t)]\, dt\, ds. \quad (4.20)$$

Then $z(x,y) > 0$, $z(x_0,y) = z(x,y_0) = \widehat{a}(X,Y)$ and (4.16) can be restated as

$$u(x,y) \leq \varphi^{-1}[z(x,y)]. \quad (4.21)$$

It is easy to observe that $z(x,y)$ is a continuous non-decreasing function for all $x \in J_1, y \in J_2$ and

$$D_1 z(x,y) = \widehat{c}(X,Y)\times$$
$$\times \sum_{i=1}^{n}\Big[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} u^q(\alpha_i(x),t)\Big[f_i(\alpha_i(x),t)\psi(u(\alpha_i(x),t))+g_i(\alpha_i(x),t)\Big]\,dt\Big]\alpha_i'(x) \leq$$
$$\leq \widehat{c}(X,Y)\big[\varphi^{-1}(z(x,y))\big]^q\times$$
$$\times \sum_{i=1}^{n}\Big[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)\psi\big(\varphi^{-1}(z(\alpha_i(x),t))\big)+g_i(\alpha_i(x),t)\Big]\,dt\Big]\alpha_i'(x).$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\big[\varphi^{-1}(z(x,y))\big]^q \geq \big[\varphi^{-1}(z(x_0,y_0))\big]^q = \big[\varphi^{-1}(\widehat{a}(X,Y))\big]^q > 0.$$

From the definition of G and the above relation, we have

$$D_1 G(z(x,y)) = \frac{D_1 z(x,y)}{[\varphi^{-1}(z(x,y))]^q} \leq \widehat{c}(X,Y)\times$$
$$\times \sum_{i=1}^{n}\Big[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)\psi\big(\varphi^{-1}(z(\alpha_i(x),t))\big)+g_i(\alpha_i(x),t)\Big]\,dt\Big]\alpha_i'(x). \quad (4.22)$$

Keeping y fixed in (4.22), setting $x=\sigma$ and integrating it with respect to σ from x_0 to $x, x \in J_1$ and making the change of variable, we obtain

$$G(z(x,y)) \leq G(z(x_0,y))+$$
$$+\,\widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big\{f_i(s,t)\psi\big[\varphi^{-1}(z(s,t))\big]+g_i(s,t)\Big\}\,dt\,ds. \quad (4.23)$$

Now, define a function $k(x,y)$ by

$$k(x,y) = G(\widehat{a}(X,Y)) + \widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(X)}\int\limits_{\beta_i(y_0)}^{\beta_i(Y)} g_i(s,t)\,dtds+$$
$$+\,\widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\psi\big[\varphi^{-1}(z(s,t))\big]\,dt\,ds.$$

Then $k(x_0,y) = G(\widehat{a}(X,Y))+\widehat{c}(X,Y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(Y)} g_i(s,t)\,dt\,ds$, and (4.23) can be restated as

$$z(x,y) \leq G^{-1}[k(x,y)]. \quad (4.24)$$

It is easy to observe that $k(x,y)$ is a continuous non-decreasing function for all $x \in J_1$, $y \in J_2$ and

$$D_1 k(x,y) = \widehat{c}(X,Y) \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)\psi\big[\varphi^{-1}(z(\alpha_i(x),t))\big]\,dt \Bigg] \alpha_i'(x) \le$$

$$\le \widehat{c}(X,Y)\psi\Big\{\varphi^{-1}\big[G^{-1}(k(x,y))\big]\Big\} \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)\,dt \Bigg] \alpha_i'(x).$$

From the definition of Φ and the above relation, we have

$$\frac{D_1 k(x,y)}{\psi\{\varphi^{-1}[G^{-1}(k(x,y))]\}} \le \widehat{c}(X,Y) \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)\,dt \Bigg] \alpha_i'(x). \quad (4.25)$$

Keeping y fixed in (4.25), setting $x = \sigma$ and integrating it with respect to σ from x_0 to $x, x \in I$ and making the change of variable, we obtain

$$\Phi(k(x,y)) \le \Phi(k(x_0,y)) + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} [f_i(s,t)]\,dt\,ds. \quad (4.26)$$

Now, using the inequalities (4.24) and (4.26) in (4.21), we get

$$u(x,y) \le \varphi^{-1}\Bigg\{ G^{-1}\Bigg[\Phi^{-1}\Bigg(\Phi(k(x_0,y)) + $$

$$+ \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} [f_i(s,t)]\,dt\,ds \Bigg) \Bigg] \Bigg\}. \quad (4.27)$$

Taking $X = x, Y = y$ in the inequality (4.27), since X and Y are arbitrary, we get the required inequality.

If $\widehat{a}(x,y) = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(x,y)$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

Corollary 4.15. *Let* u, c, $a_i \in C(\triangle, R_+)$, $\alpha_i \in C^1(J_1, J_1)$, $\beta_i \in C^1(J_2, J_2)$ *be nondecreasing with* $\alpha_i(x) \le x$ *on* J_1, $\beta_i(y) \le y$ *on* J_2, *for* $i = 1, 2, \dots, n$. *And let* $\varphi \in C(R_+, R_+)$ *be an increasing function with* $\varphi(\infty) = \infty$, *and let* c *be a nonnegative constant. Moreover, let* $w_1 \in C(R_+, R_+)$ *be a nondecreasing function with* $w_1 > 0$ *on* $(0, \infty)$. *If*

$$\varphi(u(x,y)) \le c(x,y) + \sum_{i=1}^{n} \Bigg[\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(x_0)}^{\beta_i(y)} a_i(s,t) w_1(u(s,t))\,dt\,ds \Bigg],$$

for $x \in J_1$, $y \in J_2$, *then for* $x_0 \le x \le x_1$, $y_0 \le y \le y_1$, $x_1 \in J_1$, $y_1 \in J_2$,

$$u(x,y) \le \varphi^{-1}\Big\{G^{-1}\Big[G(\widehat{c}(x,y)) + \sum_{i=1}^{n}\Big(\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(x_0)}^{\beta_i(y)} a_i(s,t)\,dt\,ds\Big)\Big]\Big\},$$

where

$$G(r) = \int\limits_{r_0}^{z} \frac{ds}{w_1[\varphi^{-1}(s)]}, \quad r \ge r_0 > 0,$$

and φ^{-1}, G^{-1} *are respectively the inverse of* φ, G, *and* $x_1 \in J_1$, $y_1 \in J_2$, *are so chosen that*

$$G(\widehat{c}(x,y)) + \sum_{i=1}^{n}\Big(\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(x_0)}^{\beta_i(y)} a_i(s,t)\,dt\,ds\Big) \in \mathrm{Dom}(G^{-1}),$$

for all x *and* y *lying in* $[x_0, x_1]$ *and* $[y_0, y_1]$.

For the special case $\varphi(u) = u^p$ ($p > q$ is a constant), Theorem 4.14 gives the following retarded integral inequality for nonlinear functions.

Corollary 4.16. *Let* u, a, c, f_i, g_i, α_i, β_i, $i = 1, \dots, n$ *and* $\psi(u)$ *be as defined in Theorem* 4.14. *Suppose that* $p > q > 0$ *are constants. If*

$$u^p(x,y) \le a(x,y) + c(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\big[f_i(s,t)\psi(u(s,t)) + g_i(s,t)\big]\,dt\,ds$$

for all $(x,y) \in \triangle$, *then*

$$u(x,y) \le$$

$$\le \Big[G_1^{-1}\Big(G_1(k_1(x_0,y)) + \frac{p-q}{p}\,\widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} [f_i(s,t)]\,dt\,ds\Big)\Big]^{\frac{1}{p-q}}$$

for all $(x,y) \in [x_0, x_1] \times [y_0, y_1]$, *where*

$$k_1(x_0,y) = [\widehat{a}(x,y)]^{\frac{p-q}{p}} + \frac{p-q}{p}\,\widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} [g_i(s,t)]\,dt\,ds,$$

$$G_1(r) = \int\limits_{r_0}^{r} \frac{ds}{\psi(s^{\frac{1}{p-q}})}, \quad r \ge r_0 > 0,$$

G_1^{-1} *denotes the inverse function of* G_1 *and* $(x_1, y_1) \in \triangle$ *is so chosen that*

$$\Big(G_1(k_1(x_0,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} [f_i(s,t)]\,dt\,ds\Big) \in \mathrm{Dom}(G_1^{-1}).$$

Corollary 4.17 (Cheung, 2006). *Let u, a, $b \in C(\triangle, R_+)$ and α, $\gamma \in C^1(J_1, J_1)$, β, $\delta \in C^1(J_2, J_2)$ be nondecreasing with $\alpha(x) \leq x$, $\gamma(x) \leq x$ on J_1, $\beta(y) \leq y$, $\delta(y) \leq y$ on J_2. Let $k \geq 0$ and $p > q > 1$ be constants and $\varphi \in C(R_+, R_+)$ is non-decreasing with $\varphi(r) > 0$ for $r > 0$. If*

$$u^p(x,y) \leq k + \frac{p}{p-q} \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(s,t) u^q(s,t)\, ds\, dt + \\ + \frac{p}{p-q} \int\limits_{\gamma(x_0)}^{\gamma(x)} \int\limits_{\delta(y_0)}^{\delta(y)} b(s,t) u^q(s,t) \varphi(u(s,t))\, ds\, dt$$

for $(x,y) \in \triangle$, then

$$u(x,y) \leq \left\{ \Phi_1^{-1} \left[\Phi_1 \left(k^{1-q/p} + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(s,t)\, ds\, dt \right) + \\ + \int\limits_{\gamma(x_0)}^{\gamma(x)} \int\limits_{\delta(y_0)}^{\delta(y)} b(s,t)\, ds\, dt \right] \right\}^{\frac{1}{p-q}}$$

for $(x,y) \in [x_0, x_1] \times [y_0, y_1]$, where Φ_1^{-1} is the inverse functions of

$$\Phi_1(r) = \int\limits_1^r \frac{ds}{\varphi\big(s^{1/(p-q)}\big)}, \quad r > 0,$$

and $x_1 \in J_1$, $y_1 \in J_2$ are so chosen that

$$\Phi_1 \left(k^{1-q/p} + \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(s,t)\, ds\, dt \right) + \int\limits_{\gamma(x_0)}^{\gamma(x)} \int\limits_{\delta(y_0)}^{\delta(y)} b(s,t) b\, ds\, dt \in \mathrm{Dom}(\Phi_1^{-1})$$

for all x and y lying in $[x_0, x_1]$ and $[y_0, y_1]$, respectively.

In the presence of a nonlinear integral term in (4.16) we can obtain other results such as the following theorems. In the same way as in Theorem 4.14 we can prove the following theorems.

Theorem 4.18. *Let u, a, c, f_i, g_i, φ, α_i, β_i, $i = 1, \ldots, n$ and $\psi(u)$ be as defined in Theorem 4.14. If*

$$\varphi(u(x,y)) \leq a(x,y) + \\ + c(x,y) \sum_{i=1}^n \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \varphi(u(s,t)) \big[f_i(s,t) \psi(u(s,t)) + g_i(s,t) \big]\, dt\, ds \quad (4.28)$$

for all $(x,y) \in \triangle$, *then*

$$u(x,y) \le \varphi^{-1}\Big\{\widehat{a}(x,y)\exp\Big[H^{-1}\Big(H(k_2(x_0,y))+ + \widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\,dt\,ds\Big)\Big]\Big\} \tag{4.29}$$

for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$, *where*

$$k_2(x_0,y) = \widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)\,dt\,ds,$$

$$H(r) = \int_{r_0}^{r}\frac{ds}{\psi[\varphi^{-1}(ce^s)]},\quad r \ge r_0 > 0, \tag{4.30}$$

$\widehat{a}$ *and* $\widehat{c}$ *are as defined in (4.15),* H^{-1} *denotes the inverse function of* H *and* $(x_1,y_1)\in\triangle$ *is so chosen that*

$$\Big(H(k_2(x_0,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\;dt\,ds\Big) \in \mathrm{Dom}(H^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 4.14 with suitable modification. We omit the details here. □

For the special case $\varphi(u) = u^p$ ($p > 0$ is a constant), Theorem 4.18 gives the following retarded integral inequality.

Corollary 4.19. *Let* u, a, c, f_i, g_i, α_i, β_i, $i = 1,\dots,n$ *and* $\psi(u)$ *be as defined in Theorem* 4.14. *Suppose that* $p > 0$ *is a constant. If*

$$u^p(x,y) \le a(x,y)+ + c(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} u^p(x,y)\big[f_i(s,t)\psi(u(s,t)) + g_i(s,t)\big]\;dt\,ds$$

for all $(x,y) \in \triangle$, *then*

$$u(x,y) \le \widehat{a}^{\frac{1}{p}}(x,y)\times \times\exp\Big[\frac{1}{p}\,H_1^{-1}\Big(H_1(k_2(x_0,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\;dt\,ds\Big)\Big]$$

for all $(x,y) \in [x_0,x_1] \times [y_0,y_1]$, *where* $k_2(x_0,y)$ *is as defined in Theorem* 4.18*,*

$$H_1(r) = \int\limits_{r_0}^{r} \frac{ds}{\psi\big[a^{\frac{1}{p}}\exp(\frac{s}{p})\big]}, \quad r \geq r_0 > 0,$$

H_1^{-1} *denotes the inverse function of* H_1 *and* $(x_1,y_1) \in \triangle$ *is so chosen that*

$$\bigg(H_1(k_2(x_0,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\,dt\,ds\bigg) \in \mathrm{Dom}(H_1^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 4.18 with suitable modification. We omit the details here. □

Theorem 4.20. *Let* u, a, c, f_i, g_i, φ, α_i, β_i, $i = 1,\ldots,n$ *and* $\psi(u)$ *be as defined in Theorem* 4.14*. If*

$$\varphi(u(x,y)) \leq a(x,y) +$$
$$+c(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} \varphi'(u(s,t))\big[f_i(s,t)\psi(u(s,t)) + g_i(s,t)\big]\,dt\,ds \quad (4.31)$$

for all $(x,y) \in \triangle$*, then*

$$u(x,y) \leq G_2^{-1}\bigg(G_2(k_5(x_0,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\,dt\,ds\bigg) \quad (4.32)$$

for all $(x,y) \in [x_0,x_1] \times [y_0,y_1]$*, where*

$$k_5(x_0,y) = \varphi^{-1}(\widehat{a}(x,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)\,dt\,ds,$$

$$G_2(r) = \int\limits_{r_0}^{r} \frac{ds}{\psi(s)}, \quad r \geq r_0 > 0, \quad (4.33)$$

G_2^{-1} *denotes the inverse function of* G_2 *and* $(x_1,y_1) \in \triangle$ *is so chosen that*

$$\bigg(G_2(k_5(x_0,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\,dt\,ds\bigg) \in \mathrm{Dom}(G_2^{-1}).$$

Proof. Fixing any numbers X and Y with $x_0 \le x \le X$ and $y_0 \le y \le Y$, we assume that $\widehat{a}(X,Y)$ is positive and define a positive function $z(x,y)$ by

$$z(x,y) = \widehat{a}(X,Y) + \\ + \widehat{c}(X,Y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \varphi'(u(x,y)) \big[f_i(s,t)\psi(u(s,t)) + g_i(s,t) \big] \, dt \, ds. \quad (4.34)$$

Then $z(x,y) > 0$, $z(x_0,y) = z(x,y_0) = \widehat{a}(X,Y)$ and (4.31) can be restated as

$$u(x,y) \le \varphi^{-1}[z(x,y)]. \quad (4.35)$$

It is easy to observe that $z(x,y)$ is a continuous non-decreasing function for all $x \in I$, $y \in J$ and

$$D_1 z(x,y) \le \widehat{c}(X,Y)\varphi'\big(\varphi^{-1}(z(\alpha_i(x),t))\big) \times \\ \times \sum_{i=1}^{n} \Big[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)\psi\big(\varphi^{-1}(z(\alpha_i(x),t))\big) + g_i(\alpha_i(x),t) \Big] \, dt \Big] \alpha_i'(x). \quad (4.36)$$

From the above relation, we have

$$\frac{D_1 z(x,y)}{\varphi'(\varphi^{-1}(z(x,y)))} \le \widehat{c}(X,Y) \times \\ \times \sum_{i=1}^{n} \Big[\int_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)\psi\big(\varphi^{-1}(z(\alpha_i(x),t))\big) + g_i(\alpha_i(x),t) \Big] \, dt \Big] \alpha_i'(x). \quad (4.37)$$

Keeping y fixed in (4.37), setting $x = \sigma$ and integrating it with respect to σ from x_0 to $x, x \in J_1$ and making the change of variable, we obtain

$$\varphi^{-1}(z(x,y)) \le \varphi^{-1}(z(x_0,y)) + \\ + \widehat{c}(X,Y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \Big\{ f_i(s,t)\psi\big[\varphi^{-1}(z(s,t))\big] + g_i(s,t) \Big\} \, dt \, ds. \quad (4.38)$$

Now, define a function $k(x,y)$ by

$$k(x,y) = \varphi^{-1}(z(x_0,y)) + \widehat{c}(X,Y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(X)} \int_{\beta_i(y_0)}^{\beta_i(Y)} g_i(s,t) \, dt \, ds + \\ + \widehat{c}(X,Y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\psi\big[\varphi^{-1}(z(s,t))\big] \, dt \, ds.$$

Then (4.38) can be restated as

$$\varphi^{-1}(z(x,y)) \le k(x,y)). \quad (4.39)$$

It is easy to observe that $k(x,y)$ is a continuous non-decreasing function for all $x \in J_1$, $y \in J_2$ and

$$D_1 k(x,y) = \widehat{c}(X,Y) \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)\psi\big[\varphi^{-1}(z(\alpha_i(x),t))\big]\,dt \Bigg] \alpha_i'(x) \leq$$

$$\leq \widehat{c}(X,Y)\psi(k(x,y)) \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)\,dt \Bigg] \alpha_i'(x).$$

From the above relation, we have

$$\frac{D_1 k(x,y)}{\psi(k(x,y))} \leq \widehat{c}(X,Y) \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)\,dt \Bigg] \alpha_i'(x). \tag{4.40}$$

Keeping y fixed in (4.40), setting $x = \sigma$, and integrating it with respect to σ from x_0 to x, $x \in J_1$, using the definition of G_2 and making the change of variable, we obtain

$$G_2(k(x,y)) \leq G_2(k(x_0,y)) + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\,dt\,ds. \tag{4.41}$$

Now, using the inequalities (4.39) and (4.41) in (4.35), we get

$$u(x,y) \leq G_2^{-1}\Bigg(G_2(k(x_0,y)) + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\,dt\,ds \Bigg). \tag{4.42}$$

Taking $X = x, Y = y$ in the inequality (4.42), since X and Y are arbitrary, we get the required inequality.

If $\widehat{a}(X,Y) = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X,Y)$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

Theorem 4.21. *Let u, a, c, f_i, $g_i \in C(\triangle, R_+)$, $i = 1, \dots, n$, and let $\alpha_i \in C^1(J_1, J_1)$ be nondecreasing with $\alpha_i(t) \leq t$, $i = 1, \dots, n$, and $\beta_i \in C^1(J_2, J_2)$ be nondecreasing with $\beta_i(t) \leq t$, $i = 1, \dots, n$. Let $\varphi \in C^1(R_+, R_+)$ be an increasing function with $\varphi(\infty) = \infty$ and φ' be a nondecreasing function. And let ψ_1, $\psi_2 \in C(R_+, R_+)$ be a non-decreasing function with $\psi_1, \psi_2 > 0$ on $(0, \infty)$. If*

$$\varphi(u(x,y)) \leq a(x,y) + c(x,y) \times$$

$$\times \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \varphi'(u(s,t)) \Big[f_i(s,t)\psi_1(u(s,t)) + g_i(s,t)\psi_2(u(s,t)) \Big]\,dt\,ds \tag{4.43}$$

for all $x \in J_1$, $y \in J_2$, *then, for* $x_0 \le x \le x_2$, $y_0 \le y \le y_2$ *with* $x_2 \in J_1$, $y_2 \in J_2$, *we have the following results.*

(1) *For the case* $\psi_2(u) \le \psi_1(u)$,

$$u(x,y) \le G_1^{-1}\Big[G_1\big(\varphi^{-1}(\widehat{a}(x,y))\big) + \\ + \widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)+g_i(s,t)\big]\,dt\,ds\Big],$$

(2) *For the case* $\psi_1(u) \le \psi_2(u)$,

$$u(x,y) \le G_2^{-1}\Big[G_2\big(\varphi^{-1}(\widehat{a}(x,y))\big) + \\ + \widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)+g_i(s,t)\big]\,dt\,ds\Big],$$

where

$$G_i(z) = \int_{z_0}^{z}\frac{ds}{\psi_i[(s)]}, \quad z \ge z_0 > 0 \ \ (i=1,2),$$

φ^{-1}, G_i^{-1} *are, respectively, the inverses of* φ, G_i *for* $i = 1,2$ *and* $x_2 \in J_1$, $y_2 \in J_2$ *are so chosen that*

$$G_i\big(\varphi^{-1}(\widehat{a}(x,y))\big)+\widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)+g_i(s,t)\big]\,dt\,ds \in \mathrm{Dom}(G_i^{-1})$$

for all $x \in [x_0, x_2]$ *and* $y \in [y_0, y_2]$.

Proof. Fixing any numbers X and Y with $x_0 \le x \le X$ and $y_0 \le y \le Y$, we assume that $\widehat{a}(X,Y)$ is positive and define a positive function $z(x,y)$ by

$$z(x,y) = \widehat{a}(X,Y) + \widehat{c}(X,Y)\times \\ \times\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\varphi'(u(s,t))\Big[f_i(s,t)\psi_1(u(s,t))+g_i(s,t)\psi_2(u(s,t))\Big]\,dt\,ds. \quad (4.44)$$

Then $z(x,y) > 0$, $z(x_0,y) = z(x,y_0) = \widehat{a}(X,Y)$ and (4.43) can be restated as

$$u(x,y) \le \varphi^{-1}[z(x,y)]. \quad (4.45)$$

It is easy to observe that $z(x,y)$ is a continuous non-decreasing function for all $x \in J_1$, $y \in J_2$ and

$$D_1 z(x,y) = \widehat{c}(X,Y) \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} \varphi'(u(\alpha_i(x),t)) \times$$

$$\times \Big[f_i(\alpha_i(x),t)\psi_1\big(u(\alpha_i(x),t)\big) + g_i(\alpha_i(x),t)\psi_2\big(u(\alpha_i(x),t)\big)\Big]\, dt \Bigg] \alpha_i'(x) \le$$

$$\le \widehat{c}(X,Y)\varphi'\big[\varphi^{-1}(z(x,y))\big] \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)\psi_1\big(\varphi^{-1}(z(\alpha_i(x),t))\big) +$$

$$+ g_i(\alpha_i(x),t)\psi_1\big(\varphi^{-1}(z(\alpha_i(x),t))\big)\Big]\, dt \Bigg] \alpha_i'(x).$$

Using the monotonicity of φ', φ^{-1}, and z, we deduce

$$\varphi'\big[\varphi^{-1}(z(x,y))\big] \ge \varphi'\big[\varphi^{-1}(z(x_0,y_0))\big] = \varphi'\big[\varphi^{-1}(\widehat{a}(X,Y))\big] > 0.$$

From the above relation, we have

$$\frac{D_1 z(x,y)}{\varphi'[\varphi^{-1}(z(x,y))]} \le \widehat{c}(X,Y) \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} \Big[f_i(\alpha_i(x),t)\psi_1\big(\varphi^{-1}(z(\alpha_i(x),t))\big) +$$

$$+ g_i(\alpha_i(x),t)\psi_1\big(\varphi^{-1}(z(\alpha_i(x),t))\big)\Big]\, dt \Bigg] \alpha_i'(x). \quad (4.46)$$

Observe that for any continuously differentiable and invertible function $\zeta(\xi)$, by a change of variable $\eta = \zeta^{-1}(\xi)$, we have

$$\int \frac{d\xi}{\zeta'[\zeta^{-1}(\xi)]} = \int \frac{\zeta'(\eta)}{\zeta'(\eta)} = \eta + c = \zeta^{-1}(\xi) + c. \quad (4.47)$$

Keeping y fixed in (4.46), setting $x = \sigma$ and integrating it with respect to σ from x_0 to x, $x \in I$, using (4.47) to the left-hand side and making the change of variable, we obtain

$$\varphi^{-1}(z(x,y)) \le$$

$$\le \varphi^{-1}(z(x_0,y)) + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \Big\{ f_i(s,t)\psi_1\big[\varphi^{-1}(z(s,t))\big] +$$

$$+ g_i(s,t)\psi_2\big[\varphi^{-1}(z(s,t))\big]\Big\}\, dt\, ds. \quad (4.48)$$

When $\psi_1(u) \geq \psi_2(u)$, from the inequality (4.48), we find

$$\varphi^{-1}(z(x,y)) \leq \varphi^{-1}(z(x_0,y)) + \widehat{c}(X,Y)\times \\ \times \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \Big\{ \big[f_i(s,t) + g_i(s,t)\big] \psi_1\big(\varphi^{-1}(z(s,t))\big) \Big\}\, dt\, ds. \quad (4.49)$$

Now, using the definition of the function G_1, and applying Corollary 4.15 to the inequality (4.49), we conclude that

$$\varphi^{-1}(z(x,y)) \leq G_1^{-1}\Bigg[G_1\big[\varphi^{-1}(z(x_0,y))\big] + \\ + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \big[f_i(s,t) + g_i(s,t)\big]\, dt\, ds \Bigg]. \quad (4.50)$$

Similarly, when $\psi_2(u) \leq \psi_1(u)$, from the inequality (4.49), we find

$$\varphi^{-1}(z(x,y)) \leq \varphi^{-1}(z(x_0,y)) + \\ + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \Big\{ \big[f_i(s,t) + g_i(s,t)\big] \psi_1\big(\varphi^{-1}(z(s,t))\big) \Big\}\, dt\, ds. \quad (4.51)$$

Now, using the definition of the function G_2, and applying Corollary 4.15 to the inequality (4.51), we conclude that

$$\varphi^{-1}(z(x,y)) \leq G_2^{-1}\Bigg[G_2\big[\varphi^{-1}(z(x_0,y))\big] + \\ + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \big[f_i(s,t) + g_i(s,t)\big]\, dt\, ds \Bigg]. \quad (4.52)$$

Now, using the inequalities (4.50) and (4.52) in (4.45), taking $X = x$, $Y = y$, since X and Y are arbitrary, we get the required inequality.

If $\widehat{a}(X,Y) = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X,Y)$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

5. Gronwall–Bellman-Type Nonlinear Inequalities II

During the past twenty years several authors have developed extensions and variants of Wendroff's inequality and exhibited applications in partial differential and integral equations. Next we deal with the Wendroff-type inequalities investigated by Pachpatte [60] and Bondge and Pachpatte [17], [18]. These inequalities which can be used in the study of certain partial differential and integral equations. Pachpatte [60] established the Wendroff-type inequalities in the following theorem.

Theorem 5.1 (Pachpatte, 1995)**.** *Let $u(x,y)$, $a(x,y)$, and $b(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$ and let $L : R_+^3 \to R_+$ be a continuous function which satisfies the condition*

$$0 \le L(x,y,v) - L(x,y,w) \le M(x,y,w)(v-w), \tag{5.1}$$

for x, $y \in R_+$ and $v \ge w \ge 0$, where $M : R_+^3 \to R_+$ is a continuous function.

(i) *If*

$$u(x,y) \le a(x,y) + b(x,y) \int_0^x \int_0^y L(s,t,u(s,t))\, ds\, dt, \tag{5.2}$$

for x, $y \in R_+$, then

$$u(x,y) \le a(x,y) + b(x,y) A(x,y) \exp\left(\int_0^x \int_0^y M(s,t,a(s,t)) b(s,t)\, ds\, dt \right), \tag{5.3}$$

for x, $y \in R_+$, where

$$A(x,y) = \int_0^x \int_0^y L(s,t,a(s,t))\, ds\, dt, \tag{5.4}$$

for x, $y \in R_+$.

(ii) *Let $F(u)$ be continuous, strictly increasing, convex and submultiplicative function for $u > 0$, $\lim_{u\to\infty} F(u) = \infty$, F^{-1} denote the inverse function of F, $\alpha(x,y)$, $\beta(x,y)$ be continuous and positive functions defined on R_+^2 and $\alpha(x,y) + \beta(x,y) = 1$. If*

$$u(x,y) \le a(x,y) + b(x,y) F^{-1}\left(\int_0^x \int_0^y L(s,t,F(u(s,t)))\, ds\, dt \right), \tag{5.5}$$

for x, $y \in R_+$, then

$$\begin{aligned} u(x,y) &\le a(x,y) + \\ &+ b(x,y) F^{-1}\Bigg[\left(\int_0^x \int_0^y L\Big(s,t,\alpha(s,t) F\big(a(s,t)\alpha^{-1}(s,t)\big)\Big)\, ds\, dt \right) \times \\ &\times \exp\Bigg(\int_0^x \int_0^y M\Big(s,t,\alpha(s,t) F\big(a(s,t)\alpha^{-1}(s,t)\big)\Big) \times \\ &\times \beta(s,t) F\big(b(s,t)\beta^{-1}(s,t)\big)\, ds\, dt \Bigg) \Bigg], \end{aligned} \tag{5.6}$$

for x, $y \in R_+$.

(iii) *Let $g(u)$, $h(u)$ be continuously differentiable functions defined for $u \geq 0$, $g(u) > 0$, $h(u) > 0$ for $u > 0$ and $g'(u) \geq 0$, $h'(u) \geq 0$ for $u \geq 0$, and let $g(u)$ be subadditive and submultiplicative for $u \geq 0$. If*

$$u(x,y) \leq a(x,y) + b(x,y)h\Big(\int_0^x \int_0^y L\big(s,t,g(u(s,t))\big)\,ds\,dt\Big), \tag{5.7}$$

for x, $y \in R_+$, then for $0 \leq x \leq x_1$, $0 \leq y \leq y_1$,

$$u(x,y) \leq a(x,y)+$$
$$+b(x,y)h\Big[G^{-1}\Big(G(B(x,y))+\int_0^x\int_0^y M\big(s,t,g(a(s,t))g(b(s,t))\big)\,ds\,dt\Big)\Big], \tag{5.8}$$

where

$$B(x,y) = \int_0^x \int_0^y L\big(s,t,g(a(s,t))\big)\,ds\,dt, \tag{5.9}$$

and G, G^{-1} are as defined in Theorem 4.2 and x_1, y_1 are so chosen that

$$G(B(x,y)) + \int_0^x \int_0^y M\big(s,t,g(a(s,t))g(b(s,t))\big)\,ds\,dt \in \mathrm{Dom}(G^{-1})$$

for all x, y lying in the subintervals $0 \leq x \leq x_1$, $0 \leq y \leq y_1$ of R_+.

The next two theorems proved by Bondge and Pachpatte [18] can be used more effectively in certain situations.

Theorem 5.2 (Bondge and Pachpatte, 1979)**.** *Let $u(x,y)$, $u_x(x,y)$, $u_x(x,y)$, and $u_{xy}(x,y)$ be nonnegative continuous functions defined for $x, y \in R_+$, $u(x,0) = u(0,y) = 0$, and $p(x,y) \geq 1$ be a continuous function defined for x, $y \in R_+$. Let $g(u)$, $g'(u)$, $a(x)$, $a'(x)$, $b(y)$, and $b'(y)$ be as in Theorem 4.1. If*

$$u_{xy}(x,y) \leq a(x) + b(y) + M\Big[u(x,y) + \int_0^x \int_0^y p(s,t)g(u_{st}(s,t))\,ds\,dt\Big], \tag{5.10}$$

for x, $y \in R_+$, then for $0 \leq x \leq x_2$, $0 \leq y \leq y_2$

$$u_{xy}(x,y) \leq H^{-1}\Big[H(a(0)+b(y))+$$
$$+\int_0^x \frac{a'(s)}{a(s)+b(0)+g(a(s)+b(0))}\,ds + M\int_0^x\int_0^y p(s,t)\,ds\,dt\Big], \tag{5.11}$$

where

$$H(r) = \int_{r_0}^r \frac{ds}{s+g(s)}, \quad r > 0, \quad r_0 > 0, \tag{5.12}$$

and H^{-1} *is the inverse function of* H, *and* x_2, y_2 *are chosen so that*

$$H(a(0)+b(y))+ \\ +\int_0^x \frac{a'(s)}{a(s)+b(0)+g(a(s)+b(0))}\,ds + M\int_0^x\int_0^y p(s,t)\,ds\,dt \in \mathrm{Dom}(\Omega^{-1}).$$

for all x, y *lying in the subintervals* $0 \le x \le x_2$, $0 \le y \le y_2$ *of* R_+.

Theorem 5.3 (Bondge and Pachpatte, 1979)**.** *Let* $u(x,y)$, $u_{xy}(x,y)$, $p(x,y)$, $g(u)$, $g'(u)$, $a(x)$, $a'(x)$, $b(y)$, *and* $b'(y)$ *be as in Theorem* 5.2. *If*

$$u_{xy}(x,y) \le a(x) + b(y) + \int_0^x\int_0^y p(s,t)g(u(s,t)+u_{st}(s,t))\,ds\,dt, \tag{5.13}$$

for x, $y \in R_+$, *then for* $0 \le x \le x_3$, $0 \le y \le y_3$

$$u_{xy}(x,y) \le a(x) + b(y) + \int_0^x\int_0^y p(s,t)g\Bigg(H^{-1}\Bigg[H(a(0)+b(y))+ \\ +\int_0^s \frac{a'(s_1)}{a(s_1)+b(0)+g(a(s_1)+b(0))}\,ds_1 + \int_0^s\int_0^t p(s_1,t_1)\,ds_1\,dt_1\Bigg]\Bigg)\,ds_1\,dt_1, \tag{5.14}$$

where H, H^{-1} *are as defined in Theorem* 5.2 *and* x_3, y_3 *are chosen so that*

$$H(a(0)+b(y)) + \int_0^s \frac{a'(s_1)}{a(s_1)+b(0)+g(a(s_1)+b(0))}\,ds_1 + \\ + \int_0^s\int_0^t p(s_1,t_1)\,ds_1\,dt_1 \in \mathrm{Dom}(\Omega^{-1}).$$

for all x, y *lying in the subintervals* $0 \le x \le x_3$, $0 \le y \le y_3$ *of* R_+.

Another interesting and useful two independent-variable inequalities given by Cheung and Ma [23] read as following from Theorem 5.4 to 5.5.

Theorem 5.4 (Cheung and Ma, 2005)**.** *Let* $u(x,y)$, $a(x,y)$ $c(x,y)$ *and* $d(x,y)$ *be nonnegative continuous functions defined for* x, $y \in R_+$ *and* $w(u)$ *be a nonnegative, nondecreasing continuous function for* $u \in R_+$ *with* $w(u) > 0$ *for* $u > 0$ *and* $e(x,y)$, $f(x,y) \in C(R_+^2, R_+)$. *Let* $\varphi(u) \in C^1(R_+, R_+)$ *with* $\varphi'(u) > 0$ *for* $u > 0$, *here* φ' *denotes the derivative of* φ. *Assume that* $b(x,y)$ *and* $d(x,y)$ *be are nondecreasing in* x *and non-increasing in* y, *and* $L, M \in C(R_+^3, R_+)$ *satisfy*

$$0 \le L(x,y,v) - L(x,y,w) \le M(x,y,w)(v-w)$$

for all $s, y, v, w \in R_+$ *with* $v \geq w$. *If*

$$\varphi(u(x,y)) \leq a(x,y) + b(x,y)\int_{\alpha}^{x} c(s,y)\varphi(u(s,y))\,ds + \\ + d(x,y)\int_{0}^{x}\int_{y}^{\infty}\varphi'(u(s,t))\big[f(s,t)L(s,t,u(s,t)) + e(s,t)\big]\,ds\,dt,$$

for $x,\ y,\ \alpha \in R_+$ *with* $\alpha \leq x$, *then*

$$u(x,y) \leq A_1(x,y) + p(x,y)d(x,y)A_2(x,y)\exp(A_3(x,y)),$$

for all $x,\ y \in R_+$, *where*

$$p(x,y) = 1 + b(x,y)\int_{\alpha}^{x} c(s,y)\exp\Big(\int_{s}^{x} b(m,y)c(m,y)\,dm\Big)ds,$$

$$E_1(x,y) = d(x,y)\int_{0}^{x}\int_{y}^{\infty} e(s,t)\,ds\,dt,$$

$$A_1(x,y) = \varphi^{-1}(p(x,y)a(x,y)) + p(x,y)E_1(x,y),$$

$$A_2(x,y) = \int_{0}^{x}\int_{y}^{\infty} f(s,t)L(s,t,A_1(s,t))\,ds\,dt,$$

$$A_3(x,y) = \int_{0}^{x}\int_{y}^{\infty} f(s,t)p(s,t)d(s,t)M(s,t,A_1(s,t))\,ds\,dt.$$

Theorem 5.5 (Cheung and Ma, 2005)**.** *Let* $u(x,y)$, $f(x,y)$, $e(x,y)$, $\varphi(u)$, $w(u)$, $L(x,y,v)$, *and* $M(x,y,v)$ *be as defined in Theorem* 5.4. *Let* $a(x,y)$, $b(x,y)$, $c(x,y)$, *and* $d(x,y)$ *be nonnegative continuous and nonincreasing in each variable* $x,\ y \in R_+$. *If*

$$\varphi(u(x,y)) \leq a(x,y) + b(x,y)\int_{x}^{\beta} c(s,y)\varphi(u(s,y))\,ds + \\ + d(x,y)\int_{x}^{\infty}\int_{y}^{\infty}\varphi'(u(s,t))\big[f(s,t)L(s,t,u(s,t)) + e(s,t)\big]\,ds\,dt,$$

for $x,\ y,\ \beta \in R_+$ *with* $x \leq \beta$, *then*

$$u(x,y) \leq \overline{A}_1(x,y) + \overline{p}(x,y)d(x,y)\overline{A}_2(x,y)\exp(\overline{A}_3(x,y))$$

for all x, $y \in R_+$, *where*

$$\overline{p}(x,y) = 1 + b(x,y)\int_{\alpha}^{x} c(s,y)\exp\Big(\int_{s}^{x} b(m,y)c(m,y)\,dm\Big)\,ds,$$

$$\overline{E}_1(x,y) = d(x,y)\int_0^x\int_y^\infty e(s,t)\,ds\,dt,$$

$$\overline{A}_1(x,y) = \varphi^{-1}\big(\overline{p}(x,y)a(x,y)\big) + \overline{p}(x,y)\overline{E}_1(x,y),$$

$$\overline{A}_2(x,y) = \int_0^x\int_y^\infty f(s,t)L(s,t,A_1(s,t))\,ds\,dt,$$

$$\overline{A}_3(x,y) = \int_0^x\int_y^\infty f(s,t)\overline{p}(s,t)d(s,t)M(s,t,\overline{A}_1(s,t))\,ds\,dt.$$

In [23], Ma and Pečarić established new explicit bounds on the solutions to a class of new nonlinear retarded integral inequalities in two independent variables as following Theorem 5.6. In what follows,

$$E = \Big\{(x,y,s,t) \in \triangle^2 :\ x_0 \le s \le x \le M,\ y_0 \le t \le y \le N\Big\}.$$

Theorem 5.6 (Ma and Pečarić, 2008)**.** *Let* $u(x,y)$, $l(x,y) \in C(\triangle, R_+)$, $a(x,y,s,t)$, $b(x,y,s,t) \in C(E, R_+)$ *with* $a(x,y,s,t)$ *and* $b(x,y,s,t)$ *be nondecreasing in* x *and* y *for each* $s \in J_1$, *and* $t \in J_2$, *and* $\alpha \in C^1(J_1,J_1)$, $\beta \in C^1(J_2,J_2)$ *be nondecreasing with* $\alpha(x) \le x$ *on* J_1, $\beta(y) \le y$ *on* J_2, *and* L, $M \in C(R_+^3, R_+)$ *satisfy*

$$0 \le L(x,y,v) - L(x,y,w) \le M(x,y,w)(v-w)$$

for all s, y, v, $w \in R_+$ *with* $v \ge w$. *If*

$$u^p(x,y) \le l(x,y) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)u^q(s,t)\,ds\,dt + \int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)L(s,t,u(s,t))\,ds\,dt,$$

for $(x,y) \in \triangle$, *where* $p \ge q \ge 0$, $p \ge r \ge 0$, p, q, *and* r *are constants and*

$$\lambda(x,y) = \frac{1}{p}\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)M\Big(s,t,\frac{p-1}{p} + \frac{1}{p}\,l(s,t)\Big)\exp(A_1(s,t))\,ds\,dt < 1$$

for $(x,y) \in \triangle$*, then*

$$u(x,y) \le \left[l(x,y) + \frac{\overline{A}_1(x,y) + L_1(x,y)}{1-\lambda(x,y)} \exp(A_1(x,y))\right]^{\frac{1}{p}},$$

for $(x,y) \in \triangle$ *and any* $K_1(x,y) \in C(\triangle, R_0)$*,* $i = 1, 2$*, where*

$$A_1(x,y) = \frac{q}{p} \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) K_1^{\frac{q-p}{p}}(s,t)\, ds\, dt,$$

$$\overline{A}_1(x,y) = \int\limits_{\alpha(x_0)}^{\alpha(x)} \int\limits_{\beta(y_0)}^{\beta(y)} a(x,y,s,t) \left[\frac{q}{p} K_1^{\frac{q-p}{p}}(s,t) l(s,t) + \frac{p-q}{p} K_1^{\frac{q}{p}}(s,t)\right] ds\, dt,$$

$$L_1(x,y) = \int\limits_{\alpha(x_0)}^{\alpha(M)} \int\limits_{\beta(y_0)}^{\beta(N)} L\Big(s,t, \frac{p-1}{p} + \frac{1}{p} l(s,t)\Big)\, ds\, dt,$$

for $(x,y) \in \triangle$*.*

We next present some new nonlinear retarded Gronwall–Bellman-type integral inequalities in two independent variables as following Theorems, which can be used as effective tools in the study of certain integral equations. In what follows, given a continuous function $a : J_1 \times J_2 \to R_+$, we write

$$\widehat{a}(x,y) = \max \Big\{a(s,t) :\ x_0 \le s \le x,\ y_0 \le s \le y\Big\}.$$

Theorem 5.7. *Let* u*,* a*,* $c \in C(\triangle, R_+)$*,* f_i*,* $g_i \in C(\triangle, R_+)$*,* $i = 1, \dots, n$*, and let* $\alpha_i \in C^1(J_1, J_1)$ *be nondecreasing with* $\alpha_i(t) \le t$*,* $i = 1, \dots, n$*, and* $\beta_i \in C^1(J_2, J_2)$ *be nondecreasing with* $\beta_i(t) \le t$*,* $i = 1, \dots, n$*. Suppose that* $q > 0$ *is a constant,* $\varphi \in C(R_+, R_+)$ *is an increasing function with* $\varphi(\infty) = \infty$ *and* $\psi(u)$ *is a nondecreasing continuous function for* $u \in R_+$ *with* $\psi(u) > 0$ *for* $u > 0$ *and let* $L : \triangle \times R_+ \to R_+$ *be a continuous function which satisfies the condition*

$$0 \le L(x,y,v) - L(x,y,w) \le M(x,y,w)(v-w), \tag{5.15}$$

for x*,* $y \in R_+$ *and* $v \ge w \ge 0$*, where* $M : \triangle \times R_+ \to R_+$ *is a continuous function. If*

$$\varphi(u(x,y)) \le a(x,y) + c(x,y) \times$$
$$\times \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t) \big[f_i(s,t) L(s,t,u(s,t)) + g_i(s,t)\big]\, dt\, ds \tag{5.16}$$

for all $(x, y) \in \triangle$, *then*

$$u(x,y) \le \varphi^{-1}\Big\{G^{-1}\Big[\Phi^{-1}\Big(\Phi(k(x_0,y))+ + \widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)M(s,t)\,dt\,ds\Big)\Big]\Big\} \quad (5.17)$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$, *where*

$$k(x_0,y) = G(\widehat{a}(x,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t)+g_i(s,t)\big]\,dt\,ds,$$

$$G(r) = \int_{r_0}^{r}\frac{ds}{[\varphi^{-1}(s)]^q}, \quad r \ge r_0 > 0,$$

$$\Phi(r) = \int_{r_0}^{r}\frac{ds}{\varphi^{-1}(G^{-1}(s))}, \quad r \ge r_0 > 0,$$

G^{-1}, Φ^{-1} *denote the inverse function of* G, Φ *and* $(x_1, y_1) \in \triangle$ *is chosen so that*

$$\Big[\Phi(k(x_0,y)) + c(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)M(s,t)\,dt\,ds\Big] \in \mathrm{Dom}(\Phi^{-1}).$$

Proof. Fixing any numbers X and Y with $x_0 \le x \le X$ and $y_0 \le y \le Y$, we assume that $\widehat{a}(X, Y)$ is positive and define a positive function $z(x, y)$ by

$$z(x,y) = \widehat{a}(X,Y) + \widehat{c}(X,Y)\times \times\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\big[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)\big]\,dt\,ds. \quad (5.18)$$

Then $z(x, y) > 0$, $z(x_0, y) = z(x, y_0) = \widehat{a}(X, Y)$, and (5.15) can be restated as

$$u(x,y) \le \varphi^{-1}[z(x,y)]. \quad (5.19)$$

It is easy to observe that $z(x, y)$ is a continuous non-decreasing function for all $x \in J_1$, $y \in J_2$ and

$$D_1 z(x,y) = \widehat{c}(X,Y)\sum_{i=1}^{n}\Big[\int_{\beta_i(y_0)}^{\beta_i(y)} u^q(\alpha_i(x),t)\times \times\Big[f_i(\alpha_i(x),t)L\big(\alpha_i(x),t,u(\alpha_i(x),t)\big) + g_i(\alpha_i(x),t)\Big]\,dt\Big]\alpha_i'(x) \le$$

$$\leq \widehat{c}(X,Y)\big[\varphi^{-1}(z(x,y))\big]^q\times$$

$$\times\sum_{i=1}^{n}\Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big[f_i(\alpha_i(x),t)L\big(\alpha_i(x),t,\varphi^{-1}(z(\alpha_i(x),t))\big)+g_i(\alpha_i(x),t)\Big]\,dt\Bigg]\alpha_i'(x).$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\big[\varphi^{-1}(z(x,y))\big]^q \geq \big[\varphi^{-1}(z(x_0,y_0))\big]^q = \big[\varphi^{-1}(\widehat{a}(X,Y))\big]^q > 0.$$

From the definition of G and the above relation, we have

$$D_1G(z(x,y)) = \frac{D_1z(x,y)}{[\varphi^{-1}(z(x,y))]^q} \leq \widehat{c}(X,Y)\sum_{i=1}^{n}\Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big[f_i(\alpha_i(x),t)\times$$

$$\times L\big(\alpha_i(x),t,\varphi^{-1}(z(\alpha_i(x),t))\big)+g_i(\alpha_i(x),t)\Big]\,dt\Bigg]\alpha_i'(x). \quad (5.20)$$

Keeping y fixed in (5.6), setting $x=\sigma$, integrating it with respect to σ from x_0 to $x, x\in J_1$, and making the change of variable, we obtain

$$G(z(x,y)) \leq G(z(x_0,y)) + \widehat{c}(X,Y)\times$$

$$\times\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big\{f_i(s,t)L\big[s,t,\varphi^{-1}(z(s,t))\big]+g_i(s,t)\Big\}\,dt\,ds. \quad (5.21)$$

Now, define a function $k(x,y)$ by

$$k(x,y) = G(\widehat{a}(X,Y)) +$$

$$+\,\widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(X)}\int\limits_{\beta_i(y_0)}^{\beta_i(Y)}\big[f_i(s,t)L(s,t)+g_i(s,t)\big]\,dtds+$$

$$+\,\widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t)\varphi^{-1}(z(s,t))\big]\,dt\,ds.$$

Then

$$k(x_0,y) = G(\widehat{a}(X,Y)) +$$

$$+\,\widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(X)}\int\limits_{\beta_i(y_0)}^{\beta_i(Y)}\big[f_i(s,t)L(s,t)+g_i(s,t)\big]\,dt\,ds,$$

and (5.7) can be restated as

$$z(x,y) \leq G^{-1}[k(x,y)]. \quad (5.22)$$

It is easy to observe that $k(x, y)$ is a continuous non-decreasing function for all $x \in J_1, y \in J_2$ and

$$D_1 k(x,y) = \widehat{c}(X,Y) \times \times \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t) M(\alpha_i(x),t) \varphi^{-1}\big(z(\alpha_i(x),t)\big)\, dt \Bigg] \alpha_i'(x) \leq$$

$$\leq \widehat{c}(X,Y) \varphi^{-1}\big[G^{-1}(k(x,y))\big] \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t) M(\alpha_i(x),t)\, dt \Bigg] \alpha_i'(x).$$

From the above relation, we have

$$\frac{D_1 k(x,y)}{\varphi^{-1}[G^{-1}(k(x,y))]} \leq \leq \widehat{c}(X,Y) \sum_{i=1}^{n} \Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t) M(\alpha_i(x),t)\, dt \Bigg] \alpha_i'(x). \quad (5.23)$$

Keeping y fixed in (5.23), setting $x = \sigma$, integrating it with respect to σ from x_0 to $x, x \in J_1$, and making the change of variable, from the definition of Φ, we obtain

$$\Phi(k(x,y)) \leq \Phi(k(x_0,y)) + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t) M(s,t)\, dt\, ds. \quad (5.24)$$

Now, using the inequalities (5.22) and (5.24) in (5.19), we get

$$u(x,y) \leq \varphi^{-1}\Bigg\{ G^{-1}\Bigg[\Phi^{-1}\Bigg(\Phi(k(x_0,y)) + + \widehat{c}(X,Y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t) M(s,t)\, dt\, ds \Bigg) \Bigg] \Bigg\}. \quad (5.25)$$

Allowing $X = x, Y = y$ in the inequality (5.25), since X and Y are arbitrary, we get the required inequality.

If $\widehat{a}(X,Y) = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X,Y)$ and subsequently let $\varepsilon \to 0$. This completes the proof. $\square$

For the special case $\varphi(u) = u^p$ ($p > q$ is a constant), Theorem 5.7 gives the following retarded integral inequality for nonlinear functions.

Corollary 5.8. *Let u, a, c, f_i, g_i, $\alpha_i(t)$, β_i, $i = 1, \dots, n$, L and M be as defined in Theorem 5.7. Suppose that $p > q > 0$ are constants. If*

$$u^p(x,y) \le a(x,y) + c(x,y)\times \\ \times \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\big[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)\big]\,dt\,ds \tag{5.26}$$

for all $(x,y) \in \triangle$, then

$$u(x,y) \le \Bigg[G_1^{-1}\Bigg(G_1(k_1(x_0,y)) + \\ + \frac{p-q}{p}\widehat{c}(x,y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)M(s,t)\,dt\,ds\Bigg)\Bigg]^{\frac{1}{p-q}} \tag{5.27}$$

for all $(x,y) \in [x_0, x_1] \times [y_0, y_1]$, where

$$k_1(x_0,y) = \big[\widehat{a}(x,y)\big]^{\frac{p-q}{p}} + \\ + \frac{p-q}{p}\,\widehat{c}(x,y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \big[f_i(s,t)L(s,t) + g_i(s,t)\big]\,dt\,ds,$$

$$G_1(r) = \int\limits_{r_0}^{r} \frac{ds}{\varphi(s^{\frac{1}{p-q}})}, \quad r \ge r_0 > 0,$$

G_1^{-1} denotes the inverse function of G_1 and $(x_1,y_1) \in \triangle$ is so chosen that

$$\Bigg(G_1(k_1(x_0,y)) + \frac{p-q}{p}\,\widehat{c}(x,y) \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)M(s,t)\,dt\,ds\Bigg) \in \mathrm{Dom}(G_1^{-1}).$$

In the presence of a nonlinear integral term in (5.16) we can obtain other results, such as the following theorems. In the same way as in Theorem 5.7 we can prove the following theorems.

Theorem 5.9. *Let u, a, c, f_i, g_i, α_i, $\beta_i(t)$, $i = 1, \dots, n$, φ, L and M be as defined in Theorem 5.7. If*

$$\varphi(u(x,y)) \le a(x,y) + c(x,y)\times \\ \times \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \varphi(u(s,t))\big[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)\big]\,dt\,ds \tag{5.28}$$

for all $(x,y) \in \triangle$, then

$$u(x,y) \le \varphi^{-1}\Bigg\{\widehat{a}(x,y)\exp\Bigg[G_2^{-1}\Bigg(G_2(k_2(x_0,y))+ + \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)M(s,t)\,dt\,ds\Bigg)\Bigg]\Bigg\} \quad (5.29)$$

for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$, *where*

$$k_2(x_0,y) = \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} \big[f_i(s,t)L(s,t)+g_i(s,t)\big]\,dt\,ds,$$

$$G_2(r) = \int\limits_{r_0}^{r}\frac{ds}{\varphi^{-1}(be^s)}, \quad r \ge r_0 > 0,$$

b *is a constant,* G_2^{-1} *denotes the inverse function of* G_2 *and* $(x_1,y_1) \in \triangle$ *is so chosen that*

$$\Bigg(G_2(k_2(x_0,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)M(s,t)dt\,ds\Bigg) \in \mathrm{Dom}(G_2^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 5.7 with suitable modification. We omit the details here. □

Theorem 5.10. *Let* u, a, c, f_i, g_i, α_i, $\beta_i(t)$, $i = 1,\ldots,n$, φ, L *and* M *be as defined in Theorem* 5.7. *If*

$$\varphi(u(x,y)) \le a(x,y) + c(x,y)\times \times\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} \varphi'(u(s,t))\big[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)\big]\,dt\,ds \quad (5.30)$$

for all $(x,y) \in \triangle$, *then*

$$u(x,y) \le k_3(x,y)\exp\Bigg(\widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)M(s,t)\,dt\,ds\Bigg)\Bigg] \quad (5.31)$$

for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$, *where*

$$k_3(x_0,y) = \varphi^{-1}(\widehat{a}(x,y)) + \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)L(s,t) + |!g_i(s,t)\big]\,dt\,ds.$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 5.7 with suitable modification. We omit the details here. □

Theorem 5.7, Theorem 5.9 and Theorem 5.10 can easily be applied to generate other useful nonlinear integral inequalities in more general situations. For example, we have the following results, Theorem 5.11, Theorem 5.13 and Theorem 5.14, respectively.

Theorem 5.11. *Let u, a, c, f_i, g_i, α_i, $\beta_i(t)$, $i = 1, \dots, n$, φ, L and M be as defined in Theorem 5.7. If*

$$\varphi(u(x,y)) \le a(x,y) + c(x,y)\times \times \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\big[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)u(s,t)\big]\,dt\,ds \quad (5.32)$$

for all $(x,y) \in \triangle$, then

$$u(x,y) \le \varphi^{-1}\bigg\{G^{-1}\bigg[\Phi^{-1}\bigg(\Phi(k(x_0,y))+ + \widehat{c}(x,y)\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \big[f_i(s,t)M(s,t) + g_i(s,t)\big]\,dt\,ds\bigg)\bigg]\bigg\} \quad (5.33)$$

for all $(x,y) \in [x_0,x_1] \times [y_0,y_1]$, where

$$(x_0,y) = G(\widehat{a}(x,y)) + \widehat{c}(x,y)\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)L(s,t)\,dt\,ds,$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{[\varphi^{-1}(s)]^q}, \quad r \ge r_0 > 0,$$

$$\Phi(r) = \int_{r_0}^{r} \frac{ds}{\varphi^{-1}(G^{-1}(s))}, \quad r \ge r_0 > 0,$$

G^{-1}, Φ^{-1} denote the inverse functions of G, Φ and $(x_1,y_1) \in \triangle$ is so chosen that

$$\bigg[\Phi(k(x_0,y)) + \widehat{c}(x,y)\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \big[f_i(s,t)M(s,t) + g_i(s,t)\big]\,dt\,ds\bigg] \in \mathrm{Dom}(\Phi^{-1}).$$

Proof. Fixing any numbers X and Y with $x_0 \le x \le X$ and $y_0 \le y \le Y$, we assume that $\widehat{a}(X,Y)$ is positive and define a positive function $z(x,y)$ by

$$z(x,y) = \widehat{a}(X,Y) + \widehat{c}(X,Y)\times \times \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\Big[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)u(s,t)\Big]\,dt\,ds. \quad (5.34)$$

Then $z(x,y) > 0$, $z(x_0,y) = z(x,y_0) = \widehat{a}(X,Y)$ and (5.32) can be restated as

$$u(x,y) \le \varphi^{-1}[z(x,y)]. \tag{5.35}$$

It is easy to observe that $z(x,y)$ is a continuous non-decreasing function for all $x \in J_1, y \in J_2$ and

$$D_1 z(x,y) = \widehat{c}(X,Y)\sum_{i=1}^{n}\Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)} u^q(\alpha_i(x),t)\times$$

$$\times\Big[f_i(\alpha_i(x),t)L\big(\alpha_i(x),t,u(\alpha_i(x),t)\big)+g_i(\alpha_i(x),t)u(\alpha_i(x),t)\Big]\,dt\Bigg]\alpha_i'(x)\le$$

$$\le \widehat{c}(X,Y)\big[\varphi^{-1}(z(x,y))\big]^q\times$$

$$\times\sum_{i=1}^{n}\Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big[f_i(\alpha_i(x),t)L\big(\alpha_i(x),t,\varphi^{-1}(z(\alpha_i(x),t))\big)+$$

$$+\,g_i(\alpha_i(x),t)u(\alpha_i(x),t)\Big]\,dt\Bigg]\alpha_i'(x).$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\big[\varphi^{-1}(z(x,y))\big]^q \ge \big[\varphi^{-1}(z(x_0,y_0))\big]^q = \big[\varphi^{-1}(\widehat{a}(X,Y))\big]^q > 0.$$

From the definition of G and the above relation, we have

$$D_1 G(z(x,y)) = \frac{D_1 z(x,y)}{[\varphi^{-1}(z(x,y))]^q} \le$$

$$\le \widehat{c}(X,Y)\sum_{i=1}^{n}\Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big[f_i(\alpha_i(x),t)L\big(\alpha_i(x),t,\varphi^{-1}(z(\alpha_i(x),t))\big)+$$

$$+\,g_i(\alpha_i(x),t)u(\alpha_i(x),t)\Big]\,dt\Bigg]\alpha_i'(x). \tag{5.36}$$

Keeping y fixed in (5.36), setting $x=\sigma$, and integrating it with respect to σ from x_0 to $x, x \in J_1$ and making the change of variable, we obtain

$$G(z(x,y)) \le G(z(x_0,y)) + \widehat{c}(X,Y)\times$$

$$\times\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big\{f_i(s,t)L\big[s,t,\varphi^{-1}(z(s,t))\big]+g_i(s,t)\varphi^{-1}(z(s,t))\Big\}dt\,ds. \tag{5.37}$$

Now, define a function $k(x,y)$ by

$$k(x,y) = G(\widehat{a}(X,Y)) + \widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(X)}\int\limits_{\beta_i(y_0)}^{\beta_i(Y)} f_i(s,t)L(s,t)\,dtds+$$

$$+\,\widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t)+g_i(s,t)\big]\varphi^{-1}(z(s,t))\,dt\,ds.$$

Then $k(x_0,y)=G(\widehat{a}(X,Y))+\widehat{c}(X,Y)\sum\limits_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(X)}\int\limits_{\beta_i(y_0)}^{\beta_i(Y)} f_i(s,t)L(s,t)\,dtds$, and (5.37) can be restated as

$$z(x,y)\le G^{-1}[k(x,y)]. \tag{5.38}$$

It is easy to observe that $k(x,y)$ is a continuous non-decreasing function for all $x\in J_1, y\in J_2$ and

$$D_1k(x,y)\le \widehat{c}(X,Y)\varphi^{-1}\big[G^{-1}(k(x,y))\big]\times$$
$$\times\sum_{i=1}^{n}\Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big[f_i(\alpha_i(x),t)M(\alpha_i(x),t)+g_i(\alpha_i(x),t)\Big]\,dt\Bigg]\alpha_i'(x). \tag{5.39}$$

From the above relation, we have

$$\frac{D_1k(x,y)}{\varphi^{-1}[G^{-1}(k(x,y))]}\le \widehat{c}(X,Y)\times$$
$$\times\sum_{i=1}^{n}\Bigg[\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\Big[f_i(\alpha_i(x),t)M(\alpha_i(x),t)+g_i(\alpha_i(x),t)\Big]\,dt\Bigg]\alpha_i'(x). \tag{5.40}$$

Keeping y fixed in (5.40), setting $x=\sigma$ and integrating it with respect to σ from x_0 to $x, x\in J_1$ and making the change of variable, from the definition of Φ, we obtain

$$\Phi(k(x,y))\le\Phi(k(x_0,y))+$$
$$+\,\widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t)+g_i(s,t)\big]\,dt\,ds. \tag{5.41}$$

Now, using the inequalities (5.38) and (5.41) in (5.35), we get

$$u(x,y)\le\varphi^{-1}\Bigg\{G^{-1}\Bigg[\Phi^{-1}\Bigg(\Phi(k(x_0,y))+$$
$$+\,\widehat{c}(X,Y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t)+g_i(s,t)\big]\,dt\,ds\Bigg)\Bigg]\Bigg\}. \tag{5.42}$$

Taking $X=x, Y=y$ in the foregoing inequality, since X and Y are arbitrary, we get the required inequality.

If $\widehat{a}(X,Y)=0$, we carry out the above procedure with $\varepsilon>0$ instead of $\widehat{a}(X,Y)$ and subsequently let $\varepsilon\to 0$. This completes the proof. □

For the special case $\varphi(u) = u^p$ ($p > q$ is a constant), Theorem 5.11 gives the following retarded integral inequality for nonlinear functions.

Corollary 5.12. *Let u, a, c, f_i, g_i, $\alpha_i(t)$, β_i, $i = 1, \ldots, n$, L and M be as defined in Theorem 5.11. Suppose that $p > q > 0$ are constants. If*

$$u^p(x,y) \le a(x,y) + c(x,y)\times$$
$$\times\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t)\Big[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)u(s,t)\Big]\,dt\,ds \quad (5.43)$$

for all $(x,y) \in \triangle$, then

$$u(x,y) \le \Bigg[G^{-1}\Bigg(G(k_1(x_0,y))+$$
$$+\frac{p-q}{p}\widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t) + g_i(s,t)\big]\,dt\,ds\Bigg)\Bigg]^{\frac{1}{p-q}} \quad (5.44)$$

for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$, where

$$k_1(x_0,y) = \big[\widehat{a}(x,y)\big]^{\frac{p-q}{p}} +$$
$$+\frac{p-q}{p}\,\widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)L(s,t)\,dt\,ds,$$

$$G(r) = \int\limits_{r_0}^{r}\frac{ds}{s^{\frac{1}{p-q}}}, \quad r \ge r_0 > 0,$$

G^{-1} denotes the inverse function of G and $(x_1,y_1) \in \triangle$ is so chosen that

$$G(k_1(x_0,y)) + \frac{p-q}{p}\widehat{c}(x,y)\times$$
$$\times\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t) + g_i(s,t)\big]\,dt\,ds \in \mathrm{Dom}(G^{-1}).$$

Theorem 5.13. *Let u, a, c, f_i, g_i, α_i, $\beta_i(t)$, $i = 1, \ldots, n$, φ, L and M be as defined in Theorem 5.11. If*

$$\varphi(u(x,y)) \le a(x,y) + c(x,y)\times$$
$$\times\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\varphi(u(s,t))\Big[f_i(s,t)L(s,t,u(s,t))+g_i(s,t)u(s,t)\Big]\,dt\,ds \quad (5.45)$$

for all $(x, y) \in \triangle$, *then*

$$u(x,y) \leq \varphi^{-1}\Bigg\{\widehat{a}(x,y)\exp\Bigg[G_2^{-1}\Bigg(G_2(k_2(x_0,y))+ \\ +\widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t)+g_i(s,t)\big]\,dt\,ds\Bigg)\Bigg]\Bigg\} \quad (5.46)$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$, *where*

$$k_2(x_0,y) = \widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)L(s,t)\,dt\,ds,$$

$$G_2(r) = \int\limits_{r_0}^{r}\frac{ds}{\varphi^{-1}(be^s)},\quad r \geq r_0 > 0,$$

b *is a constant,* G_2^{-1} *denotes the inverse function of* G_2 *and* $(x_1, y_1) \in \triangle$ *is so chosen that*

$$G_2(k_2(x_0,y)) + \widehat{c}(x,y)\times \\ \times\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t)+g_i(s,t)\big]\,dt\,ds \in \mathrm{Dom}(G_2^{-1}).$$

Proof. The proof follows from an argument similar to that found in the proofs of Theorems!5.9 and 5.11 with suitable modification. We omit the details here. □

Theorem 5.14. *Let* u, a, c, f_i, g_i, α_i, $\beta_i(t)$, $i = 1, \ldots, n$, φ, L *and* M *be as defined in Theorem* 5.11. *If*

$$\varphi(u(x,y)) \leq a(x,y) + c(x,y)\times \\ \times\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\varphi'(u(s,t))\Big[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)u(s,t)\Big]\,dt\,ds \quad (5.47)$$

for all $(x, y) \in \triangle$, *then*

$$u(x,y) \leq \\ \leq k_3(x,y)\exp\Bigg(\widehat{c}(x,y)\sum_{i=1}^{n}\int\limits_{\alpha_i(x_0)}^{\alpha_i(x)}\int\limits_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t)+g_i(s,t)\big]\,dt\,ds\Bigg)\Bigg] \quad (5.48)$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$, *where*

$$k_3(x_0, y) = \varphi^{-1}(\widehat{a}(x, y)) + \widehat{c}(x, y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i} f_i(s, t) L(s, t)\, dt\, ds.$$

Proof. he proof here also follows from an argument similar to that found in the proofs of Theorems 5.9 and 5.11 with suitable modification. We omit the details here. □

6. Gronwall–Bellman-Type Nonlinear Inequalities III

Pachpatte [48], [49] investigated some nonlinear Bihari-type integral inequalities which are applicable in certain general situations. In this section we present some two-independent-variable generalizations of the certain inequalities in Pachpatte [48], [49] obtained by Bondge and Pachpatte [18], [19]. These inequalities can be used as tools in the study of certain partial integro-differential and integral equations. Bondge and Pachpatte [18] established the following generalization of the inequality given by Pachpatte [48].

Theorem 6.1 (Bondge and Pachpatte, 1979)**.** *Let* $u(x, y)$ *and* $p(x, y)$ *be nonnegative continuous functions defined for* x, $y \in R_+$. *Let* $g(u)$ *be a continuously differentiable function defined for* $u \ge 0$, $g(u) > 0$ *for* $u > 0$ *and* $g'(u) \ge 0$ *for* $u \ge 0$, *and let* $a(x) > 0$, $b(y) > 0$, $a'(x) \ge 0$, $b'(y) \ge 0$ *are continuous functions defined for* x, $y \in R_+$. *If*

$$u(x, y) \le a(x) + b(y) + \int_0^x \int_0^y f(s, t) \left(u(s, t) + \int_0^s \int_0^t p(s_1, t_1) g(u(s_1, t_1))\, ds_1\, dt_1 \right) ds\, dt, \tag{6.1}$$

for x, $y \in R_+$, *then for* $0 \le x \le x_1$, $0 \le y \le y_1$,

$$u(x, y) \le a(x) + b(y) + \int_0^x \int_0^y p(s, t) H^{-1}\Bigg[H(a(0) + b(t)) + \int_0^s \frac{a'(s_1)}{a(s_1) + b(0) + g(a(s_1) + b(0))}\, ds_1 + \int_0^s \int_0^t p(s_1, t_1)\, ds_1\, dt_1 \Bigg] ds\, dt, \tag{6.2}$$

where

$$H(r) = \int_{r_0}^{r} \frac{ds}{s + g(s)}, \quad r \ge r_0 > 0, \tag{6.3}$$

H^{-1} denotes the inverse function of H and x_1, y_1 are so chosen that

$$H(a(0)+b(y)) + \int_0^x \frac{a'(s_1)}{a(s_1)+b(0)+g(a(s_1)+b(0))}\,ds_1 + \int_0^x\int_0^y p(s_1,t_1)\,ds_1\,dt_1 \in \mathrm{Dom}(H^{-1}).$$

for all x, y lying in the subintervals $0 \le x \le x_1$, $0 \le y \le y_1$ of R_+.

Bondge and Pachpatte [19] established the following three theorems which deal with two-independent-variable generalizations of certain inequalities given by Pachpatte [48], [49].

Theorem 6.2 (Bondge and Pachpatte, 1980)**.** *Let $u(x,y)$, $a(x,y)$, and $b(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$. Let $g(u)$ be a continuously differentiable function defined for $u \ge 0$, $g(u) > 0$ for $u > 0$ and $g'(u) \ge 0$ for $u \ge 0$, and in addition $g(u)$ be subadditive on R_+. If*

$$u(x,y) \le a(x,y) + \int_0^x\int_0^y b(s,t)\left(u(s,t) + \int_0^s\int_0^t b(s_1,t_1)g(u(s_1,t_1))\,ds_1\,dt_1\right)ds\,dt, \quad (6.4)$$

for x, $y \in R_+$, then for $0 \le x \le x_2$, $0 \le y \le y_2$,

$$u(x,y) \le a(x,y) + A(x,y) + \int_0^x\int_0^y b(s,t)\left\{H^{-1}\left[H(A(s,t)) + \int_0^s\int_0^t p(s_1,t_1)\,ds_1\,dt_1\right]\right\}ds\,dt, \quad (6.5)$$

where

$$A(x,y) = \int_0^x\int_0^y b(s_1,t_1)\left(a(s_1,t_1) + \int_0^{s_1}\int_0^{t_1} b(s_2,t_2)g(a(s_2,t_2))\,ds_2\,dt_2\right)ds_1\,dt_1,$$

H, H^{-1} are as defined in Theorem 6.1 and x_2, y_2 are chosen so that

$$H(A(x,y)) + \int_0^x\int_0^y p(s_1,t_1)\,ds_1\,dt_1 \in \mathrm{Dom}(H^{-1}).$$

for all x, y lying in the subintervals $0 \le x \le x_2$, $0 \le y \le y_2$ of R_+.

Theorem 6.3 (Bondge and Pachpatte, 1980)**.** *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, and $c(x,y)$ be nonnegative continuous functions defined for x, $y \in R_+$. Let $g(u)$ be a continuously differentiable function defined for $u \ge 0$, $g(u) > 0$*

for $u > 0$ *and* $g'(u) \geq 0$ *for* $u \geq 0$, *and in addition* $g(u)$ *be subadditive on* R_+. *If*

$$u(x,y) \leq a(x,y) + \int_0^x \int_0^y b(s,t)\Big[u(s,t) + \int_0^s \int_0^t c(s_1,t_1)u(s_1,t_1)\,ds_1\,dt_1 + \int_0^s \int_0^t \big[b(s_1,t_1) + c(s_1,t_1)\big] g(u(s_1,t_1))\,ds_1\,dt_1\Big]\,ds\,dt, \quad (6.6)$$

for x, $y \in R_+$, *then for* $0 \leq x \leq x_3$, $0 \leq y \leq y_3$,

$$u(x,y) \leq a(x,y) + B(x,y) + \int_0^x \int_0^y b(s,t)\Big\{H^{-1}\Big[H(B(s,t)) + \int_0^s \int_0^t \big[b(s_1,t_1) + c(s_1,t_1)\big]\,ds_1\,dt_1\Big]\Big\}\,ds\,dt, \quad (6.7)$$

where

$$B(x,y) = \int_0^x \int_0^y b(s_1,t_1)\Big(a(s_1,t_1) + \int_0^{s_1} \int_0^{t_1} b(s_2,t_2)c(s_2,t_2)\,ds_2\,dt_2 + \int_0^{s_1} \int_0^{t_1} \big[b(s_2,t_2) + c(s_2,t_2)\big] g(a(s_2,t_2))\,ds_2\,dt_2\Big)\,ds_1\,dt_1,$$

H, H^{-1} *are as defined in Theorem 6.1 and* x_3, y_3 *are so chosen that*

$$H(B(x,y)) + \int_0^x \int_0^y \big[b(s_1,t_1) + c(s_1,t_1)\big]\,ds_1\,dt_1 \in \mathrm{Dom}(H^{-1}).$$

for all x, y *lying in the subintervals* $0 \leq x \leq x_3$, $0 \leq y \leq y_3$ *of* R_+.

Theorem 6.4 (Bondge and Pachpatte, 1980)**.** *Let* $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, *and* $k(x,y)$ *be nonnegative continuous functions defined for* x, $y \in R_+$. *Let* $g(u)$ *be a continuously differentiable function defined for* $u \geq 0$, $g(u) > 0$ *for* $u > 0$ *and* $g'(u) \geq 0$ *for* $u \geq 0$, *and in addition* $g(u)$ *be subadditive on* R_+. *If*

$$u(x,y) \leq a(x,y) + b(x,y) \times \int_0^x \int_0^y c(s,t) g\Big(u(s,t) + b(s,t)\int_0^s \int_0^t k(s_1,t_1) g(u(s_1,t_1))\,ds_1\,dt_1\Big)\,ds\,dt, \quad (6.8)$$

for $x,\ y \in R_+$, *then for* $0 \le x \le x_4$, $0 \le y \le y_4$,

$$u(x,y) \le a(x,y) + \\ + b(x,y)\Big[L(x,y) + \int_0^x\int_0^y c(s,t)g\Big(b(s,t)\Big\{\Omega^{-1}\Big[\Omega(L(s,t)) + \\ + \int_0^s\int_0^t \big[c(s_1,t_1)+k(s_1,t_1)\big]g(b(s_1,t_1))\,ds_1\,dt_1\Big]\Big\}\Big)\,ds\,dt\Big], \quad (6.9)$$

where

$$L(x,y) = \int_0^x\int_0^y c(s_1,t_1)\times \\ \times g\Big(a(s_1,t_1) + b(s_1,t_1)\int_0^{s_1}\int_0^{t_1} k(s_2,t_2)g(a(s_2,t_2))\,ds_2\,dt_2\Big)\,ds_1\,dt_1,$$

Ω, Ω^{-1} *are as defined in Theorem* 4.1 *and* x_4, y_4 *are so chosen that*

$$\Omega(L(x,y)) + \int_0^x\int_0^y \big[c(s_1,t_1)+k(s_1,t_1)\big]\,ds_1\,dt_1 \in \mathrm{Dom}(\Omega^{-1}).$$

for all $x,\ y$ *lying in the subintervals* $0 \le x \le x_4$, $0 \le y \le y_4$ *of* R_+.

In view of wider applications, Wendroff's inequality given in Beckenbach and Belllman [9] has been generalized and extended in various directions. The current article is devoted to the Wendroff-like inequalities investigated by Pachpatte [59], [60], [62] in order to apply them in the study of certain higher order partial differential equations. Pachpatte [59] established the Wendroff-like inequalities in the following two theorems.

Theorem 6.5 (Pachpatte, 1988)**.** *Let* $u(x,y)$ *and* $h(x,y)$ *be nonnegative continuous functions defined for* $x,\ y \in R_+$. *Let* $a(x)$, $b(y)$, $p(x)$, $q(y)$ *be positive and twice continuously differentiable functions defined for* $x,\ y \in R_+$; *also let* $a'(x)$, $b'(y)$, $p'(x)$, $q'(y)$ *be nonnegative for* $x,\ y \in R_+$. *Define* $c(x,y) = a(x)+b(y)+yp(x)+xq(y)$, *for* $x,\ y \in R_+$. *Let* g *be a continuously differentiable function defined on* R_+ *and* $g(u) > 0$ *on* $(0,\infty)$, $g'(u) \ge 0$ *on* R_+ *and*

$$u(x,y) \le c(x,y) + A\big[x,y,h(s_1,t_1)g(u(s_1,t_1))\big], \quad (6.10)$$

holds for $x,\ y \in R_+$.

(i) *If* $a''(x)$, $p''(x)$ *are nonnegative for* $x \ge 0$, *then for* $0 \le x \le x_5$, $0 \le y \le y_5$,

$$u(x,y) \le \Omega^{-1}\Big[\Omega(c(0,y)) + x\Big(\frac{c_x(0,y)}{g(c(0,y))}\Big) +$$

$$+\int_0^x\int_0^s \frac{a''(s_1)+yp''(s_1)}{g(c(s_1,0))}\,ds_1\,ds + A\big[a,y,h(s_1,t_1)\big]\Bigg], \quad (6.11)$$

where

$$\Omega(r)=\int_{r_0}^{r}\frac{ds}{s+g(s)}, \quad r>0, \quad r_0>0, \quad (6.12)$$

Ω^{-1} *denotes the inverse function of* Ω *and* x_5, y_5 *are so chosen that*

$$\Omega(c(0,y))+x\Big(\frac{c_x(0,y)}{g(c(0,y))}\Big)+ \\ +\int_0^x\int_0^s \frac{a''(s_1)+yp''(s_1)}{g(c(s_1,0))}\,ds_1\,ds + A\big[a,y,h(s_1,t_1)\big]\in \mathrm{Dom}(H^{-1}).$$

for all x,y *lying in the subintervals* $0\le x\le x_5$, $0\le y\le y_5$ *of* R_+.

(ii) *If* $b''(x)$, $q''(x)$ *are nonnegative for* $y\ge 0$, *then for* $0\le x\le x_6$, $0\le y\le y_6$,

$$u(x,y)\le \Omega^{-1}\Bigg[\Omega(c(x,0))+y\Big(\frac{c_y(x,0)}{g(c(x,0))}\Big)+ \\ +\int_0^y\int_0^t \frac{b''(t_1)+xq''(t_1)}{g(c(0,t_1))}\,dt_1\,dt + A\big[y,x,h(s_1,t_1)\big]\Bigg], \quad (6.13)$$

where Ω, Ω^{-1} *are as defined in* (i) *and* x_6, y_6 *are so chosen that*

$$\Omega(c(x,0))+y\Big(\frac{c_y(x,0)}{g(c(x,0))}\Big)+ \\ +\int_0^y\int_0^t \frac{b''(t_1)+xq''(t_1)}{g(c(0,t_1))}\,dt_1\,dt + A\big[y,x,h(s_1,t_1)\big]\in \mathrm{Dom}(H^{-1}).$$

for all x, y *lying in the subintervals* $0\le x\le x_6$, $0\le y\le y_6$ *of* R_+.

Theorem 6.6 (Pachpatte, 1988)**.** *Let* $u(x,y)$ *and* $h(x,y)$ *be nonnegative continuous functions defined for* x, $y\in R_+$. *let* $a(x)$, $b(y)$, $p(x)$, $q(y)$ *be positive and twice continuously differentiable functions defined for* x, $y\in R_+$; *also let* $a'(x)$, $b'(y)$, $p'(x)$, $q'(y)$ *be nonnegative for* x, $y\in R_+$ *Define* $c(x,y)=a(x)+b(y)+yp(x)+xq(y)$, *for* x, $y\in R_+$. *Let* g *be a continuously differentiable function defined on* R_+ *and* $g(u)>0$ *on* $(0,\infty)$, $g'(u)\ge 0$ *on* R_+. *Also,*

$$u(x,y)\le c(x,y)+ \\ +A\Big[x,y,h(s_1,t_1)\Big[u(s_1,t_1)+A\big[s_1,t_1,h(s_2,t_2)g(u(s_2,t_2))\big]\Big]\Big], \quad (6.14)$$

holds for x, $y\in R_+$.

(i) *If* $a''(x)$, $p''(x)$ *are nonnegative for* $x \in R_+$, *then for* $0 \le x \le x_7$, $0 \le y \le y_7$,

$$u(x,y) \le c(0,y)) + xc_x(0,y) + \int_0^x \int_0^s \big[a''(s_1) + yp''(s_1)\big]\, ds_1\, ds + A\big[a, y, h(s_1,t_1)Q_1(s_1,t_1)\big], \quad (6.15)$$

in which

$$Q_1(x,y) \le H^{-1}\Bigg[H(c(0,y)) + x\Big(\frac{c_x(0,y)}{c(0,y) + g(c(0,y))}\Big) + \int_0^x \int_0^{s_2} \frac{a''(s_3) + yp''(s_3)}{c(s_3,0) + g(c(s_3,0))}\, ds_3\, ds_2 + A\big[x, y, h(s_3,t_3)\big]\Bigg], \quad (6.16)$$

where

$$H(r) = \int_{r_0}^r \frac{ds}{s + g(s)}, \quad r > 0, \quad r_0 > 0,$$

H^{-1} *denotes the inverse function of* H *and* x_7, y_7 *are so chosen that*

$$H(c(0,y)) + x\Big(\frac{c_x(0,y)}{c(0,y) + g(c(0,y))}\Big) + \int_0^x \int_0^{s_2} \frac{a''(s_3) + yp''(s_3)}{c(s_3,0) + g(c(s_3,0))}\, ds_3\, ds_2 + A\big[x, y, h(s_3,t_3)\big] \in \mathrm{Dom}(H^{-1}).$$

for all x, y *lying in the subintervals* $0 \le x \le x_7$, $0 \le y \le y_7$ *of* R_+.

(ii) *If* $b''(x)$, $q''(x)$ *are nonnegative for* $y \in R_+$, *then for* $0 \le x \le x_8$, $0 \le y \le y_8$,

$$u(x,y) \le c(x,0)) + yc_y(x,0) + \int_0^y \int_0^t \big[b''(t_1) + xq''(t_1)\big]\, dt_1\, dt + A\big[y, xh(s_1,t_1)Q_2(s_1,t_1)\big], \quad (6.17)$$

in which

$$Q_2(x,y) \le H^{-1}\Bigg[H(c(x,0)) + x\Big(\frac{c_y(x,0)}{c(x,0) + g(c(x,0))}\Big) + \int_0^y \int_0^{t_2} \frac{b''(t_3) + xq''(t_3)}{c(0,t_3) + g(c(0,t_3))}\, dt_3\, dt_2 + A\big[y, x, h(s_3,t_3)\big]\Bigg], \quad (6.18)$$

where H, H^{-1} *are as defined in* (i) *and* x_8, y_8 *are so chosen that*

$$H(c(x,0)) + y\Big(\frac{c_y(x,0)}{c(x,0)+g(c(x,0))}\Big) +$$
$$+ \int_0^y \int_0^{t_2} \frac{b''(t_3)+xq''(t_3)}{c(0,t_3)+g(c(0,t_3))}\,dt_3\,dt_2 + A\big[y,x,h(s_3,t_3)\big] \in \mathrm{Dom}(H^{-1}).$$

for all x, y *lying in the subintervals* $0 \le x \le x_8$, $0 \le y \le y_8$ *of* R_+.

Pachpatte [63] established the following inequality which can be used in the study of certain partial differential and integral equations.

Theorem 6.7 (Pachpatte, 1998)**.** *Let* $u(x,y)$, $f(x,y)$, *and* $p(x,y)$ *be nonnegative continuous functions defined for* x, $y \in R_+$. *Let* g *be a continuously differentiable function defined on* R_+ *and* $g(u) > 0$ *on* $(0,\infty)$, $g'(u) \ge 0$ *on* R_+. *If*

$$u(x,y) \le c + \int_0^x \int_0^s f(s_1,y)u(s_1,y)\,ds_1\,ds + A\big[x,y,p(s_1,t_1)g(u(s_1,t_1))\big], \quad (6.19)$$

for x, $y \in R_+$, *where* $c \ge 0$ *is a constant, then for* $0 \le x \le x_9$, $0 \le y \le y_9$,

$$u(x,y) \le Q(x,y)\Big\{\Omega^{-1}\Big[\Omega(c) + A\big[x,y,p(s_1,t_1)g(Q(s_1,t_1))\big]\Big]\Big\}, \quad (6.20)$$

where

$$Q(x,y) = \exp\Big(\int_0^x \int_0^s f(s_1,y)u(s_1,y)\,ds_1\,ds\Big),$$

Ω, Ω^{-1} *are as defined in Theorem* 5.5 *and* x_9, y_9 *are so chosen that*

$$\Omega(c) + A\big[x,y,p(s_1,t_1)g(Q(s_1,t_1))\big] \in \mathrm{Dom}(H^{-1})$$

for all x, y *lying in the subintervals* $0 \le x \le x_9$, $0 \le y \le y_9$ *of* R_+.

The inequalities given in the following theorem have been recently established by Pachpatte [60], [62] and are motivated by the study of certain higher order partial differential equations.

Theorem 6.8 (Pachpatte, 1993, 1996)**.** *Let* $u(x,y)$, $a(x,y)$, *and* $b(x,y)$ *be nonnegative continuous functions defined for* x, $y \in R_+$ *and* $h : R_+^3 \to R_+$ *be a continuous function which satisfies the condition*

$$0 \le h(x,y,v_1) - h(x,y,v_2) \le k(x,y,v_2)(v_1 - v_2), \quad (6.21)$$

for x, $y \in R_+$ *and* $v_1 \ge v_2 \ge 0$, *where* $k : R_+^3 \to R_+$ *is a continuous function.*

(i) *If*

$$u(x,y) \le a(x,y) + b(x,y)B\big[x,y,h(s,t,u(s,t))\big], \quad (6.22)$$

for x, $y \in R_+$, *then*

$$u(x,y) \le a(x,y) + b(x,y)p(x,y)\exp\Big(B\big[x,y,k(s,t,a(s,t))b(s,t)\big]\Big), \quad (6.23)$$

for $x,\ y \in R_+$, *where*

$$p(x,y) = B\big[x, y, h(s,t,a(s,t))\big] \tag{6.24}$$

for $x, y \in R_+$.

(ii) *Let* $F(u)$ *be a continuous, strictly increasing, convex, submultiplicative function for* $u > 0$, $\lim_{u\to\infty} F(u) = \infty$, F^{-1} *denote the inverse function of* F, *and* $\alpha(x,y)$, $\beta(x,y)$ *be continuous and positive functions for* $x,\ y \in R_+$ *and* $\alpha(x,y) + \beta(x,y) = 1$. *If*

$$u(x,y) \le a(x,y) + b(x,y)F^{-1}\Big(B\big[x, y, h(s,t,F(u(s,t)))\big]\Big), \tag{6.25}$$

for $x,\ y \in R_+$, *then*

$$\begin{aligned} u(x,y) \le{}& a(x,y) + \\ &+ b(x,y)F^{-1}\Big(B\Big[x, y, h\Big(s, t, \alpha(s,t)F\big(a(s,t)\alpha^{-1}(s,t)\big)\Big)\Big]\times \\ &\times \exp\Big(B\Big[x, y, k\Big(s, t, \alpha(s,t)F\big(a(s,t)\alpha^{-1}(s,t)\big)\Big)\times \\ &\times \beta(s,t)F\big(b(s,t)\beta^{-1}(s,t)\big)\Big]\Big)\Big), \end{aligned} \tag{6.26}$$

for $x,\ y \in R_+$.

(iii) *Let* $g(u)$ *be a continuously differentiable function defined for* $u \ge 0$, $g(u) > 0$ *for* $u > 0$ *and* $g'(u) \ge 0$ *for* $u \ge 0$ *and* $g(u)$ *is subadditive and submultiplicative for* $u \ge 0$. *If*

$$u(x,y) \le a(x,y) + b(x,y)B\big[x, y, h(s,t,g(u(s,t)))\big], \tag{6.27}$$

for $x,\ y \in R_+$, *then for* $0 \le x \le x_0$, $0 \le y \le y_0$,

$$\begin{aligned} u(x,y) \le{}& a(x,y) + \\ &+ b(x,y)\Omega^{-1}\Big[\Omega(q(x,y)) + B\big[x, y, k(s,t,g(a(s,t)))g(b(s,t))\big]\Big], \end{aligned} \tag{6.28}$$

where

$$q(x,y) = B\big[x, y, h(s,t,g(a(s,t)))\big], \tag{6.29}$$

Ω, Ω^{-1} *are as defined in Theorem* 6.5 *and* x_0, y_0 *are so chosen that*

$$\Omega(q(x,y)) + B\big[x, y, k(s,t,g(a(s,t)))g(b(s,t))\big] \in \mathrm{Dom}(H^{-1}),$$

for $x,\ y$ *lying in the subintervals* $0 \le x \le x_0$, $0 \le y \le y_0$ *of* R_+.

We next present some new nonlinear retarded Gronwall–Bellman-type integral inequalities in two independent variables as following Theorems. These inequalities can be used as effective tools in the study of certain integral equations. In what follows, given a continuous function $a : J_1\times J_2 \to R_+$, we write

$$\widehat{a}(x,y) = \max\Big\{a(s,t) :\ x_0 \le s \le x,\ y_0 \le s \le y\Big\}.$$

Theorem 6.9. *Let u, a, c, f, $g \in (\triangle, R_+,)$ and $\psi(u)$ be a nondecreasing continuous function for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. If*

$$u(x,y) \le a(x,y) + \\ + c(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(u(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\psi(u(s_1,t_1))\,ds_1\,dt_1\right) ds\,dt, \quad (6.30)$$

for x, $y \in R_+$, then

(i) *in the case $\psi(u) \le u$,*

$$u(x,y) \le \widehat{a}(x,y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} \left[\widehat{c}(x,y)f(s,t) + g(s,t)\right] ds\,dt\right), \quad (6.31)$$

for all $(x,y) \in \triangle$ and

(ii) *in the case $\psi(u) > u$,*

$$u(x,y) \le H^{-1}\left(H(\widehat{a}(x,y)) + \int_{x_0}^{x}\int_{y_0}^{y} \left[\widehat{c}(x,y)f(s,t) + g(s,t)\right] ds\,dt\right), \quad (6.32)$$

for all $x_0 \le x \le x_1$, $y_0 \le y \le y_1$, where

$$H(r) = \int_{r_0}^{r} \frac{ds}{\psi(s)}, \quad r \ge r_0 > 0, \quad (6.33)$$

H^{-1} denotes the inverse function of H and x_1, y_1 are so chosen that

$$H(\widehat{a}(x,y)) + \int_{x_0}^{x}\int_{y_0}^{y} \left[\widehat{c}(x,y)f(s,t) + g(s,t)\right] ds\,dt \in \mathrm{Dom}(H^{-1})$$

for all x, y lying in the subintervals $0 \le x \le x_1$, $0 \le y \le y_1$ of R_+.

Proof. Fixing any numbers X and Y with $x_0 \le x \le X$ and $y_0 \le y \le Y$, we assume that $\widehat{a}(X,Y)$ is positive and define a positive function $z(x,y)$ by

$$z(x,y) = \widehat{a}(X,Y) + \\ + \widehat{c}(X,Y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(u(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\psi(u(s_1,t_1))\,ds_1\,dt_1\right) ds\,dt, \quad (6.34)$$

then $z(x,y_0) = \widehat{a}(X,Y)$, $z(x_0,y) = \widehat{a}(X,Y)$, $u(x,y) \le z(x,y)$ and

$$z_{xy}(x,y) = \widehat{c}(X,Y)f(x,y)\left(u(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s_1,t_1)\psi(u(s_1,t_1))\,ds_1\,dt_1\right) \le \\ \le \widehat{c}(X,Y)f(x,y)\left(z(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s_1,t_1)\psi(z(s_1,t_1))\,ds_1\,dt_1\right). \quad (6.35)$$

If we put

$$v(x,y) = z(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s_1,t_1)\psi(z(s_1,t_1))\,ds_1\,dt_1$$

then

$$v(x,y_0) = \widehat{a}(X,Y), \quad v(x_0,y) = \widehat{a}(X,Y),$$
$$z_{xy}(x,y) \le \widehat{c}(X,Y)f(x,y)v(x,y), \quad z(x,y) \le v(x,y)$$

and

$$v_{xy}(x,y) = z_{xy}(x,y) + g(x,y)\psi(z(x,y)) \le$$
$$\le \widehat{c}(X,Y)f(x,y)v(x,y) + g(x,y)\psi(v(x,y)). \tag{6.36}$$

When $\psi(u) \le u$, from the inequality (6.36), we find

$$\frac{v_{xy}(x,y)}{v(x,y)} \le \widehat{c}(X,Y)f(x,y) + g(x,y). \tag{6.37}$$

From (6.37) and by using the facts that $v_x(x,y) \ge 0$, $v_y(x,y) \ge 0$, $v(x,y) > 0$ for x, $y \in R_+$, we observe that

$$\frac{v_{xy}(x,y)}{v(x,y)} \le \widehat{c}(X,Y)f(x,y) + g(x,y) + \frac{v_x(x,y)v_y(x,y)}{[v(x,y)]^2},$$

i.e.

$$\frac{\partial}{\partial y}\left(\frac{v_x(x,y)}{v(x,y)}\right) \le \widehat{c}(X,Y)f(x,y) + g(x,y). \tag{6.38}$$

Keeping x fixed in (6.38), we set $y = t$; then, integrating with respect to t from y_0 to y and using the fact that $v_x(x,y_0) = 0$, we have

$$\frac{v_x(x,y)}{v(x,y)} \le \int_{y_0}^{y} \big[\widehat{c}(X,Y)f(x,t) + g(x,t)\big]\,dt. \tag{6.39}$$

Keeping y fixed in (6.39), we set $x = s$; then, integrating with respect to s from x_0 to x and using the fact that $v(x_0,y) = \widehat{a}(X,Y)$, we have

$$v(x,y) \le \widehat{a}(X,Y)\exp\biggl(\int_{x_0}^{x}\int_{y_0}^{y} \big[\widehat{c}(X,Y)f(s,t) + g(s,t)\big]\,dt\,ds\biggr). \tag{6.40}$$

Taking $X = x$, $Y = y$ and using $u(x,y) \le z(x,y) \le v(x,y)$ in the inequality (6.40), since X and Y are arbitrary, we get the required inequality (6.31).

When $\psi(u) > u$, from the inequality (6.36), we find

$$\frac{v_{xy}(x,y)}{\psi(v(x,y))} \le \widehat{c}(X,Y)f(x,y) + g(x,y). \tag{6.41}$$

From (6.41) and by using the facts that $v_x(x,y) \geq 0$, $v_y(x,y) \geq 0$, $v(x,y) > 0$, $\psi'(v(x,y)) \geq 0$ for x, $y \in R_+$, we observe that

$$\frac{v_{xy}(x,y)}{\psi(v(x,y))} \leq \widehat{c}(X,Y)f(x,y) + g(x,y) + \frac{v_x(x,y)\psi'(v(x,y))v_y(x,y)}{[\psi(v(x,y))]^2},$$

i.e.

$$\frac{\partial}{\partial y}\left(\frac{v_x(x,y)}{\psi(v(x,y))}\right) \leq \widehat{c}(X,Y)f(x,y) + g(x,y). \tag{6.42}$$

Keeping x fixed in (6.42), we set $y = t$; then, integrating with respect to t from y_0 to y and using the fact that $v_x(x,y_0) = 0$, we have

$$\frac{v_x(x,y)}{\psi(v(x,y))} \leq \int_{y_0}^{y} \left[\widehat{c}(X,Y)f(x,t) + g(x,t)\right] dt. \tag{6.43}$$

Keeping x fixed in (6.43), we set $x = s$; then, integrating with respect to s from x_0 to x and using the fact that $v(x_0,y) = \widehat{a}(X,Y)$ and the definition of the function H, we have

$$v(x,y) \leq H^{-1}\left(H(\widehat{a}(X,Y)) + \int_{x_0}^{x}\int_{y_0}^{y} \left[\widehat{c}(X,Y)f(s,t) + g(s,t)\right] dt\, ds\right). \tag{6.44}$$

Taking $X = x$, $Y = y$ and using $u(x,y) \leq z(x,y) \leq v(x,y)$ in the inequality (6.44), since X and Y are arbitrary, we get the required inequality (6.32).

If $\widehat{a}(X,Y) = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X,Y)$ and subsequently let $\varepsilon \to 0$. This completes the proof. ☐

Theorem 6.9 can easily be applied to generate other useful nonlinear integral inequalities in more general situations. For example, we have the following result (Theorems 6.10–6.11).

Theorem 6.10. *Let u, a, c, f, $g \in (\triangle, R_+,)$ and $\psi(u)$ be a nondecreasing continuous function for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. Suppose that $\varphi \in C^1(R_+, R_+)$ is an increasing function with $\varphi(\infty) = \infty$ and $\varphi'(u)$ is a nondecreasing continuous function for $u \in R_+$. If*

$$\varphi(u(x,y)) \leq a(x,y) + c(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Bigg(u(s,t)\varphi'(u(s,t)) + \\ + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\varphi'(u(s_1,t_1))\psi(u(s_1,t_1))\, ds_1\, dt_1\Bigg)\, ds\, dt, \tag{6.45}$$

for x, $y \in R_+$, then

(i) *in the case $\psi(\varphi^{-1}(z)) \leq \varphi^{-1}(z)$ for $z \in R_+$,*

$$u(x,y) \leq \varphi^{-1}(\widehat{a}(x,y)) \exp\left(\int_{x_0}^{x}\int_{y_0}^{y} \left[\widehat{c}(x,y)f(s,t) + g(s,t)\right] ds\, dt\right), \tag{6.46}$$

for all $(x,y)\in\triangle$.

(ii) *in the case* $\psi(\varphi^{-1}(z))>\varphi^{-1}(z)$ *for* $z\in R_+$,

$$u(x,y)\le H^{-1}\left(H(\varphi^{-1}(\widehat{a}(x,y)))+\int_{x_0}^{x}\int_{y_0}^{y}\big[\widehat{c}(x,y)f(s,t)+g(s,t)\big]\,ds\,dt\right),\quad (6.47)$$

for all $x_0\le x\le x_1$, $y_0\le y\le y_1$, *where*

$$H(r)=\int_{r_0}^{r}\frac{ds}{\psi(s)},\quad r\ge r_0>0,\qquad (6.48)$$

H^{-1} *denotes the inverse function of* H *and* x_1, y_1 *are so chosen that*

$$H(\varphi^{-1}(\widehat{a}(x,y)))+\int_{x_0}^{x}\int_{y_0}^{y}\big[\widehat{c}(x,y)f(s,t)+g(s,t)\big]\,ds\,dt\in \mathrm{Dom}(H^{-1}).$$

for all x, y *lying in the subintervals* $0\le x\le x_1$, $0\le y\le y_1$ *of* R_+.

Proof. Fixing any numbers X and Y with $x_0\le x\le X$ and $y_0\le y\le Y$, we assume that $\widehat{a}(X,Y)$ is positive and define a positive function $z(x,y)$ by

$$z(x,y)=\widehat{a}(X,Y)+\widehat{c}(X,Y)\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)\Bigg(u(s,t)\varphi'(u(s,t))+ \\ +\int_{x_0}^{s}\int_{y_0}^{t}g(s_1,t_1)\varphi'(u(s_1,t_1))\psi(u(s_1,t_1))\,ds_1\,dt_1\Bigg)\,ds\,dt;\quad (6.49)$$

then $z(x,y_0)=\widehat{a}(X,Y)$, $z(x_0,y)=\widehat{a}(X,Y)$, $u(x,y)\le\varphi^{-1}(z(x,y))$ and

$$z_x(x,y)=\widehat{c}(X,Y)\int_{y_0}^{y}f(x,t)\Bigg(u(x,t)\varphi'(u(x,t))+ \\ +\int_{x_0}^{x}\int_{y_0}^{t}g(s_1,t_1)\varphi'(u(s_1,t_1))\psi(u(s_1,t_1))\,ds_1\,dt_1\Bigg)\,dt\le \\ \le\varphi'(\varphi^{-1}(z(x,y)))\widehat{c}(X,Y)\int_{y_0}^{y}f(x,t)\Bigg(\varphi^{-1}(z(x,t))+ \\ +\int_{x_0}^{x}\int_{y_0}^{t}g(s_1,t_1)\psi(\varphi^{-1}(z(s_1,t_1)))\,ds_1\,dt_1\Bigg)\,dt,$$

i.e.

$$\frac{z_x(x,y)}{\varphi'(\varphi^{-1}(z(x,y)))} \le \widehat{c}(X,Y)\int_{y_0}^{y} f(x,t)\Big(\varphi^{-1}(z(x,t))+ + \int_{x_0}^{x}\int_{y_0}^{t} g(s_1,t_1)\psi(\varphi^{-1}(z(s_1,t_1)))\,ds_1\,dt_1\Big)\,dt. \quad (6.50)$$

Keeping y fixed in (6.50), we set $x = s$; then, integrating with respect to s from x_0 to x and using the fact that $z(x_0,y) = \widehat{a}(X,Y)$, we have

$$\varphi^{-1}(z(x,y)) \le \varphi^{-1}(\widehat{a}(X,Y)) + \widehat{c}(X,Y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(\varphi^{-1}(z(s,t))+ + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\psi(\varphi^{-1}(z(s_1,t_1)))\,ds_1\,dt_1\Big)\,ds\,dt. \quad (6.51)$$

Taking $X = x$, $Y = y$, since X and Y are arbitrary, by applying Theorem 6.9 to (6.51), we get the required inequalities (6.46) and (6.47) from the inequality $u(x,y) \le \varphi^{-1}(z(x,y))$.

If $\widehat{a}(X,Y) = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X,Y)$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

Theorem 6.11. *Let u, a, c, f, $g \in (\triangle, R_+,)$, and $\psi_i(u), i = 1,2$ be nondecreasing continuous functions for $u \in R_+$ with $\psi_i(u) > 0$ for $u > 0$. Suppose that $\varphi \in C^1(R_+,R_+)$ is an increasing function with $\varphi(\infty) = \infty$. If*

$$\varphi(u(x,y)) \le a(x,y) + c(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(\psi_1(u(s,t))+ + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\psi_2(u(s_1,t_1))\,ds_1\,dt_1\Big)\,ds\,dt, \quad (6.52)$$

for x, $y \in R_+$, then

(i) *in the case $\psi_1(\varphi^{-1}(z)) \le \psi_2(\varphi^{-1}(z))$ for $z \in R_+$,*

$$u(x,y) \le \varphi^{-1}\Big[H_2^{-1}\Big(H_2(\widehat{a}(x,y))+ + \widehat{c}(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(1 + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,ds\,dt\Big)\Big], \quad (6.53)$$

for all $x_0 \le x \le x_2$, $y_0 \le y \le y_2$, *where*

$$H_2(r) = \int_{r_0}^{r} \frac{ds}{\psi_2(\varphi^{-1}(s))}, \quad r \ge r_0 > 0, \tag{6.54}$$

H_2^{-1} *denotes the inverse function of* H_2, x_2 *and* y_2 *are so chosen that*

$$H_2(\widehat{a}(x,y)) + \widehat{c}(x,y) \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(1 + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,ds\,dt \in \mathrm{Dom}(H_2^{-1})$$

for all x, y *lying in the subintervals* $0 \le x \le x_2$, $0 \le y \le y_2$ *of* R_+.

(ii) *in the case* $\psi_1(\varphi^{-1}(z)) > \psi_2(\varphi^{-1}(z))$ *for* $z \in R_+$,

$$u(x,y) \le \varphi^{-1}\Big[H_1^{-1}\Big(H_1(\widehat{a}(x,y)) + \widehat{c}(x,y) \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(1 + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,ds\,dt\Big)\Big], \tag{6.55}$$

for all $x_0 \le x \le x_3$, $y_0 \le y \le y_3$, *where*

$$H_1(r) = \int_{r_0}^{r} \frac{ds}{\psi_1(\varphi^{-1}(s))}, \quad r \ge r_0 > 0, \tag{6.56}$$

H_1^{-1} *denotes the inverse function of* H_1, x_3 *and* y_3 *are so chosen that*

$$H_1(\widehat{a}(x,y)) + \widehat{c}(x,y) \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(1 + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,ds\,dt \in \mathrm{Dom}(H_1^{-1})$$

for all x, y *lying in the subintervals* $0 \le x \le x_3$, $0 \le y \le y_3$ *of* R_+.

Proof. Fixing any numbers X and Y with $x_0 \le x \le X$ and $y_0 \le y \le Y$, we assume that $\widehat{a}(X,Y)$ is positive and define a positive function $z(x,y)$ by

$$z(x,y) = \widehat{a}(X,Y) + \widehat{c}(X,Y) \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(\psi_1(u(s,t)) + \int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\psi_2(u(s_1,t_1))\,ds_1\,dt_1\Big)\,ds\,dt, \tag{6.57}$$

then $z(x,y_0) = \widehat{a}(X,Y)$, $z(x_0,y) = \widehat{a}(X,Y)$, $u(x,y) \le \varphi^{-1}(z(x,y))$ and

$$z_x(x,y) = \widehat{c}(X,Y)\int_{y_0}^{y} f(x,t)\Big(\psi_1(u(x,t))+ + \int_{x_0}^{x}\int_{y_0}^{t} g(s_1,t_1)\psi_2(u(s_1,t_1))\,ds_1\,dt_1\Big)\,dt \le$$
$$\le \widehat{c}(X,Y)\int_{y_0}^{y} f(x,t)\Big(\psi_1(\varphi^{-1}(z(x,t)))+ + \int_{x_0}^{x}\int_{y_0}^{t} g(s_1,t_1)\psi_2(\varphi^{-1}(z(s_1,t_1)))\,ds_1\,dt_1\Big)\,dt. \quad (6.58)$$

When $\psi_1(\varphi^{-1}(z)) \le \psi_2(\varphi^{-1}(z))$, from the inequality (6.58), we find

$$\frac{z_x(x,y)}{\psi_2(\varphi^{-1}(z(x,y)))} \le \widehat{c}(X,Y)\int_{y_0}^{y} f(x,t)\Big(1+\int_{x_0}^{x}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,dt. \quad (6.59)$$

Keeping y fixed in (6.50), we set $x = s$; then, integrating with respect to s from x_0 to x and using the definition of H_2, we have

$$H_2(z(x,y)) \le H_2(\widehat{a}(X,Y)) + \widehat{c}(X,Y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(1+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,ds\,dt. \quad (6.60)$$

Taking $X = x$, $Y = y$ in (6.60), since X and Y are arbitrary, we get the required inequality (6.53) from the inequality $u(x,y) \le \varphi^{-1}(z(x,y))$.

When $\psi_1(\varphi^{-1}(z)) > \psi_2(\varphi^{-1}(z))$, by following the same argument as in the proof below the inequality (6.59), we get the required inequality (6.55).

If $\widehat{a}(X,Y) = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X,Y)$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

For the special case $\psi_1(u) = \varphi(u)$, Theorem 6.11 gives the following integral inequality for nonlinear functions.

Corollary 6.12. *Let u, a, c, f, $g \in (\triangle, R_+,)$ and $\psi(u)$ be a nondecreasing continuous function for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. Suppose that $\varphi \in C(R_+,R_+)$ is an increasing function with $\varphi(\infty) = \infty$. If*

$$\varphi(u(x,y)) \le a(x,y) + c(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(\varphi(u(s,t))+$$

$$+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\psi(u(s_1,t_1))\,ds_1\,dt_1\Big)\,ds\,dt, \quad (6.61)$$

for x, $y \in R_+$, *then*

(i) *in the case* $\psi(\varphi^{-1}(z)) \le z$ *for* $z \in R_+$,

$$u(x,y) \le$$

$$\le \varphi^{-1}\Big[\widehat{a}(x,y)\exp\Big(\widehat{c}(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(1+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,ds\,dt\Big)\Big],$$

for all $(x,y) \in \triangle$.

(ii) *in the case* $\psi(\varphi^{-1}(z)) > z$ *for* $z \in R_+$,

$$u(x,y) \le \varphi^{-1}\Big[H^{-1}\Big(H(\widehat{a}(x,y))+$$

$$+\widehat{c}(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(1+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,ds\,dt\Big)\Big], \quad (6.62)$$

for all $x_0 \le x \le x_4$, $y_0 \le y \le y_4$, *where*

$$H(r) = \int_{r_0}^{r} \frac{ds}{\psi(\varphi^{-1}(s))}, \quad r \ge r_0 > 0, \quad (6.63)$$

H^{-1} *denotes the inverse function of* H, x_4 *and* y_4 *are so chosen that*

$$H(\widehat{a}(x,y))+$$

$$+\widehat{c}(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Big(1+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Big)\,ds\,dt \in \mathrm{Dom}(H^{-1})$$

for all x, y *lying in the subintervals* $0 \le x \le x_4$, $0 \le y \le y_4$ *of* R_+.

7. Multidimensional Linear Integral Inequalities

Wendroff's inequality has received considerable attention and several papers have appeared in the literature dealing with various extensions, generalizations and applications. During the past few years, many papers have appeared in the literature which deal with integral inequalities in n independent variables. See Agarwal at al. [1], [4], [6], Akinyele [7], Beesack [11], [12]–[14], Chandra and Davis [21], Conlan and Wang [25], [26], Fink [32], Ghoshal at al. [35], Pachpatte [58], Singare and Pachpatte [71], Yang [77], [78], Yeh [79], and Young [80], [81].

We present in this section some inequalities of Wendroff's type in n independent variables investigated by Pachpatte alone, Pachoatte and his co-workers, as well as others.

In what follows, we adopt the following definitions and notational conventions. Let Ω be an open bounded set in R^n and a point $(x_1, \dots, x_n)$ in Ω be denoted by x. Let x^0 and $x(x^0 < x)$ be any two points in Ω and $\int\limits_{x^0}^{x} \cdots d\xi$ denote the n-fold integral $\int\limits_{x_1^0}^{x_1} \cdots \int\limits_{x_n^0}^{x_n} \cdots d\xi_n \cdots d\xi_1$, $D_i = \partial/\partial x_i$, $1 \le i \le n$. For any pair x, s of points of Ω with $x < s$ we denote $D(x,s) = \{x \in R^n : \ x \le \xi \le s\} \subset \Omega$ and $\int\limits_{x}^{s} \cdots d\xi$ denotes the n-fold integral $\int\limits_{x_1}^{s_1} \cdots \int\limits_{x_n}^{s_n} \cdots d\xi_n \cdots d\xi_1$.

Bondge and Pachpatte [18] investigated the inequalities in the following theorem.

Theorem 7.1 (Bondge and Pachpatte, 1979)**.** *Let $u(x)$, $p(x)$, $q(x)$ be nonnegative continuous functions defined on Ω and $a_i(x_i) > 0$, $a_i'(x_i) \ge 0$ for $1 \le i \le n$, be continuous defined for $x_i \ge x_i^0$.*

(i) *If*

$$u(x) \le \sum_{i=1}^{n} a_i(x_i) + \int\limits_{x^0}^{x} p(y)u(y)\,dy, \tag{7.1}$$

for $x \in \Omega$, then

$$u(x) \le E(x) \exp\left(\int\limits_{x^0}^{x} p(y)\,dy\right), \tag{7.2}$$

for $x \in \Omega$, where

$$E(x) = \frac{\left[\sum\limits_{i=1}^{n} a_i(x_i) + a_1(x_1^0) - a_1(x_1)\right]\left[\sum\limits_{i=1}^{n} a_i(x_i) + a_2(x_2^0) - a_2(x_2)\right]}{\left[\sum\limits_{i=3}^{n} a_i(x_i) + a_1(x_1^0) - a_2(x_2)\right]}. \tag{7.3}$$

(ii) *If*

$$u(x) \le \sum_{i=1}^{n} a_i(x_i) + \int\limits_{x^0}^{x} p(y)u(y)\,dy + \int\limits_{x^0}^{x} p(y)\left(\int\limits_{x^0}^{y} q(s)u(s)\,ds\right)dy, \tag{7.4}$$

for $x \in \Omega$, then

$$u(x) \le \sum_{i=1}^{n} a_i(x_i) + \int\limits_{x^0}^{x} p(y)E(y)\exp\left(\int\limits_{x^0}^{y} [p(s) + q(s)]\,ds\right)dy, \tag{7.5}$$

for $x \in \Omega$, where $E(x)$ is defined by (7.3)*.*

A slight variant of Theorem 7.1 appeared in [63, pp. 400–401].

Theorem 7.2 (Pachpatte, 1998)**.** *Let $u(x)$, $p(x)$, and $q(x)$ be nonnegative continuous functions defined on Ω and let $a(x)$, $x \in \Omega$, be a continuous function which is nonnegative and nondecreasing in each component x_i of x.*

(i) *If*

$$u(x) \le a(x) + \int_{x^0}^{x} p(y)u(y)\,dy, \tag{7.6}$$

for $x \in \Omega$, *then*

$$u(x) \le a(x) \exp\biggl(\int_{x^0}^{x} p(y)\,dy\biggr), \tag{7.7}$$

for $x \in \Omega$.

(ii) *If*

$$u(x) \le a(x) + \int_{x^0}^{x} p(y)\biggl(u(y) + \int_{x^0}^{y} q(s)u(s)\,ds\biggr)\,dy, \tag{7.8}$$

for $x \in \Omega$, *then*

$$u(x) \le a(x)\biggl[\int_{x^0}^{x} p(y) \exp\biggl(\int_{x^0}^{y} [p(s) + q(s)]\,ds\biggr)\,dy\biggr], \tag{7.9}$$

for $x \in \Omega$.

The following theorem deals with general inequalities which appeared in [63, pp. 401–402].

Theorem 7.3 (Pachpatte, 1998)**.** *Let* $u(x)$, $a(x)$, $b(x)$, $p(x)$, *and* $q(x)$ *be nonnegative continuous functions defined on* Ω.

(i) *If*

$$u(x) \le a(x) + b(x)\int_{x^0}^{x} p(y)u(y)\,dy, \tag{7.10}$$

for $x \in \Omega$, *then*

$$u(x) \le a(x) + b(x)m(x) \exp\biggl(\int_{x^0}^{x} b(y)p(y)\,dy\biggr), \tag{7.11}$$

for $x \in \Omega$, *where*

$$m(x) = \int_{x^0}^{x} a(y)p(y)\,dy, \tag{7.12}$$

for $x \in \Omega$.

(ii) *If*

$$u(x) \le a(x) + b(x)\int_{x^0}^{x} p(y)\biggl(u(y) + \int_{x^0}^{y} q(s)u(s)\,ds\biggr)\,dy, \tag{7.13}$$

for $x \in \Omega$, *then*

$$u(x) \le a(x) + b(x)r(x)\left[1 + \int_{x^0}^{x} p(y)b(y)\exp\left(\int_{x^0}^{y} [p(s)+q(s)]\,ds\right)dy\right], \quad (7.14)$$

for $x \in \Omega$, *where*

$$r(x) = \int_{x^0}^{x} p(y)\left(a(y) + b(y)\int_{x^0}^{y} q(s)a(s)\,ds\right)dy, \quad (7.15)$$

for $x \in \Omega$.

Inspired by the inequalities given by Gollwitzer [36], Pachpatte [48], and Singare and Pachpatte [72] established the inequalities in the following theorem.

Theorem 7.4 (Singare and Pachpatte, 1981)**.** *Let* $\phi(x)$, $a(x)$, $b(x)$, *and* $c(x)$ *be nonnegative continuous functions defined on* Ω *and* $u(x)$ *be a positive continuous function defined on* Ω.

(i) *If*

$$u(s) \ge \phi(x) - a(s)\int_{x}^{s} b(\xi)\phi(\xi)\,d\xi, \quad (7.16)$$

for $x \le s$, x, $s \in \Omega$, *then*

$$u(s) \ge \phi(x)\exp\left(-a(s)\int_{x}^{s} b(\xi)\,d\xi\right), \quad (7.17)$$

for $x \le s$, x, $s \in \Omega$.

(ii) *If*

$$u(s) \ge \phi(x) - a(s)\left[\int_{x}^{s} b(\xi)\phi(\xi)\,d\xi + \int_{x}^{s} b(\xi)\left(\int_{\xi}^{s} c(\rho)\phi(\rho)\,d\rho\right)d\xi\right], \quad (7.18)$$

for $x \le s$, x, $s \in \Omega$, *then*

$$u(s) \ge \phi(x)\left[1 + a(s)\left(\int_{x}^{s} b(\xi)\exp\left(\int_{x}^{s} \left[a(s)b(\rho) + c(\rho)\right]d\rho\right)d\xi\right)\right], \quad (7.19)$$

for $x \le s$, x, $s \in \Omega$.

Remark 7.1. Note that the method employed in the proofs of the foregoing theorems can also be used to obtain n-independent-variable versions of various inequalities given in earlier section. Since this translation is quite straightforward in view of the results given in this section, it is left to the reader to fill in the details where needed.

Young [80] extended Snow's technique to n independent variables. Bondge and Pachpatte [19] gave more general integral inequalities in n independent variables by using Young's technique. The inequalities given in Bondge and Pachpatte [19] are further generalizations of the inequalities given by Pachpatte [18]. Next, we deal with the inequalities given by Young [80] and Bondge and Pachpatte [19], which have many important applications in the theory of partial differential and integro-differential equations in n independent variables.

The following inequality is established by Young [80].

Theorem 7.5 (Young, 1973)**.** *Let $\phi(x), a(x),$ and $b(x) \geq 0$ be nonnegative continuous functions in $\Omega \subset R^n$. Let $v(\xi; x)$ be a solution of the characteristic initial value problem*

$$\begin{gathered} (-1)^n v_{\xi_1,\ldots,\xi_n}(\xi; x) - b(\xi)v(\xi; x) = 0 \;\; \text{in} \;\; \Omega, \\ v(\xi; x) = 1 \;\; \text{on} \;\; \xi_i = x_i, \;\; i = 1, \ldots, n \end{gathered} \tag{7.20}$$

and let D^+ be a connected subdomain of Ω containing x such that $v \geq 0$ for all $\xi \in D^+$. If $D \subset D^+$ and

$$\phi(x) \leq a(x) + \int_{x^0}^{x} b(\xi)\phi(\xi)\, d\xi, \tag{7.21}$$

then

$$\phi(x) \leq a(x) + \int_{x^0}^{x} a(\xi)b(\xi)v(\xi; x)\, d\xi. \tag{7.22}$$

Remark 7.2. The existence and regularity property of v can be deduced from Courant and Hilbert [28](see also Copson [27]; Garbedian, [33]). Indeed, the problem (7.20) is equivalent to the integral equation

$$v(\xi; x) = 1 + \int_{\xi}^{x} b(\eta)v(\eta; x)\, d\eta.$$

The following two theorems given by Bondge and Pachpatte [19] provide an extension to the case of n independent variables – the quite general results established by Pachpatte [51].

Theorem 7.6 (Bondge and Pachpatte, 1980)**.** *Let $\phi(x)$, $a(x)$, $b(x)$, $c(x)$, and $\sigma(x)$ be nonnegative continuous functions in $\Omega \subset R^n$. Let $v(\xi; x)$ be a solution of the characteristic initial value problem*

$$\begin{gathered} (-1)^n v_{\xi_1,\ldots,\xi_n}(\xi; x) - [b(\xi) + c(\xi)]v(\xi; x) = 0 \;\; \text{in} \;\; \Omega, \\ v(\xi; x) = 1 \;\; \text{on} \;\; \xi_i = x_i, \;\; 1 \leq i \leq n, \end{gathered} \tag{7.23}$$

and let D^+ be a connected subdomain of Ω containing x such that $v \geq 0$ for all $\xi \subset D^+$. If $D \in D^+$ and

$$\phi(x) \leq a(x) + \int_{x^0}^{x} b(\rho)\phi(\rho)\,d\rho + \int_{x^0}^{x} b(\rho)\Big[\sigma(\rho) + \int_{x^0}^{\rho} c(\xi)\phi(\xi)\,d\xi\Big]\,d\rho, \quad (7.24)$$

then

$$\phi(x) \leq a(x) + \int_{x^0}^{x} b(\rho)\Big[a(\rho)+\sigma(\rho)+\int_{x^0}^{\rho}\Big\{a(\xi)c(\xi)+b(\xi)\big[a(\xi)+\sigma(\xi)\big]\Big\}v(\xi;\rho)\,d\xi\Big]\,d\rho. \quad (7.25)$$

Theorem 7.7 (Bondge and Pachpatte, 1980)**.** *Let $\phi(x)$, $a(x)$, $b(x)$, $c(x)$, and $k(x)$ be nonnegative continuous functions defined on $\Omega \subset R^n$. Let $v(\xi;x)$ and $w(\xi;x)$ be the solutions of the characteristic initial value problems*

$$\begin{aligned}(-1)^n v_{\xi_1,\dots,\xi_n}(\xi;x) - \big[b(\xi)+c(\xi)+k(\xi)\big]v(\xi;x) = 0 \ \ \textit{in} \ \ \Omega,\\ v(\xi;x) = 1 \ \ \textit{on} \ \ \xi_i = x_i, \ \ 1 \leq i \leq n,\end{aligned} \quad (7.26)$$

and

$$\begin{aligned}(-1)^n w_{\xi_1,\dots,\xi_n}(\xi;x) - \big[b(\xi)-c(\xi)\big]w(\xi;x) = 0 \ \ \textit{in} \ \ \Omega,\\ w(\xi;x) = 1 \ \ \textit{on} \ \ \xi_i = x_i, \ \ 1 \leq i \leq n,\end{aligned} \quad (7.27)$$

respectively and let D^+ be a connected subdomain of Ω containing x such that $v \geq 0, w \geq 0$ for all $\xi \in D^+$. If $D \subset D^+$ and

$$\phi(x) \leq a(x) + \int_{x^0}^{x} b(\rho)\phi(\rho)\,d\rho + \int_{x^0}^{x} c(\rho)\Big(\int_{x^0}^{\rho} k(\xi)\phi(\xi)\,d\xi\Big)\,d\rho, \quad (7.28)$$

then

$$\phi(x) \leq a(x) + \int_{x^0}^{x} w(\rho;x)\Big[a(\rho)b(\rho)+c(\rho)\int_{x^0}^{\rho} a(\xi)[b(\xi)+k(\xi)]v(\xi;\rho)\,d\xi\Big]\,d\rho. \quad (7.29)$$

The inequalities established in the following two theorems by Bondge and Pachpatte [19] can be used in certain applications.

Theorem 7.8 (Bondge and Pachpatte, 1980)**.** *Let $\phi(x)$, $a(x)$, $b(x)$, $c(x)$, and $d(x)$ be nonnegative continuous functions defined on $\Omega \subset R^n$. Let $v(\xi;x)$ and $w(\xi;x)$ be the solutions of the characteristic initial value problems*

$$\begin{aligned}(-1)^n v_{\xi_1,\dots,\xi_n}(\xi;x) - \big[b(\xi)+c(\xi)+d(\xi)\big]v(\xi;x) = 0 \ \ \textit{in} \ \ \Omega,\\ v(\xi;x) = 1 \ \ \textit{on} \ \ \xi_i = x_i, \ \ 1 \leq i \leq n,\end{aligned} \quad (7.30)$$

and

$$\begin{aligned}(-1)^n w_{\xi_1,\dots,\xi_n}(\xi;x) - b(\xi)w(\xi;x) = 0 \ \ \textit{in} \ \ \Omega,\\ w(\xi;x) = 1 \ \ \textit{on} \ \ \xi_i = x_i, \ \ 1 \leq i \leq n,\end{aligned} \quad (7.31)$$

respectively and let D^+ *be a connected subdomain of* Ω *containing* x *such that* $v \geq 0$, $w \geq 0$ *for all* $\xi \in D^+$. *If* $D \subset D^+$ *and*

$$\phi(x) \leq a(x) + \int_{x^0}^{x} b(\eta)\phi(\eta)\,d\eta + \int_{x^0}^{x} b(\eta)\Big(\int_{x^0}^{\eta} c(\rho)\phi(\rho)\,d\rho\Big)\,d\eta + \\ + \int_{x^0}^{x} b(\eta)\Big(\int_{x^0}^{\eta} c(\rho)\Big(\int_{x^0}^{\rho} d(\xi)\phi(\xi)\,d\xi\Big)\,d\rho\Big)\,d\eta, \tag{7.32}$$

then

$$\phi(x) \leq a(x) + \int_{x^0}^{x} b(\eta)\Big[a(\eta) + \int_{x^0}^{\eta} w(\rho;\eta)\Big(a(\rho)\big[b(\rho)+c(\rho)\big] + \\ + c(\rho)\int_{x^0}^{\rho} a(\xi)\big[b(\xi)+c(\xi)+d(\xi)\big]v(\xi;\rho)\,d\xi\Big)\,d\rho\Big]\,d\eta. \tag{7.33}$$

Theorem 7.9 (Bondge and Pachpatte, 1980)**.** *Let* $\phi(x)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$, $p(x)$ *and* $q(x)$ *be nonnegative continuous functions defined on* $\Omega \subset R^n$. *Let* $v(\xi;x)$, $w(\xi;x)$ *and* $e(\xi;x)$ *be the solutions of the characteristic initial value problems*

$$(-1)^n v_{\xi_1,\ldots,\xi_n}(\xi;x) - \big[b(\xi)+c(\xi)+d(\xi)+p(\xi)+q(\xi)\big]v(\xi;x) = 0 \ \text{in}\ \Omega, \\ v(\xi;x) = 1 \ \text{on}\ \xi_i = x_i,\ 1 \leq i \leq n, \tag{7.34}$$

and

$$(-1)^n w_{\xi_1,\ldots,\xi_n}(\xi;x) - \big[b(\xi)+c(\xi)+d(\xi)-p(\xi)\big]w(\xi;x) = 0 \ \text{in}\ \Omega, \\ w(\xi;x) = 1 \ \text{on}\ \xi_i = x_i,\ 1 \leq i \leq n, \tag{7.35}$$

and

$$(-1)^n e_{\xi_1,\ldots,\xi_n}(\xi;x) - \big[b(\xi)-c(\xi)\big]e(\xi;x) = 0 \ \text{in}\ \Omega, \\ e(\xi;x) = 1 \ \text{on}\ \xi_i = x_i,\ 1 \leq i \leq n, \tag{7.36}$$

respectively and let D^+ *be a connected subdomain of* Ω *containing* x *such that* $v \geq 0$, $w \geq 0$ *for all* $\xi \in D^+$. *If* $D \subset D^+$ *and*

$$\phi(x) \leq a(x) + \int_{x^0}^{x} b(\eta)\phi(\eta)\,d\eta + \int_{x^0}^{x} c(\eta)\Big(\int_{x^0}^{\eta} d(\rho)\phi(\rho)\,d\rho\Big)\,d\eta + \\ + \int_{x^0}^{x} c(\eta)\Big(\int_{x^0}^{\eta} p(\rho)\Big(\int_{x^0}^{\rho} q(\xi)\phi(\xi)\,d\xi\Big)\,d\rho\Big)\,d\eta, \tag{7.37}$$

then

$$\phi(x) \le a(x) + \int_{x^0}^{x} e(\eta;x)\Big[a(\eta)b(\eta) + c(\eta)\int_{x^0}^{\eta} w(\rho;\eta)\Big(a(\rho)[b(\rho)+d(\rho)] +$$

$$+ p(\rho)\int_{x^0}^{\rho} a(\xi)\big[b(\xi)+c(\xi)+d(\xi)+q(\xi)\big]v(\xi;\rho)\,d\xi\Big)\,d\rho\Big]\,d\eta. \quad (7.38)$$

8. Multidimensional Nonlinear Integral Inequalities

The integral inequalities involving functions of many independent variables, which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of partial differential equations. The last few years have witnessed a great deal of research concerning such inequalities and their applications in the theory of partial differential equations. This section deals with some basic inequalities established in Pelczar [70], Headley [40], Beesack [10] and Pachpatte [58] which provide a very useful and important device in the study of many qualitative properties of the solutions of various types of partial differential, integral, and integro-differential equations.

Pelczar [70] initiated the study of some inequalities for a broad class of operators. In order to present the main results in Pelczar [70] we need the following definitions found there.

We call a set P *partly ordered* if, for some pairs of elements x, $y \in P$, a relation $x \le y$ is defined in such a way that:

(a) for each $x \in P$, $x \le x$,

(b) if $x \le y$ and $y \le x$ then $x = y$ and

(c) if $x \le y$ and $y \le z$, then $x \le z$.

Let P be a partly ordered set and $Q \subset P$. We call z the *upper bound of Q in P* if $z \in P$, $x \in Q$, and $x \le \widehat{z}$. We call $\widehat{z}$ the *supremum of the set Q* (abbreviated $\sup Q$) if $\widehat{z}$ is an upper bound of Q in P and $x \le \widehat{z}$. Each partially ordered set can have at most one supremum.

The set P will be said to satisfy the condition (II) if the difference $x - y \in P$ is defined for each $x, y \in P$ in such a way that

(d) if $x \le y$, then for each $x \in P$, $x - z \le y - z$,

(e) there exists an element $0 \in P$, such that for each $x \in p$, $x - 0 = x$ and

(f) $x = y$ if and only if $x - y = 0$.

The set P will be said to satisfy the condition (II*) if, for each x, $y \in P$, there exists in P, $z = \sup\{x, y\}$.

The main result established by Pelczar [70] in embodied in the following theorem.

Theorem 8.1 (Pelczar, 1963)**.** *Assume that*

(a_1) *The set P is not empty, partly ordered and fulfils the conditions* (II) *and* (II*),

(a_2) *the functions* $W(x)$ *and* $L(x)$ *are defined in the set* P *and are such that* $W(p) \subset P$ *and* $L(P) \subset P$,
(a_3) *if* $x \le L(x)$, *then* $x \le 0$,
(a_4) *if* $x \le y$, *then* $W(x) \le W(y)$,
(a_5) *if* $0 \le W(x) - W(y)$, *then* $W(x) - W(y) \le L(x - y)$,
(a_6) w *is a solution of the equation*

$$w = W(w), \tag{8.1}$$

(a_7) $v \in P$ *is such that*

$$v \le W(v). \tag{8.2}$$

Then we have

$$v \le w. \tag{8.3}$$

As an application, consider the following equation

$$u(x) = f(x) + \int\limits_E F(x, y, u(y))\, dy, \tag{8.4}$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and E is an n-dimensional set, and the inequality

$$v(x) \le f(x) + \int\limits_E F(x, y, v(y))\, dy. \tag{8.5}$$

Using Theorem 8.1, Pelczar [70] proved the following important result.

Theorem 8.2 (Pelczar, 1963). *Assume that*
(b_1) $F(x, y, z)$ *is defined and is continuous in* $\overline{E} \times E \times R$, $R = (-\infty, \infty,)$
(b_2) $f(x)$ *is defined and is continuous in* $\overline{E}$,
(b_3) *if* $x \le \widehat{z}$, *then* $F(x, y, z) \le F(x, y, \widehat{z})$,
(b_4) *if* $|F(x, y, z) - F(x, y, \widehat{z})| \le l(x, y, |z - \widehat{z}|)$, *where the function* $l(x, y, z)$ *is defined in* $\overline{E} \times E \times R$ *and is such that if*

$$w(x) \le \int\limits_E l(x, y, w(y))\, dy,$$

then

$$W(x) \le 0,$$

(b_5) $u(x)$ *is a solution of equation* (8.4) *in* $\overline{E}$,
(b_6) $v(x)$ *is a continuous function defined in* $\overline{E}$ *and fulfils the inequality* (8.5). *Then in the set* $\overline{E}$ *we have*

$$v(x) \le u(x). \tag{8.6}$$

In order to establish the next theorem, given by Headly [40], we require the following result, given in Beesack [10, p. 88].

Theorem 8.3 (Beesack, 1975). *Let G be an open set in R^N, and for $x,\, y \in G$ let*

$$G(x,y) = \Big\{z \in R^N :\ z_j = \lambda_j x_j + (1-\lambda_j)y_j,\ 0 \le \lambda_j \le 1,\ 1 \le j \le N\Big\},$$

denote the rectangular parallelepiped with one diagonal joining the points x, y. Let the points x^0, $y \in G$ be such that $G_0 = G(x^0, y) \subset G$, and let the functions $a(x)$, $k(x,t,z)$ be real-valued and continuous on G_0 and on $G_T \times R$ respectively, where

$$G_T = \Big\{(x,t) :\ x \in G_0,\ t \in G(x^0, x)\Big\}.$$

Suppose also that k is nondecreasing in z for each $(x,t) \in G_T$ and that

$$|k(x,t,z)| \le h(t) g(|z|), \tag{8.7}$$

for $(x,t,z) \in G_T \times R$, where $h \in L(G_0)$ and g is continuous and nondecreasing on R_+ with $\int_1^\infty ds/g(s) = \infty$. Then the integral equation

$$u(x) = a(x) + \int\limits_{G(x^0,x)} k(x,t,u(t))\, dt, \tag{8.8}$$

has a solution which is continuous on G_0. Moreover, if $\{\epsilon_n\}$ is a strictly decreasing sequence with $\lim \epsilon_n = 0$, and if u_n is a continuous solution on G_0 of the integral equation

$$u(x) = a(x) + \epsilon_n + \int\limits_{G(x^0,x)} k(x,t,u_n(t))\, dt, \tag{8.9}$$

then $U(x) = \lim u_n(x)$ exists uniformly on G_0, and $U(x)$ is the maximal solution of (8.11)*.*

Headley [40, Theorem 1] considered the case of Theorem 8.3 with $k = k(t,z)$ continuous on $G_0 \times R$ and nondecreasing in z. Hypothesis (8.7) was overlooked in [40] and in the case $k = k(t,z)$ of the following theorem given by Headley [40, Theorem 2].

Theorem 8.4 (Headley, 1974). *Let G, $G_0 = G(x^0, y)$ and the functions $a(x)$, $k(x,t,z)$ be as in Theorem* 8.3*, and let the function v be continuous on G_0 and satisfy the inequality*

$$v(x) \le a(x) + \int\limits_{G(x^0,x)} k(x,t,v(t))\, dt, \quad x \in G_0. \tag{8.10}$$

Then

$$v(x) \le U(x), \quad x \in G_0, \tag{8.11}$$

where U is the maximal solution of (8.11) *on G_0.*

In our further discussion, some useful integral inequalities in n independent variables given by Pachpatte [58] are presented. These are motivated by a well-known integral inequality due to Ważeski [76]. We use the same notation as given in Section 3.7 without further mention. Pachpatte [58] gave the following general version of Ważeski's inequality given in Ważeski [76].

Theorem 8.5 (Pachpatte, 1981)**.** *Let $u(x)$ and $a(x)$ be nonnegative continuous functions defined on Ω. Let $k(x,y,z)$ and $W(x,x)$ be nonnegative continuous functions defined on $\Omega^2 \times R$ and $\Omega \times R$, respectively, and nondecreasing in the last variables, and $k(x,y,z)$ be uniformly Lipschitz in the last variable. If*

$$u(x) \leq a(x) + W\Big(x, \int_{x^0}^{x} k(x,t,u(y))\,dy\Big), \tag{8.12}$$

then

$$u(x) \leq a(x) + W(x, r(x)), \tag{8.13}$$

for $x \in \Omega$, where $r(x)$ is the solution of the equation

$$r(x) = \int_{x^0}^{x} k\big(x,t,a(y) + W(y, r(y))\big)\,dy, \tag{8.14}$$

existing on Ω.

The following inequality established by Pachpatte [58] combines the features of two inequalities, namely, the $n-$ independent-variable generalization of Wendroff's inequality, and the integral inequality given by Headley [40, Theorem 2]. This inequality can be used more effectively in the theory of certain integral and integro-differential equations involving n independent variables.

Theorem 8.6 (Pachpatte, 1981)**.** *Let $u(x)$, $f(x)$, $g(x)$, $q(x)$, and $c(x)$ be nonnegative continuous functions defined on Ω, with $f(x) > 0$ and nondecreasing in x and $q(x) \geq 1$. Let $k(x,y,z)$ and $W(x,x)$ be nonnegative continuous functions defined on $\Omega^2 \times R$ and $\Omega \times R$, respectively; let $k(x,y,z)$ be nondecreasing in x and z and is uniformly Lipschitz in z and $W(x,z)$ is nondecreasing in both x and z. If*

$$\begin{aligned} u(x) \leq f(x) + q(x)\Big[\int_{x^0}^{x} g(y)u(y)\,dy + \int_{x^0}^{x} g(y)q(y)\Big(\int_{x^0}^{y} c(s)u(s)\,ds\Big)\,dy\Big] + \\ + W\Big(x, \int_{x^0}^{x} k(x,t,u(y))\,dy\Big), \end{aligned} \tag{8.15}$$

for $x \in \Omega$, then

$$u(x) \leq E_0(x)\big[f(x) + W(x, r(x))\big], \tag{8.16}$$

for $x \in \Omega$, *where*

$$E_0(x) = q(x)\left[1 + \int_{x^0}^{x} g(y)q(y)\exp\left(\int_{x^0}^{y} q(s)[g(s) + c(s)]\,ds\right)dy\right], \tag{8.17}$$

and $r(x)$ *is the solution of the equation*

$$r(x) = \int_{x^0}^{x} k\Big(x, y, E_0(y)\big[f(y) + W(y, r(y))\big]\Big)\,dy, \tag{8.18}$$

existing on Ω.

Another interesting and useful integral inequality given by Pachpatte [58] in n independent variables involving two nonlinear functions on the right-hand side of the inequality is embodied in the following theorem.

Theorem 8.7 (Pachpatte, 1981)**.** *Let* $u(x)$*,* $f(x)$*,* $g(x)$*,* $q(x)$*, and* $c(x)$ *be nonnegative continuous functions defined on* Ω*, with* $f(x) \geq 1$ *and nondecreasing in* x *and* $q(x) \geq 1$*. Let* $k(x, y, z)$ *and* $W(x, x)$ *be nonnegative continuous functions defined on* $\Omega^2 \times R$ *and* $\Omega \times R$*, respectively;* $k(x, y, z)$ *is nondecreasing in* x *and* z *and is uniformly Lipschitz in* z *and* $W(x, z)$ *is nondecreasing in both* x *and* z*. Let* $H : R_+ \to R_+$ *be continuously differentiable function with* $H(u) > 0$ *for* $u > 0$*,* $H'(u) \geq 0$ *for* $u \geq 0$ *and satisfy* $(1/v)H(u) \leq H(u/v)$ *for* $v \geq 1$*,* $u \geq 0$ *and* $H(u)$ *is submultiplicative for* $u \geq 0$*. If*

$$u(x) \leq f(x) + q(x)\int_{x^0}^{x} g(y)H(u(y))\,dy + W\left(x, \int_{x^0}^{x} k(x, t, u(y))\,dy\right), \tag{8.19}$$

for $x \in \Omega$*, then for* $x \in \Omega_1 \subset \Omega$*, then*

$$u(x) \leq E_1(x)\big[f(x) + W(x, r(x))\big], \tag{8.20}$$

where

$$E_1(x) = q(x)G^{-1}\left[G(1) + \int_{x^0}^{x} g(y)H(q(y))\,dy\right], \tag{8.21}$$

in which

$$G(v) = \int_{v_0}^{v} \frac{ds}{H(s)}, \quad v > 0, \quad v_0 > 0, \tag{8.22}$$

G^{-1} *is the inverse of* G *and*

$$G(1) + \int_{x^0}^{x} g(y)H(q(y))\,dy \in \mathrm{Dom}(G^{-1}),$$

for $x \in \Omega$, *and* $r(x)$ *is the solution of the equation*

$$r(x) = \int_{x^0}^{x} k(x, y, E_1(y)[f(y) + W(y, r(y))])\, dy, \tag{8.23}$$

existing on Ω.

Pachpatte [58] gave the following inequality, which can be used in more general situations.

Theorem 8.8 (Pachpatte, 1981)**.** *Let* $u(x)$, $f(x)$, *and* $g(x)$ *be nonnegative continuous functions defined on* Ω, *with* $f(x) \geq 1$ *and nondecreasing in* x *and* $q(x) \geq 1$. *Let* $k(x, y, z)$ *and* $w(x, x)$ *be nonnegative continuous functions defined on* $\Omega^2 \times R$ *and* $\Omega \times R$, *respectively;* $k(x, y, z)$ *is nondecreasing in* x *and* z *and is uniformly Lipschitz in* z *and* $W(x, z)$ *is nondecreasing in both* x *and* z. *Let* $H : R_+ \to R_+$ *be continuously differentiable function with* $H(u) > 0$ *for* $u > 0, H'(u) \geq 0$ *for* $u \geq 0$ *and satisfies* $(1/v)H(u) \leq H(u/v)$ *for* $v \geq 1$, $u \geq 0$. *If*

$$u(x) \leq f(x) + \int_{x^0}^{x} g(y)\Big(u(y) + \int_{x^0}^{y} g(s)H(u(s))\, ds\Big)\, dy + \\ + W\Big(x, \int_{x^0}^{x} k(x, t, u(y))\, dy\Big), \tag{8.24}$$

for $x \in \Omega$, *then for* $x \in \Omega_2 \subset \Omega$,

$$u(x) \leq E_2(x)\big[f(x) + W(x, r(x))\big], \tag{8.25}$$

where

$$E_2(x) = 1 + \int_{x^0}^{x} q(y)F^{-1}\Big[F(1) + \int_{x^0}^{y} g(s)\, ds\Big]\, dy, \tag{8.26}$$

in which

$$F(\sigma) = \int_{\sigma_0}^{\sigma} \frac{ds}{s + H(s)}, \quad \sigma > 0, \quad \sigma_0 > 0, \tag{8.27}$$

F^{-1} *is the inverse of* F *and*

$$F(1) + \int_{x^0}^{y} g(s)\, ds \in \mathrm{Dom}(F^{-1}),$$

for $x \in \Omega_2$, *and* $r(x)$ *is the solution of the equation*

$$r(x) = \int_{x^0}^{x} k(x, y, E_2(y)[f(y) + W(y, r(y))])\, dy, \tag{8.28}$$

existing on Ω.

The details of the proof of this theorem follow by an argument similar to that in the proof of Theorem 8.7 and the details are omitted here.

Pachpatte[58] has established the following generalization of the integral inequality given by Young [80].

Theorem 8.9 (Pachpatte, 1981)**.** *Let $u(x)$, $a(x)$, $b(x)$, $c(x)$, $f(x)$, and $g(x)$ be nonnegative continuous functions defined on Ω with $f(x) > 0$ and nondecreasing in x. Let $k(x, y, z)$ and $W(x, x)$ be nonnegative continuous functions defined on $\Omega^2 \times R$ and $\Omega \times R$, respectively; let $k(x, y, z)$ be nondecreasing in x and z and be uniformly Lipschitz in z and $W(x, z)$ be nondecreasing in both x and z. Let $v(y; x)$ and $e(y; x)$ be the solutions of the characteristic initial value problems*

$$\begin{gathered}(-1)^n v_{y_1\cdots y_n}(y;x) - \big[a(y)b(y) + a(y)g(y) + c(y)\big]v(y;x) = 0 \text{ in } \Omega,\\ v(y;x) = 1 \text{ on } y_i = x_i,\ \ 1 \le i \le n,\end{gathered} \tag{8.29}$$

and

$$\begin{gathered}(-1)^n e_{y_1\cdots y_n}(y;x) - \big[a(y)b(y) - c(y)\big]e(y;x) = 0 \text{ in } \Omega,\\ e(y;x) = 1 \text{ on } y_i = x_i,\ \ 1 \le i \le n,\end{gathered} \tag{8.30}$$

respectively, and let D^+ be a connected subdomain of Ω containing x such that $v \ge 0$, $e \ge 0$ for all $y \in D^+$. If $D \subset D^+$ and

$$u(x) \le f(x) + a(x)\Big[\int_{x^0}^{x} b(y)u(y)\,dy + \int_{x^0}^{x} c(y)\Big(\int_{x^0}^{y} g(s)u(s)\,ds\Big)\,dy\Big] + + W\Big(x, \int_{x^0}^{x} k(x, t, u(y))\,dy\Big), \tag{8.31}$$

for $x \in \Omega$, then

$$u(x) \le E_3(x)\big[f(x) + W(x, r(x))\big], \tag{8.32}$$

where

$$E_3(x) = 1 + a(x)\Big[\int_{x^0}^{x} e(y;x)\Big\{b(y) + c(y)\Big(\int_{x^0}^{y}[b(s)+g(s)]v(s;y)\,ds\Big)\Big\}\,dy\Big], \tag{8.33}$$

and $r(x)$ is the solution of the equation

$$r(x) = \int_{x^0}^{x} k\Big(x, y, E_3(y)\big[f(y) + W(y, r(y))\big]\Big)\,dy, \tag{8.34}$$

existing on Ω.

We now present some new nonlinear retarded Gronwall–Bellman-type integral inequalities in many independent variables as following Theorems, which can be used as effective tools in the study of certain integral equations.

In what follows, we adopt the following definitions and notational conventions. Let Ω be an open bounded set in R^n and a point $(x_1, \ldots, x_n)$ in Ω be denoted by x. Let x^0 and $x(x^0 < x)$ be any two points in Ω and $\int\limits_{x^0}^{x} \cdots d\xi$ denote the n-fold integral $\int\limits_{x_1^0}^{x_1} \cdots \int\limits_{x_n^0}^{x_n} \cdots d\xi_n \cdots d\xi_1$, $D_i = \partial/\partial x_i$, $1 \le i \le n$. For any pair x, s of points of Ω with $x < s$, $D(x,s) = \{x \in R^n : \ x \le \xi \le s\} \subset \Omega$ and $\int\limits_{x}^{s} \cdots d\xi$ the n-fold integral $\int\limits_{x_1}^{s_1} \cdots \int\limits_{x_n}^{s_n} \cdots d\xi_n \cdots d\xi_1$. Given a continuous function $a : \Omega \to R_+$, we write

$$\widehat{a}(x) = \max\big\{a(y) : \ x^0 \le y \le x\big\}.$$

Theorem 8.10. *Let u, a, c, f, $g \in (\Omega, R_+,)$ and $\psi(u)$ be nondecreasing continuous functions for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. If*

$$u(x) \le a(x) + c(x) \int\limits_{x^0}^{x} f(y)\bigg(u(y) + \int\limits_{x^0}^{y} g(s)\psi(u(s))\,ds\bigg)\,dy,$$

for $x \in \Omega$, then

(i) *in the case $\psi(u) \le u$,*

$$u(x) \le \widehat{a}(x) \exp\bigg(\int\limits_{x^0}^{x} \big[\widehat{c}(x) f(s) + g(s)\big]\,ds\bigg), \tag{8.35}$$

for all $x \in \Omega$ and

(ii) *in the case $\psi(u) > u$,*

$$u(x) \le H^{-1}\bigg(H(\widehat{a}(x)) + \int\limits_{x^0}^{x} \big[\widehat{c}(x) f(s) + g(s)\big]\,ds\bigg), \tag{8.36}$$

for all $x_0 \le x \le X^1$, where

$$H(r) = \int\limits_{r_0}^{r} \frac{ds}{\psi(s)}, \quad r \ge r_0 > 0, \tag{8.37}$$

H^{-1} denotes the inverse function of H and X^1 is chosen so that

$$H(\widehat{a}(x)) + \widehat{c}(x) \int\limits_{x^0}^{x} [f(s) + g(s)]\,ds \in \mathrm{Dom}(H^{-1}).$$

for all x lying in the subintervals $0 \le x \le X^1$ of Ω.

Proof. Fixing any number $X \in \Omega$ with $x_0 \le x \le X$, we assume that $\widehat{a}(X)$ is positive and define a positive function $z(x)$ by

$$z(x) = \widehat{a}(X) + \widehat{c}(X) \int\limits_{x^0}^{x} f(y)\bigg(u(y) + \int\limits_{x^0}^{y} g(s)\psi(u(s))\,ds\bigg)\,dy,$$

then $z(x^0) = \widehat{a}(X)$, $u(x) \le z(x)$ and

$$D_1 \cdots D_n z(x) = \widehat{c}(X) f(x) \Bigg(u(x) + \int\limits_{x^0}^{x} g(s) \psi(u(s))\, ds \Bigg) \le$$

$$\le \widehat{c}(X) f(x) \Bigg(z(x) + \int\limits_{x^0}^{x} g(s) \psi(z(s))\, ds \Bigg). \tag{8.38}$$

If we put

$$v(x) = z(x) + \int\limits_{x^0}^{x} g(s) \psi(z(s))\, ds,$$

$v(x) = z(x)$ on $x_j = x_j^0$, $1 \le j \le n$, then $D_1 \cdots D_n z(x) \le \widehat{c}(X) f(x) v(x)$, $z(x) \le v(x)$ and

$$D_1 \cdots D_n v(x) \le \widehat{c}(X) f(x) v(x) + g(x) \psi(v(x)). \tag{8.39}$$

When $\psi(v) \le v$, from the inequality (8.39), we find

$$\frac{D_1 \cdots D_n v(x)}{v(x)} \le \widehat{c}(X) f(x) + g(x). \tag{8.40}$$

From (8.40) and by using the facts that $D_n v(x) \ge 0$, $D_1 \cdots D_{n-1} v(x) \ge 0$, $v(x) > 0$ for $x \in \Omega$, we observe that

$$\frac{D_1 \cdots D_n v(x)}{v(x)} \le \widehat{c}(X) f(x) + g(x) + \frac{[D_n v(x)][D_1 \cdots D_{n-1} v(x)]}{[v(x)]^2},$$

i.e.

$$D_n \Bigg(\frac{D_1 \ldots D_{n-1} v(x)}{v(x)} \Bigg) \le \widehat{c}(X) f(x) + g(x). \tag{8.41}$$

Keeping $x_1, \ldots, x_{n-1}$ fixed in (8.41), we set $x_n = s_n$; then, integrating with respect to s_n from x_n^0 to x_n we have

$$\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \le$$

$$\le \int\limits_{x_n^0}^{x_n} \Big[\widehat{c}(X) f(x_1, \ldots, x_{n-1}, s_n) + g(x_1, \ldots, x_{n-1}, s_n) \Big]\, ds_n. \tag{8.42}$$

Again as above, from (8.42) we observe that

$$D_{n-1} \Bigg(\frac{D_1 \ldots D_{n-2} v(x)}{v(x)} \Bigg) \le$$

$$\le \int\limits_{x_n^0}^{x_n} \Big[\widehat{c}(X) f(x_1, \ldots, x_{n-1}, s_n) + g(x_1, \ldots, x_{n-1}, s_n) \Big]\, ds_n. \tag{8.43}$$

Keeping $x_1, \dots, x_{n-2}$ and x_n fixed in (8.43), we set $x_{n-1} = s_{n-1}$; then, integrating with respect to s_{n-1} from x^0_{n-1} to x_{n-1} we have

$$\frac{D_1 \dots D_{n-2} v(x)}{v(x)} \leq$$

$$\leq \int_{x^0_{n-1}}^{x_{n-1}} \int_{x^0_n}^{x_n} \Big[\widehat{c}(X) f(x_1, \dots, x_{n-2}, s_{n-1}, s_n) + g(x_1, \dots, x_{n-2}, s_{n-1}, s_n)\Big]\, ds_n\, ds_{n-1}.$$

Continuing in this way we have

$$\frac{D_1 v(x)}{v(x)} \leq$$

$$\leq \int_{x^0_2}^{x_2} \cdots \int_{x^0_n}^{x_n} \Big[\widehat{c}(X) f(x_1, s_2, \dots, s_n) + g(x_1, s_2, \dots, s_n)\Big]\, ds_n \cdots ds_2. \quad (8.44)$$

Keeping $x_2, \dots, x_n$ fixed in (8.44), we set $x_1 = s_1$; then, integrating with respect to s_1 from x^0_1 to x_1 we have

$$v(x) \leq \widehat{a}(X) \exp\bigg(\int_{x^0}^{x} \big[\widehat{c}(X) f(s) + g(s)\big]\, ds \bigg). \quad (8.45)$$

Taking $X = x$ and using $u(x) \leq z(x) \leq v(x)$ in the inequality (8.45), since X is arbitrary, we get the required inequality (8.35).

When $\psi(v) > v$, from the inequality (8.39), we find

$$\frac{D_1 \cdots D_n v(x)}{\psi(v(x))} \leq \widehat{c}(X) f(x) + g(x). \quad (8.46)$$

The rest of the proof is immediate by analogy with the last argument when $\psi(v) \leq v$, together with the definition of the function H.

If $\widehat{a}(X) = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X)$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

Theorem 8.10 can easily be applied to generate other useful nonlinear integral inequalities in more general situations. For example, we have the following result (Theorems 8.11–12).

Theorem 8.11. *Let u, a, c, f, $g \in (\Omega, R_+,)$ and $\psi(u)$ be a nondecreasing continuous function for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. Suppose that $\varphi \in C^1(R_+, R_+)$ is an increasing function with $\varphi(\infty) = \infty$ and $\varphi'(u)$ is a nondecreasing continuous function for $u \in R_+$. If*

$$\varphi(u(x)) \leq a(x) +$$

$$+ c(x) \int_{x^0}^{x} f(y) \bigg(u(y)\varphi'(u(y)) + \int_{x^0}^{y} g(s)\varphi'(u(s))\psi(u(s))\, ds \bigg)\, dy, \quad (8.47)$$

for $x \in \Omega$, *then*

(i) *in the case* $\psi(\varphi^{-1}(z)) \le \varphi^{-1}(z)$ *for* $z \in R_+$,

$$u(x) \le \varphi^{-1}(\widehat{a}(x)) \exp\left(\int_{x^0}^{x} \left[\widehat{c}(x) f(y) + g(y)\right] dy\right), \tag{8.48}$$

for all $x,\, y \in \Omega$,

(ii) *in the case* $\psi(\varphi^{-1}(z)) > \varphi^{-1}(z)$ *for* $z \in R_+$,

$$u(x) \le H^{-1}\left(H(\varphi^{-1}(\widehat{a}(x))) + \int_{x^0}^{x} \left[\widehat{c}(x) f(y) + g(y)\right] dy\right), \tag{8.49}$$

for all $x_0 \le x \le X$, *where*

$$H(r) = \int_{r_0}^{r} \frac{ds}{\psi(s)}, \quad r \ge r_0 > 0, \tag{8.50}$$

H^{-1} *denotes the inverse function of* H *and* X *is so chosen so that*

$$H(\varphi^{-1}(\widehat{a}(x))) + \widehat{c}(x) \int_{x^0}^{x} [f(y) + g(y)]\, dy \in \mathrm{Dom}(H^{-1}).$$

for all $x,\, y$ *lying in the subintervals* $0 \le x \le X$ *of* Ω.

Proof. Fixing any numbers $X \in \Omega$ with $x_0 \le x \le X$, we assume that $\widehat{a}(X)$ is positive and define a positive function $z(x)$ by

$$z(x) = \widehat{a}(X) + \widehat{c}(X) \int_{x^0}^{x} f(y)\left(u(y)\varphi'(u(y)) + \int_{x^0}^{y} g(s)\varphi'(u(s))\psi(u(s))\, ds\right) dy,$$

then $z(x_1^0, x_2, \ldots, x_n) = \widehat{a}(X)$, $u(x) \le \varphi^{-1}(z(x))$ and

$$\begin{aligned} D_1 z(x) &= \\ = \widehat{c}(X) &\int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} f(x_1, y_2, \ldots, y_n)\Bigg(u(x_1, y_2, \ldots, y_n)\varphi'(u(x_1, y_2, \ldots, y_n)) + \\ &+ \int_{x_1^0}^{x_1} \int_{x_2^0}^{y_2} \cdots \int_{x_n^0}^{y_n} g(s)\varphi'(u(s))\psi(u(s))\, ds\Bigg) dy_n \cdots dy_2 \le \\ \le \varphi'(\varphi^{-1}(z(x)))\widehat{c}(X) &\int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} f(x_1, y_2, \ldots, y_n)\Bigg(\varphi^{-1}(z(x_1, y_2, \ldots, y_n)) + \\ &+ \int_{x_1^0}^{x_1} \int_{x_2^0}^{y_2} \cdots \int_{x_n^0}^{y_n} g(s)\psi(\varphi^{-1}(z(s)))\, ds\Bigg) dy_n \cdots dy_2, \end{aligned}$$

i.e.

$$\frac{D_1 z(x)}{\varphi'(\varphi^{-1}(z(x)))} \leq$$

$$\leq \widehat{c}(X) \int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} f(x_1, y_2, \ldots, y_n) \Big(\varphi^{-1}(z(x_1, y_2, \ldots, y_n)) +$$

$$+ \int_{x_1^0}^{x_1} \int_{x_2^0}^{y_2} \cdots \int_{x_n^0}^{y_n} g(s) \psi(\varphi^{-1}(z(s)))\, ds \Big)\, dy_n \cdots dy_2. \quad (8.51)$$

Keeping $x_2, \ldots, x_n$ fixed in (8.51), we set $x_1 = s_1$; then, integrating with respect to s_1 from x_1^0 to x_1 we have

$$\varphi^{-1}(z(x)) \leq \varphi^{-1}(\widehat{a}(X)) +$$

$$+ \widehat{c}(X) \int_{x^0}^{x} f(y) \Big(\varphi^{-1}(z(y)) + \int_{x^0}^{y} g(s) \psi(\varphi^{-1}(z(s)))\, ds \Big)\, dy. \quad (8.52)$$

Taking $X = x$, since X is arbitrary, by applying Theorem 8.10 to (8.52), we get the required inequalities (8.48) and (8.49) from the inequality $u(x) \leq \varphi^{-1}(z(x))$.

If $\widehat{a}(X) = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X)$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

Theorem 8.12. *Let u, a, c, f, $g \in (\Omega, R_+,)$ and $\psi_i(u)$, $i = 1, 2$ be nondecreasing continuous functions for $u \in R_+$ with $\psi_i(u) > 0$ for $u > 0$. Suppose that $\varphi \in C(R_+, R_+)$ is an increasing function with $\varphi(\infty) = \infty$. If*

$$\varphi(u(x)) \leq a(x) + c(x) \int_{x^0}^{x} f(y) \Big(\psi_1(u(y)) + \int_{x^0}^{y} g(s) \psi_2(u(s))\, ds \Big)\, dy, \quad (8.53)$$

for x, $y \in \Omega$, then

(i) *in the case $\psi_1(\varphi^{-1}(z)) \leq \psi_2(\varphi^{-1}(z))$ for $z \in R_+$,*

$$u(x) \leq \varphi^{-1} \Big[H_2^{-1} \Big(H_2(\widehat{a}(x)) + \widehat{c}(x) \int_{x^0}^{x} f(y) \Big(1 + \int_{x^0}^{y} g(s)\, ds \Big)\, dy \Big) \Big], \quad (8.54)$$

for all $x^0 \leq x \leq \overline{X}$, where

$$H_2(r) = \int_{r_0}^{r} \frac{ds}{\psi_2(\varphi^{-1}(s))}, \quad r \geq r_0 > 0, \quad (8.55)$$

H_2^{-1} *denotes the inverse function of* H_2, $\overline{X}$ *is chosen so that*

$$H_2(\widehat{a}(x)) + \widehat{c}(x) \int\limits_{x^0}^{x} f(y)\Big(1 + \int\limits_{x^0}^{y} g(s)\,ds\Big)\,dy \in \mathrm{Dom}(H_2^{-1})$$

for all x *lying in the subintervals* $x^0 \le x \le \overline{X}$ *of* Ω,

(ii) *in the case* $\psi_1(\varphi^{-1}(z)) > \psi_2(\varphi^{-1}(z))$ *for* $z \in R_+$,

$$u(x) \le \varphi^{-1}\Big[H_1^{-1}\Big(H_1(\widehat{a}(x)) + \widehat{c}(x) \int\limits_{x^0}^{x} f(y)\Big(1 + \int\limits_{x^0}^{y} g(s)\,ds\Big)\,dy\Big)\Big], \quad (8.56)$$

for all $x_0 \le x \le Y$, *where*

$$H_1(r) = \int\limits_{r_0}^{r} \frac{ds}{\psi_1(\varphi^{-1}(s))}, \quad r \ge r_0 > 0, \quad (8.57)$$

H_1^{-1} *denotes the inverse function of* H_1, Y *is so chosen that*

$$H_1(\widehat{a}(x)) + \widehat{c}(x) \int\limits_{x^0}^{x} f(y)\Big(1 + \int\limits_{x^0}^{y} g(s)\,ds\Big)\,dy \in \mathrm{Dom}(H_1^{-1})$$

for all x *lying in the subintervals* $x^0 \le x \le Y$ *of* Ω.

Proof. Fixing any numbers X with $x^0 \le x \le X$, we assume that $\widehat{a}(X)$ is positive and define a positive function $z(x)$ by

$$z(x) = \widehat{a}(X) + \widehat{c}(X) \int\limits_{x^0}^{x} f(y)\Big(\psi_1(u(y)) + \int\limits_{x^0}^{y} g(s)\psi_2(u(s))\,ds\Big)\,dy,$$

then $z(x_1^0, x_2, \dots, x_n) = \widehat{a}(X)$, $u(x) \le \varphi^{-1}(z(x))$ and

$$D_1 z(x) = \widehat{c}(X) \int\limits_{x_2^0}^{x_2} \cdots \int\limits_{x_n^0}^{x_n} f(x_1, y_2, \dots, y_n)\Big(\psi_1(u(x_1, y_2, \dots, y_n)) + $$
$$+ \int\limits_{x_1^0}^{x_1} \int\limits_{x_2^0}^{y_2} \cdots \int\limits_{x_n^0}^{y_n} g(s)\psi_2(u(s))\,ds\Big)\,dy_n \cdots dy_2 \le$$
$$\le \widehat{c}(X) \int\limits_{x_2^0}^{x_2} \cdots \int\limits_{x_n^0}^{x_n} f(x_1, y_2, \dots, y_n)\Big(\psi_1\big(\varphi^{-1}(z(x_1, y_2, \dots, y_n))\big) + $$
$$+ \int\limits_{x_1^0}^{x_1} \int\limits_{x_2^0}^{y_2} \cdots \int\limits_{x_n^0}^{y_n} g(s)\psi_2(\varphi^{-1}(z(s)))\,ds\Big)\,dy_n \cdots dy_2. \quad (8.58)$$

When $\psi_1(\varphi^{-1}(z)) \le \psi_2(\varphi^{-1}(z))$, from the inequality (8.58), we find

$$\frac{D_1 z(x)}{\psi_2(\varphi^{-1}(z(x)))} \le$$
$$\le \widehat{c}(X) \int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} f(x_1, y_2, \dots, y_n)\Big(1 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{y_2} \cdots \int_{x_n^0}^{y_n} g(s)\,ds\Big)\,dy_n \cdots dy_2. \tag{8.59}$$

Keeping $x_2, \dots, x_n$ fixed in (8.51), we set $x_1 = s_1$; then, integrating with respect to s_1 from x_1^0 to x_1 and using the definition of H_2, we have

$$H_2(z(x)) \le H_2(\widehat{a}(X)) + \widehat{c}(X) \int_{x^0}^{x} f(y)\Big(1 + \int_{x^0}^{y} g(s)\,ds\Big)\,dy. \tag{8.60}$$

Taking $X = x$ in (8.60), since X was arbitrary, we get the required inequality (8.54) from the inequality $u(x) \le \varphi^{-1}(z(x))$.

When $\psi_1(\varphi^{-1}(z)) > \psi_2(\varphi^{-1}(z))$, by following the same argument as in the proof below the inequality (8.58), we get the required inequality (8.56).

If $\widehat{a}(X) = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of $\widehat{a}(X)$ and subsequently let $\varepsilon \to 0$. This completes the proof. □

For the special case $\psi_1(u) = \varphi(u)$, Theorem 6.12 gives the following integral inequality for nonlinear functions.

Corollary 8.13. *Let u, a, c, f, $g \in (\Omega, R_+,)$ and $\psi(u)$ be a nondecreasing continuous function for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. Suppose that $\varphi \in C(R_+, R_+)$ is an increasing function with $\varphi(\infty) = \infty$. If*

$$\varphi(u(x)) \le a(x) + c(x) \int_{x^0}^{x} f(y)\Big(\varphi(u(y)) + \int_{x^0}^{y} g(s)\psi(u(s))\,ds\Big)\,dy,$$

for $x \in \Omega$, then

(i) *in the case $\psi(\varphi^{-1}(z)) \le z$ for $z \in R_+$,*

$$u(x) \le \varphi^{-1}\Big[\widehat{a}(x) \exp\Big(\widehat{c}(x) \int_{x^0}^{x} f(y)\Big(1 + \int_{x^0}^{y} g(s)\,ds\Big)\,dy\Big)\Big],$$

for all $x \in \Omega$.

(ii) *in the case $\psi(\varphi^{-1}(z)) > z$ for $z \in R_+$,*

$$u(x) \le \varphi^{-1}\Big[H^{-1}\Big(H(\widehat{a}(x)) + \widehat{c}(x) \int_{x^0}^{x} f(y)\Big(1 + \int_{x^0}^{y} g(s)\,ds\Big)\,dy\Big)\Big],$$

for all $x^0 \leq x \leq \overline{Y}$, *where*

$$H(r) = \int\limits_{r_0}^{r} \frac{ds}{\psi(\varphi^{-1}(s))}, \quad r \geq r_0 > 0, \tag{8.61}$$

H^{-1} *denotes the inverse function of* H, $\overline{Y}$ *is so chosen that*

$$H(\widehat{a}(x)) + \widehat{c}(x) \int\limits_{x^0}^{x} f(y)\biggl(1 + \int\limits_{x^0}^{y} g(s)\, ds\biggr)\, dy \in \operatorname{Dom}(H^{-1})$$

for all x *lying in the subintervals* $0 \leq x \leq \overline{Y}$ *of* Ω.

9. Applications

Most of the inequalities given here are recently investigated and can be used as tools in the study of various branches of partial differential, integral, and integro-differential equations. Akinyele [7] applied inequalities to a certain nonlinear hyperbolic functional integro-differential equations; Bondge and Pachpatte [17], [18] applied these to some differential and integral equations as well as nonlinear hyperbolic partial differential equations; Ghoshal and Masood [34] and Ghoshal at al. [35] applied them to a non-linear non-self-adjoint vector hyperbolic partial differential equations; Pachpatte [51], [54], [56]–[58] applied inequalities to nonlinear hyperbolic integro-differential equations and to nonlinear non-self-adjoint hyperbolic partial differential equations; Shastri and Kasture [70] applied them to scalar hyperbolic differential equations; Singare and Pachpatte [71] applied these to nonlinear integral equations; Snow [74] applied inequalities to nonlinear vector hyperbolic partial differential equation; Yang [77], [78] applied them to Volterra integral equations and hyperbolic integro-partial differential equations.

In this section we present application of some of the inequalities given in earlier sections to study the qualitative behavior of the solutions of certain partial differential and integro-differential equations. We first consider a nonlinear hyperbolic partial differential equation of the form

$$D_2 D_1 z(x,y) = \sum_{i=1}^{n} F_i[x, y, z(x,y)], \tag{9.1}$$

with the given boundary conditions

$$z(x,0) = e_1(x), \quad z(0,y) = e_2(y), \quad e_1(0) = e_2(0) = 0, \tag{9.2}$$

where e_1, $e_2 \in C(R_+, R)$, $F_i \in C(R_+^2 \times R, R)$, $i = 1, \dots, n$.

The following theorem deals with boundedness on the solution of the problem (9.1).

Theorem 9.1. *Assume that* $F_i : R_+^2 \times R \to R, i = 1, \dots, n$ *are continuous functions for which there exist continuous non-negative functions*

$a(x,y)$, $f_i(x,y)$, $g_i(x,y)$, $i=1,\dots,n$ for x, $y \in R_+$ such that

$$\begin{cases} \big|F_i(x,y,u)\big| \le |u|^q \big\{f_i(x,y)\psi(|u|) + g_i(x,y)\big\}, \\ \big|e_1(x)+e_2(y)\big| \le a(x,y), \end{cases} \tag{9.3}$$

where $0 < q < 1$ is a constant, and $\psi(u)$ is a nondecreasing continuous function for $u \in R_+$ with $\psi(u) > 0$ for $u > 0$. If $z(x,y)$ is any solution of the problem (9.1) with the condition (9.2), then

$$|z(x,y)| \le \left[G_1^{-1}\left(G_1(k_1(0,y)) + (1-q)\sum_{i=1}^{n}\int_0^x\int_0^y f_i(s,t)\,dt\,ds\right)\right]^{\frac{1}{1-q}} \tag{9.4}$$

for all $(x,y) \in [0,x_1]\times[0,y_1]$, where

$$\begin{aligned} k_1(0,y) &= [a(x,y)]^{1-q} + (1-q)\sum_{i=1}^{n}\int_0^x\int_0^y g_i(s,t)\,dt\,ds, \\ G_1(r) &= \int_{r_0}^{r} \frac{ds}{\psi(s^{\frac{1}{1-q}})}, \quad r \ge r_0 > 0, \end{aligned} \tag{9.5}$$

G_1^{-1} denotes the inverse function of G_1 and x_1, $y_1 \in R_+$ are so chosen that

$$G_1(k_1(0,y)) + (1-q)\sum_{i=1}^{n}\int_0^x\int_0^y f_i(s,t)\,dt\,ds \in \mathrm{Dom}(G_1^{-1}).$$

Proof. It is easy to see that the solution $z(x,y)$ of the problem (9.1) satisfies the equivalent integral equation:

$$z(x,y) = e_1(x) + e_2(y) + \sum_{i=1}^{n}\int_0^x\int_0^y F_i(s,t,z(s,t))\,dt\,ds. \tag{9.6}$$

From (9.3) and making the change of variables, we have

$$|z(x,y)| \le |e_1(x)+e_2(y)| + \sum_{i=1}^{n}\int_0^x\int_0^y \big|F_i(s,t,z(s,t))\big|\,dt\,ds \le$$

$$\le a(x,y) + \sum_{i=1}^{n}\int_0^x\int_0^y |z(s,t)|^q\Big\{f_i(s,t)\psi(|z(s,t)|) + g_i(s,t)\Big\}\,dt\,ds. \tag{9.7}$$

Now, a suitable application of the inequality given in Theorem 4.14 or Corollary 4.16 to (9.7) yields the desired result. □

Theorem 9.2. *Assume that $F_i : \triangle \times R \to R$, $i=1,\dots,n$ are continuous functions for which there exist continuous non-negative functions $a(x,y)$,*

$f_i(x,y)$, $g_i(x,y)$ $i=1,\dots,n$ for $x \in J_1$, $y \in J_2$ such that

$$\begin{cases} \left|F_i(x,y,u)\right| \le |u|^q \left\{f_i(x,y)L(x,y,|u|) + g_i(x,y)\right\}, \\ \left|e_1(x)+e_2(y)\right| \le a(x,y), \end{cases}$$

where $0 < q < 1$ is constants, and $L : \triangle \times R_+ \to R_+$ be a continuous function which satisfies the condition

$$0 \le L(x,y,v) - L(x,y,w) \le M(x,y,w)(v-w),$$

for $v \ge w \ge 0$ with $M : \triangle \times R_+ \to R_+$ is a continuous function. If $z(x,y)$ is any solution of the problem (9.1) with the condition (9.2), then

$$|z(x,y)| \le$$

$$\le \left[G_1^{-1}\left(G_1(k_2(x_0,y)) + (1-q)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)M(s,t)\,dt\,ds\right)\right]^{\frac{1}{1-q}} \tag{9.8}$$

for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$, where

$$\begin{aligned} k_2(x_0,y) &= [a(x,y)]^{1-q} + (1-q)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y}\left[f_i(s,t)L(s,t)+g_i(s,t)\right]dt\,ds, \\ G_1(r) &= \int_{r_0}^{r}\frac{ds}{\psi(s^{\frac{1}{1-q}})}, \quad r \ge r_0 > 0, \end{aligned} \tag{9.9}$$

G_1^{-1} denotes the inverse function of G_1 and $(x_1,y_1) \in \triangle$ is chosen so that

$$G_1(k_2(x_0,y)) + (1-q)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)M(s,t)\,dt\,ds \in \mathrm{Dom}(G_1^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 9.1 and by using Theorem 5.7 or Corollary 5.8 with suitable modification. We omit the details here. □

We now present an application of Theorem 6.9, Theorem 6.10, or Theorem 6.11 to obtain bounds on the solutions of a nonlinear hyperbolic partial integro-differential equation of the form

$$D_2\big(z^{p-1}(x,y)D_1 z(x,y)\big) = $$

$$= F\left(x,y,z(x,y),\int_{x_0}^{x}\int_{y_0}^{y} k(x,y,s,t,z(s,t))\,dt\,ds\right), \tag{9.10}$$

with the given boundary conditions

$$z^p(x,y_0) = e_3(x), \quad z^p(x_0,y) = e_4(y), \quad e_3(0)=e_4(0)=0, \tag{9.11}$$

where $e_j \in C(J_i,R)$, $j=3,4$ and $F \in C(\triangle \times R^2, R)$.

The following theorem deals with a boundedness on the solution of the problem (9.10) with condition (9.11).

Theorem 9.3. *Assume that* $F : \triangle \times R^2 \to R$ *is a continuous function for which there exists continuous non-negative functions* $a(x, y)$, $f(x, y)$, $g(x, y)$ *for* $x \in J_1$, $y \in J_2$ *such that*

$$\begin{cases} \big|e_1(x) + e_2(y)\big| \le a(x, y), \\ \big|F(x, y, u, v)\big| \le f(x, y)\big(|u^p| + |v|\big), \\ \big|k(x, y, s, t, u)\big| \le g(s, t)|u^{p-1}|\psi(|u|), \end{cases} \tag{9.12}$$

where $p > 1$ *is a constant,* $\psi(u)$ *is a nondecreasing continuous function for* $u \in R_+$ *with* $\psi(u) > 0$ *for* $u > 0$. *If* $z(x, y)$ *is any solution of the problem* (9.10) *with the condition* (9.11), *then*

(i) *in the case* $\psi(z^{\frac{1}{p}}) \le z^{\frac{1}{p}}$ *for* $z \in R_+$,

$$|z(x, y)| \le (\widehat{a}(x, y))^{\frac{1}{p}} \exp\bigg(\int_{x_0}^{x}\int_{y_0}^{y} \big[pf(s, t) + g(s, t)\big]\, ds\, dt\bigg), \tag{9.13}$$

for all $(x, y) \in \triangle$,

(ii) *in the case* $\psi(z^{\frac{1}{p}}) > z^{\frac{1}{p}}$ *for* $z \in R_+$,

$$|z(x, y)| \le H^{-1}\bigg(H((\widehat{a}(x, y))^{\frac{1}{p}}) + \int_{x_0}^{x}\int_{y_0}^{y} \big[pf(s, t) + g(s, t)\big]\, ds\, dt\bigg), \tag{9.14}$$

for all $x_0 \le x \le x_1$, $y_0 \le y \le y_1$, *where*

$$H(r) = \int_{r_0}^{r} \frac{ds}{\psi(s)}, \quad r \ge r_0 > 0, \tag{9.15}$$

H^{-1} *denotes the inverse function of* H *and* x_1, y_1 *are so chosen that*

$$H\big((\widehat{a}(x, y))^{\frac{1}{p}}\big) + \int_{x_0}^{x}\int_{y_0}^{y} \big[pf(s, t) + g(s, t)\big]\, ds\, dt \in \mathrm{Dom}(H^{-1}).$$

for all x, y *lying in the subintervals* $0 \le x \le x_1$, $0 \le y \le y_1$ *of* R_+.

Proof. It is easy to see that the solution $z(x, y)$ of the problem (9.10) satisfies the equivalent integral equation:

$$z^p(x, y) = e_3(x) + e_4(y) +$$
$$+ p\int_{x_0}^{x}\int_{y_0}^{y} F\bigg(s, t, z(s, t), \int_{x_0}^{s}\int_{y_0}^{t} k\big(s, t, s_1, t_1, z(s_1, t_1)\big)\, ds_1\, dt_1\bigg)\, dt\, ds. \tag{9.16}$$

From (9.14), (9.16), and making the change of variables, we have

$$\left|z^p(x,y)\right| \le a(x,y) + p\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Bigg(|z^p(s,t)|+$$
$$+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)|z^{p-1}(s,t)|\psi(|z(s_1,t_1)|)\,dt_1\,ds_1\Bigg)\,dt\,ds. \quad (9.17)$$

Now, a suitable application of the inequality given in Theorem 6.10 to (9.17) yields the desired result. This completes the proof. □

Theorem 9.4. *Assume that $F : \triangle\times R^2 \to R$ is a continuous function for which there exists continuous non-negative functions $a(x,y), f(x,y), g(x,y)$ for $x \in J_1, y \in J_2$ such that*

$$\begin{cases} \left|e_1(x)+e_2(y)\right| \le a(x,y), \\ |F(x,y,u,v)| \le f(x,y)(\psi_1(|u|)+|v|), \\ |k(x,y,s,t,u)| \le g(s,t)|\psi_2(|u|), \end{cases} \quad (9.18)$$

where $\psi_i(u), i = 1,2$ are nondecreasing continuous functions for $u \in R_+$ with $\psi_i(u) > 0$ for $u > 0$. If $z(x,y)$ is any solution of the problem (9.10) with the condition (9.11), then

(i) *in the case $\psi_1(z^{\frac{1}{p}}) \le \psi_2(z^{\frac{1}{p}})$ for $z \in R_+$,*

$$|z(x,y)| \le \Bigg[H_2^{-1}\Bigg(H_2(\widehat{a}(x,y))+$$
$$+p\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Bigg(1+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Bigg)\,ds\,dt\Bigg)\Bigg]^{\frac{1}{p}}, \quad (9.19)$$

for all $x_0 \le x \le x_2$, $y_0 \le y \le y_2$, where $p > 0$ is a constant,

$$H_2(r) = \int_{r_0}^{r} \frac{ds}{\psi_2(s^{\frac{1}{p}})}, \quad r \ge r_0 > 0, \quad (9.20)$$

H_2^{-1} denotes the inverse function of H_2, x_2 and y_2 are so chosen that

$$H_2(\widehat{a}(x,y))+p\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\Bigg(1+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\Bigg)\,ds\,dt \in \mathrm{Dom}(H_2^{-1})$$

for all x, y lying in the subintervals $0 \le x \le x_2$, $0 \le y \le y_2$ of R_+,

(ii) *in the case* $\psi_1(z^{\frac{1}{p}}) > \psi_2(z^{\frac{1}{p}})$ *for* $z \in R_+$,

$$|z(x,y)| \le \left[H_1^{-1}\left(H_1(\widehat{a}(x,y)) + p\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(1+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\right)ds\,dt\right)\right]^{\frac{1}{p}}, \quad (9.21)$$

for all $x_0 \le x \le x_3$, $y_0 \le y \le y_3$, *where* $p > 0$ *is a constant,*

$$H_1(r) = \int_{r_0}^{r} \frac{ds}{\psi_1(s^{\frac{1}{p}})}, \quad r \ge r_0 > 0, \quad (9.22)$$

H_1^{-1} *denotes the inverse function of* H_1, x_3 *and* y_3 *are so chosen that*

$$H_1(\widehat{a}(x,y)) + p\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(1+\int_{x_0}^{s}\int_{y_0}^{t} g(s_1,t_1)\,ds_1\,dt_1\right)ds\,dt \in \mathrm{Dom}(H_1^{-1})$$

for all x, y *lying in the subintervals* $0 \le x \le x_3$, $0 \le y \le y_3$ *of* R_+.

Proof. The proof follows by an argument similar to that in the proof of Theorem 9.3 and using Theorem 6.11 with suitable modification. We omit the details here. □

We now present application of Theorem 8.10, Theorem 8.11, or Theorem 8.12 to the boundedness of the solutions of some multivariate nonlinear hyperbolic partial integro-differential equation of the form

$$D_1 \dots D_n z^p(x) = F\left(x, z(x), \int_{x_0}^{x} k(x,y,z(y))\,dy\right), \quad (9.23)$$

with the conditions prescribed on $x_i = x_i^0$, $1 \le i \le n$, where $k \in C(\Omega^2 \times R, R)$ and $F \in C(\Omega \times R^2, R)$.

The following theorem deals with a boundedness on the solution of the problem (9.23) with condition $h(x)$, $x \in \Omega$.

Theorem 9.5. *Assume that* $F : \Omega \times R^2 \to R$ *is a continuous function for which there exists continuous non-negative functions* $a(x)$, $f(x)$, $g(x)$ *for* $x \in \Omega$ *such that*

$$\begin{cases} |h(x)| \le a(x), \\ |F(x,u,v)| \le f(x)\big(|u^p| + |v|\big), \\ |k(x,y,u)| \le g(y)|u^{p-1}|\psi(|u|), \end{cases} \quad (9.24)$$

where $p > 1$ *is a constant,* $\psi(u)$ *is a nondecreasing continuous function for* $u \in R_+$ *with* $\psi(u) > 0$ *for* $u > 0$. *If* $z(x)$ *is any solution of the problem*

(9.23) *where the boundary conditions are such that the given equation* (9.23) *is equivalent to the integral equation*

$$z^p(x) = h(x) + \int_{x_0}^{x} F\left(y, z(y), \int_{x_0}^{y} k(y,s,z(s))\,ds\right) dy, \tag{9.25}$$

where $h(x)$ *depends on the given boundary conditions, then*

(i) *in the case* $\psi(z^{\frac{1}{p}}) \le z^{\frac{1}{p}}$ *for* $z \in R_+$,

$$|z(x)| \le (\widehat{a}(x))^{\frac{1}{p}} \exp\left(\int_{x_0}^{x} [f(y) + g(y)]\,dy\right), \tag{9.26}$$

for all $x \in \Omega$,

(ii) *in the case* $\psi(z^{\frac{1}{p}}) > z^{\frac{1}{p}}$ *for* $z \in R_+$,

$$|z(x)| \le H^{-1}\left(H((\widehat{a}(x))^{\frac{1}{p}}) + \int_{x_0}^{x} [f(y) + g(y)]\,dy\right), \tag{9.27}$$

for all $x_0 \le x \le X$, $X \in \Omega$, *where*

$$H(r) = \int_{r_0}^{r} \frac{ds}{\psi(s)}, \quad r \ge r_0 > 0, \tag{9.28}$$

H^{-1} *denotes the inverse function of* H *and* X *is so chosen that*

$$H((\widehat{a}(x))^{\frac{1}{p}}) + \int_{x_0}^{x} [f(y) + g(y)]\,dy \in \mathrm{Dom}(H^{-1}),$$

for all x *lying in the subintervals* $0 \le x \le X$ *of* Ω.

Proof. It is easy to see that the solution $z(x)$ of the problem (9.23) satisfies the equivalent integral equation:

$$z^p(x) = h(x) + \int_{x_0}^{x} F\left(y, z(y), \int_{x_0}^{y} k(y,s,z(s))\,ds\right) dy. \tag{9.29}$$

From (9.24), (9.29), and making the change of variables, we have

$$|z^p(x)| \le a(x) + \int_{x_0}^{x} f(y)\left(|z||z^{p-1}(y)| + \int_{x_0}^{s} g(s)|z^{p-1}(s)|\psi(|z(s)|)\,ds\right) dy. \tag{9.30}$$

Now, a suitable application of the inequality given in Theorem 8.11 to (9.30) yields the desired result. □

Theorem 9.6. *Assume that $F : \Omega \times R^2 \to R$ is a continuous function for which there exist continuous non-negative functions $a(x)$, $f(x)$, $g(x)$ for $x \in \Omega$ such that*

$$\begin{cases} |h(x)| \le a(x), \\ |F(x,u,v)| \le f(x)\big(\psi_1(|u|) + |v|\big), \\ |k(x,y,u)| \le g(y)\psi_2(|u|), \end{cases} \tag{9.31}$$

where $\psi_i(u)$, $i = 1, 2$ are nondecreasing continuous functions for $u \in R_+$ with $\psi_i(u) > 0$ for $u > 0$. If $z(x)$ is any solution of the problem (9.23) where the boundary conditions are such that the given equation (9.23) is equivalent to the integral equation (9.25), then

(i) *in the case $\psi_1(z^{\frac{1}{p}}) \le \psi_2(z^{\frac{1}{p}})$ for $z \in R_+$,*

$$|z(x)| \le \left[H_2^{-1}\left(H_2(\widehat{a}(x)) + \int_{x^0}^{x} f(y)\left(1 + \int_{x^0}^{y} g(s)\,ds\right)dy\right)\right]^{\frac{1}{p}}, \tag{9.32}$$

for all $x^0 \le x \le \overline{X}$, where $p > 0$ is constant,

$$H_2(r) = \int_{r_0}^{r} \frac{ds}{\psi_2(s^{\frac{1}{p}})}, \quad r \ge r_0 > 0, \tag{9.33}$$

H_2^{-1} denotes the inverse function of H_2, $\overline{X}$ is so chosen that

$$H_2(\widehat{a}(x)) + \int_{x^0}^{x} f(y)\left(1 + \int_{x^0}^{y} g(s)\,ds\right)dy \in \mathrm{Dom}(H_2^{-1}),$$

for all x lying in the subintervals $x^0 \le x \le \overline{X}$ of Ω,

(ii) *in the case $\psi_1(z^{\frac{1}{p}}) > \psi_2(z^{\frac{1}{p}})$ for $z \in R_+$,*

$$|z(x)| \le \left[H_1^{-1}\left(H_1(\widehat{a}(x)) + \int_{x^0}^{x} f(y)\left(1 + \int_{x^0}^{y} g(s)\,ds\right)dy\right)\right]^{\frac{1}{p}}, \tag{9.34}$$

for all $x_0 \le x \le Y$, where where $p > 0$ is constant,

$$H_1(r) = \int_{r_0}^{r} \frac{ds}{\psi_1(s^{\frac{1}{p}})}, \quad r \ge r_0 > 0, \tag{9.35}$$

H_1^{-1} denotes the inverse function of H_1, Y is so chosen that

$$H_1(\widehat{a}(x)) + \int_{x^0}^{x} f(y)\left(1 + \int_{x^0}^{y} g(s)\,ds\right)dy \in \mathrm{Dom}(H_1^{-1}),$$

for all x lying in the subintervals $x^0 \le x \le Y$ of Ω.

Proof. The proof follows by an argument similar to that in the proof of Theorem 9.6 and using Theorem 8.12 with suitable modification. We omit the details here. □

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