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ON SOME NONLOCAL BOUNDARY VALUE PROBLEMS FOR LINEAR SINGULAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Abstract. For a higher order linear singular differential equation a class of nonlocal problems having the Fredholm property is described, and unimprovable in a sense sufficient conditions of unique solvability of those problems are established.

[illegible]

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Consider the boundary value problem

$$u^{(n)} = \sum_{k=1}^{n} h_k(t)u^{(k-1)} + h_0(t), \tag{1}$$

$$u^{(i-1)}(a) = c_i \quad (i = 1, \dots, n-1), \qquad \sum_{j=1}^{m} \int_{a_0}^{b} u^{(j-1)}(s)\, d\varphi_j(s) = c_n, \tag{2}$$

where $n \geq 2$, $c_i \in \mathbb{R}$ $(i = 1, \dots, n)$, $h_k : (a, b) \to \mathbb{R}$ $(k = 0, \dots, n)$ are locally integrable functions, $a_0 \in (a, b)$, $m \in \{1, \dots, n\}$, and $\varphi_j : [a_0, b] \to \mathbb{R}$ $(j = 1, \dots, m)$ are functions satisfying the conditions

$$\begin{gathered} \varphi_j(t) \geq \varphi_j(s) \quad \text{for} \quad a_0 \leq s \leq t \leq b \quad (j = 1, \dots, m), \\ \sum_{j=1}^{m} \big(\varphi_j(b) - \varphi_j(a)\big) > 0. \end{gathered} \tag{3}$$

In the regular case, where

$$\int_a^b |h_k(t)|\, dt < +\infty \qquad (k = 0, \dots, n)$$

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problems of type (1), (2) and similar problems for nonlinear differential equations were studied rather thoroughly (see [5], [6], [11], [23] and the literature cited therein). Mainly we concentrate our attention on singular case, where at least one of the coefficients of the differential equation (1) is nonintegrable on (a,b) having singularities at the endpoints of that interval.

In the present paper we prove that problem (1), (2) has the Fredholm property if h_k $(k=0,1,\dots,n)$ are integrable with certain weights. These conditions are provided in Theorems 1 and 3 below. As for Theorems 2 and 4, they contain new unimprovable conditions of unique solvability of singular problem (1), (2) different from previously known ones (see, e.g., [1–4], [7–10], [12–22]).

In the present paper the following notations will be used.

$$[x]_+ = \frac{|x|+x}{2}, \qquad [x]_- = \frac{|x|-x}{2}.$$

$\widetilde{C}^{n-1}_{loc}((a,b);\mathbb{R})$ is a space of $(n-1)$–times continuously differentiable functions $u:(a,b)\to\mathbb{R}$ whose derivative of $(n-1)$–th order is absolutely continuous on $[a+\varepsilon, b-\varepsilon]$ for arbitrary $\varepsilon\in(0,\frac{b-a}{2})$.

If $u\in\widetilde{C}^{n-1}_{loc}((a,b);\mathbb{R})$, then for any $i\in\{1,\dots,n\}$ by $u^{(i-1)}(a)$ (by $u^{(i-1)}(b)$) we understand the right (left) limit of $u^{(i-1)}$ at point a (point b) provided that such limit exists.

A function $u\in\widetilde{C}^{n-1}_{loc}((a,b);\mathbb{R})$ is called a solution of equation (1), if it satisfies (1) almost everywhere on (a,b).

A solution u of equation (1) is called a solution of problem (1), (2), if there exist $u^{(i-1)}(a)$ $(i=1,\dots,n-1)$ and $u^{(j-1)}(b)$ $(j=1,\dots,m)$, and equalities (2) hold.

Along with problem (1), (2) consider its corresponding homogeneous problem

$$u^{(n)} = \sum_{k=1}^{n} h_k(t)u^{(k-1)}, \tag{1_0}$$

$$u^{(i-1)}(a)=0 \quad (i=1,\dots,n-1), \qquad \sum_{j=1}^{m}\int_{a_0}^{b} u^{(j-1)}(s)\,d\varphi_j(s)=0. \tag{2_0}$$

Theorem 1. *Let conditions* (3) *hold and*

$$\int_a^b (s-a)^{n_k}(b-s)^{m_k}|h_k(s)|\,ds < +\infty \quad (k=0,\dots,n), \tag{4}$$

where $n_k = 1-[k+1-n]_+$, $m_k = n-m-[k-m]_+$ $(k=1,\dots,n)$. *Then problem* (1), (2) *is uniquely solvable if and only if the homogeneous problem* (1_0), (2_0) *has only a trivial solution.*

Theorem 2. *If along with* (3) *and* (4) *the condition*

$$\sum_{k=1}^{n}\int_a^b \frac{(s-a)^{n_k}}{(n-k)!}[h_k(s)]_-\,ds \le 1 \tag{5}$$

holds, then problem (1), (2) *is uniquely solvable.*

Corollary 1. *Let condition* (3) *hold,* $c_i > 0$ $(i = 1, \dots, n-1)$,

$$h_k(t) \ge 0 \quad \textit{for} \quad a < t < b \quad (k = 0, \dots, n), \tag{6}$$

$$\int_{t_0}^{b}(b-s)^{m_k}h_k(s)\,ds<+\infty\ (k=0,\dots,n),\ \int_a^{t_0}h_n(s)\,ds<+\infty,\ t_0\in(a,b).$$

Then problem (1), (2) *is uniquely solvable if and only if*

$$\int_a^{t_0}(s-a)h_k(s)\,ds < +\infty \qquad (k = 0, \dots, n-1).$$

Now consider the case where problem (1), (2) is not solvable for certain $c_i \in \mathbb{R}$ $(i = 1, \dots, n)$ but is uniquely solvable if only

$$c_i = 0 \qquad (i = 1, \dots, n_0), \tag{7}$$

where $n_0 \in \{1, \dots, n-1\}$.

Theorem 3. *Let along with* (3) *and* (7) *the conditions*

$$\begin{gathered}\int_a^b (b-s)^{n-m}|h_0(s)|\,ds < +\infty,\\ \int_a^b (s-a)^{n_{0k}}(b-s)^{m_k}|h_k(s)|\,ds < +\infty \quad (k = 1, \dots, n)\end{gathered} \tag{8}$$

hold, where $n_0 \in \{1, \dots, n-1\}$, $n_{0k} = [n_0+1-k]_+$, $m_k = n-m-[k-m]_+$ $(k = 1, \dots, n)$. *Then problem* (1), (2) *is uniquely solvable if and only if the homogeneous problem* $(1_0), (2_0)$ *has only a trivial solutions.*

Theorem 4. *If conditions* (3), (5), (7) *and* (8) *hold, then problem* (1), (2) *has one and only one solution.*

Corollary 2. *Let conditions* (3), (6), (7) *hold,* $n \ge 3$, $n_0 \in \{1, \dots, n-2\}$, $c_i > 0$ $(i = n_0+1, \dots, n-1)$,

$$\int_a^b(b-s)^{n-m}h_0(s)\,ds<+\infty, \quad \int_{t_0}^b(b-s)^{m_k}h_k(s)\,ds<+\infty\ (k=1,\dots,n),$$

$$\int_a^{t_0}h_k(s)\,ds < +\infty \quad (k = n_0+1, \dots, n),\ t_0 \in (a,b).$$

Then problem (1), (2) *is uniquely solvable if and only if*

$$\int_a^{t_0} (s-a)^{n_0+1-k} h_k(s)\, ds < +\infty \qquad (k = 1, \dots, n_0).$$

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