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ON NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Necessary and sufficient conditions are established for stability in the Lyapunov sense of solutions of the linear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t),$$

where $A : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ and $f : \mathbb{R}_+ \to \mathbb{R}^n$ ($\mathbb{R}_+ = [0, +\infty[$) are, respectively, continuous from the left matrix-and vector-functions with bounded total variation components on every closed interval from $\mathbb{R}_+$.

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[illegible]

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[illegible] $A : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ [illegible] $f : \mathbb{R}_+ \to \mathbb{R}^n$ ($\mathbb{R}_+ = [0, +\infty[$), [illegible] $\mathbb{R}_+$-[illegible]

1. Statement of the Problem and Formulation of the Results

Let $A = (a_{ik})_{i,k=1}^{n} : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $f = (f_i)_{i=1}^{n} : \mathbb{R}_+ \to \mathbb{R}^n$ ($\mathbb{R}_+ = [0, +\infty[$) be, respectively, continuous from the left matrix-and vector-functions with bounded total variation components on every closed interval from $\mathbb{R}_+$.

In this paper, necessary and sufficient conditions for stability in the Lyapunov sense with respect to small perturbations are established for the solutions of the linear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \ \text{ for } \ t \in \mathbb{R}_+. \tag{1.1}$$

To a considerable extent, the interest to the theory of generalized ordinary differential equations has been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified viewpoint.

Quite a few questions of the theory of generalized ordinary differential equations (both linear and nonlinear) have been studied sufficiently well (see [1]–[9] and the references therein). In particular, some sufficient (among them effective) conditions for stability in the Lyapunov sense of solutions of the system (1.1) have been investigated, e.g., in [3]–[8] (see also the references therein). Analogous questions, as well as some other ones, are investigated for example in [9], [10] for linear systems of ordinary differential equations, and in [12]–[14] for linear systems of both impulsive and difference equations.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $[a, b]$ and $]a, b[$ ($a, b \in \mathbb{R}$) is, respectively, a closed and an open interval.

$\mathbb{I}$ is an arbitrary interval from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ – matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the components x_{ij} ($i = 1, \dots, n$; $j = 1, \dots, m$) and the norm

$$\|X\| = \max\Big\{ \sum_{i=1}^{n} |x_{ij}| : \ j = 1, \dots, m \Big\}.$$

If $X = (x_{ij})_{i,j=1}^{n,m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$O_{n\times m}$ is the zero $n \times m$-matrix.

$\mathbb{R}_+ = \big\{ (x_{i,j})_{i,j=1}^{n,m} : \ x_{i,j} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m) \big\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $\mathbb{R}^n_+ = \mathbb{R}^{n\times 1}_+$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} is the matrix inverse to X; $\det X$ is the determinant of X; $r(X)$ is the spectral radius of X.

I_n is the identity $n \times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix function is said to be continuous, integrable, nondecreasing, etc., if such is every its component.

If $X:\mathbb{R}_+\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]\subset\mathbb{R}_+$ of its components x_{ij} $(i=1,\dots,n;\ j=1,\dots,m)$; $V(X)(t)=(v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(0)=0$, $v(x_{ij})(t)=\bigvee\limits_0^t(x_{ij})$ for $t>0$ $(i=1,\dots,n;\ j=1,\dots,m)$;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits at the point $t\in\mathbb{R}_+$ $(X(0-)=X(0))$; $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ such that $\bigvee\limits_a^b(X)<+\infty$.

$\mathrm{BV}_{loc}(\mathbb{I};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{I}\to\mathbb{R}^{n\times m}$ such that $\bigvee\limits_a^b(X)<+\infty$ for $a,\,b\in\mathbb{I}$.

$s_0:\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})\to\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R})$ is the operator defined by

$$\begin{aligned}&s_0(x)(0)=x(0),\\&s_0(x)(t)=x(t)-\sum_{0\le\tau<t}d_2x(\tau).\end{aligned}$$

If $g:\mathbb{R}_+\to\mathbb{R}$ is a continuous from the left nondecreasing function, $x:\mathbb{R}_+\to\mathbb{R}$ and $0\le s<t$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t]}x(\tau)\,ds_0(g)_(\tau)+\sum_{s\le\tau<t}x(\tau)\,d_2g(\tau),$$

where $\int_{]s,t]}x(\tau)\,ds_0g(\tau)$ is the Lebesgue–Stiltjes integral over the interval $]s,t]$ with respect to the measure corresponding to the function $s_0(g)$;

If $s=t$, then

$$\int_s^t x(\tau)\,dg(\tau)=0.$$

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1, g_2 are continuous from the left nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_s^t x(\tau)\,dg_1(\tau)-\int_s^t x(\tau)\,dg_2(\tau)\quad\text{for}\quad 0\le s\le t.$$

If $G=(g_{ik})_{i,k=1}^{l,n}\in\mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{l\times n})$ is a continuous from the left matrix-function and $X=(x_{kj})_{k,j=1}^{n,m}:\mathbb{R}_+\to\mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}\quad\text{for}\quad 0\le s\le t.$$

$\mathcal{A},\mathcal{B},\mathcal{L} : \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n}) \times \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times m}) \to \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times m})$ are the operators defined by the equalities (for corresponding X):

$$\mathcal{A}(X,Y)(0) = Y(0),$$

$$\mathcal{A}(X,Y)(t) = Y(t) - \sum_{0\le\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\, d_2Y(\tau) \ \text{ for } \ t>0;$$

$$\mathcal{B}(X,Y)(t) \equiv X(t)Y(t) - X(0)Y(0) - \int_0^t dX(\tau)\cdot Y(\tau)$$

and

$$\mathcal{L}(X,Y)(t) \equiv \int_0^t d\big(X(\tau) + \mathcal{B}(X,Y)\big)(\tau)\cdot X^{-1}(\tau) \ \text{ for } \ n=m.$$

A vector-function $x \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ is said to be a solution of the system (1.1) if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \quad \text{for } \ 0\le s\le t.$$

We will assume that $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, $f \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^n)$ and

$$\det\big(I_n + d_2A(t)\big) \neq 0 \ \text{ for } \ t\in\mathbb{R}_+). \tag{1.2}$$

Moreover, we assume that $A(0) = O_{n\times n}$ without loss of generality.

The condition (1.2) guarantees the unique solvability of the Cauchy problem $x(t_0) = c_0$ for the system (1.1) (see [9, Theorem III.1.4]).

We will use the following formulae (see [9, Proposition III.1.25])

$$X^{-1}(t) = X^{-1}(s) - X^{-1}(t)A(t) + X^{-1}(s)A(s) + \\ + \int_s^t dA(\tau)\cdot X^{-1}(\tau) \ \text{ for } \ 0\le s\le t, \tag{1.3}$$

where $X \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is a fundamental matrix of the homogeneous system

$$dx(t) = dA(t)\cdot x(t). \tag{1.1$_0$}$$

Definition 1.1. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty.$$

A solution x_0 of the system (1.1) is called ξ-exponentially asymptotically stable if there exists a positive number η such that for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that an arbitrary solution x of the system (1.1) satisfying the inequality

$$\big\|x(t_0) - x_0(t_0)\big\| < \delta \ \text{ for some } \ t_0\in\mathbb{R}_+$$

admits the estimate

$$\big\|x(t)-x_0(t)\big\| < \varepsilon\exp\big(-\eta\big(\xi(t)-\xi(t_0)\big)\big) \ \text{ for } \ t\geq t_0.$$

Stability, uniform stability, asymptotical stability and exponential asymptotically stability of the solution x_0 of system (1.1) are defined in the same way as for systems of ordinary differential equations (see, e.g., [9], [10]). Note that the exponential asymptotical stability is a particular case of the ξ-exponential asymptotical stability if we assume $\xi(t)\equiv t$.

Definition 1.2. The system (1.1) is called stable in this or that sense if every its solution is stable in the same sense.

It is evident that the stability of the system (1.1) is equivalent both to the stability of some solution of this system and the stability of the zero solution of the system (1.1_0).

Therefore the stability is not the property of some solution of the system (1.1). It is the common property of all solutions, and the vector-function f does not affect this property. Hence it is the property of only the matrix-function A. Thus, the following definition is natural.

Definition 1.3. The matrix-function $A\in BV_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is called stable in this or that sense if the system (1.1_0) is stable in the same sense.

Below, in Theorems 1.1–1.5, we assume that $H(0)=I_n$ without loss of generality.

Theorem 1.1. *The matrix-function $A\in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is stable if and only if there exists a nonsingular continuous from the left matrix-function $H\in BV_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that*

$$\sup\big\{\|H^{-1}(t)\| :\ t\in\mathbb{R}_+\big\} < +\infty \tag{1.4}$$

and

$$\bigvee_0^{+\infty}\big(H+\mathcal{B}(H,A)\big) < +\infty. \tag{1.5}$$

Theorem 1.2. *The matrix-function $A\in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$is uniformly stable if and only if there exists a nonsingular continuous from the left matrix-function $H\in BV_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that the conditions* (1.5) *and*

$$\sup\Big\{\big\|H^{-1}(t)H(\tau)\big\| :\ t\geq\tau\geq 0\Big\} < +\infty \tag{1.6}$$

hold.

Theorem 1.3. *The matrix-function $A\in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is asymptotically stable if and only if there exists a nonsingular continuous from the left matrix-function $H\in BV_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ such that the conditions* (1.5) *and*

$$\lim_{t\to+\infty}\|H^{-1}(t)\| = 0 \tag{1.7}$$

hold.

Theorem 1.4. *The matrix-function* $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *is* ξ*- exponentially asymptotically stable if and only if there exist a positive number* η *and a nonsingular continuous from the left matrix-function* $H \in BV_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *such that*

$$\sup\Big\{\exp\big(\eta\big(\xi(t)-\xi(\tau)\big)\big)\cdot\big\|H^{-1}(t)H(\tau)\big\| :\ t\geq\tau\geq 0\Big\} < +\infty \tag{1.8}$$

and

$$\bigvee_0^{+\infty}\mathcal{B}_\eta(H,A) < +\infty, \tag{1.9}$$

where

$$\mathcal{B}_\eta(H,A)(t) \equiv \int_0^t \exp(-\eta\xi(\tau))\, d\bigg(H(\tau)+H(\tau)A(\tau)-\int_0^\tau dH(s)\cdot A(s)\bigg).$$

Corollary 1.1. *Let a continuous from the left matrix-function* $Q \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that*

$$\det\big(I_n + d_2Q(t)\big)\neq 0 \ \ \textit{for}\ \ t\in\mathbb{R}_+$$

and

$$\bigvee_0^{+\infty}\mathcal{B}(Y^{-1}, A-Q) < +\infty, \tag{1.10}$$

where Y $(Y(0)=I_n)$ *is the fundamental matrix of the system*

$$dy(t) = dQ(t)\cdot y(t) \ \ \textit{for}\ \ t\in\mathbb{R}_+.$$

Then the stability in one or another sense of the matrix-function Q *guarantees the stability of the matrix-functions* A *in the same sense.*

Theorem 1.5. *Let a continuous from the left matrix-function* $A_0 \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be uniformly stable and*

$$\det\big(I_n + d_2A_0(t)\big)\neq 0 \ \ \textit{for}\ \ t\in\mathbb{R}_+.$$

Let, moreover, the matrix-function $A \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *be such that*

$$\bigvee_0^{+\infty}\mathcal{A}(A_0, \mathcal{L}(H,A)-A_0) < +\infty, \tag{1.11}$$

where $H \in \mathrm{BV}_{loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ *is a nonsingular continuous from the left matrix-function satisfying the condition* (1.6). *Then the matrix-function* A *is uniformly stable as well.*

2. Proof of the Main Results

To proves the theorems we will use the following two lemmas.

Lemma 2.1. *Let* $h\in\mathrm{BV}([a,b];\mathbb{R}^n)$, *and let* $H\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ *be a nonsingular matrix-function. Then the mapping*

$$x \to y = Hx + h$$

establishes a one-to-one correspondence between the solutions of the systems

$$dx(t) = dA(t) \cdot x(t) + df(t)$$

and

$$dy(t) = dA^*(t) \cdot y(t) + df^*(t),$$

respectively, where

$$A^*(t) \equiv \mathcal{L}(H, A)(t),$$

and

$$f^*(t) \equiv h(t) - h(a) + \mathcal{B}(H, f)(t) - \int\limits_a^t dA^*(\tau) \cdot h(\tau).$$

Besides,

$$I_n + (-1)^j d_j A^*(t) = \\ = \big(H(t) + (-1)^j d_j H(t)\big) \cdot \big(I_n + (-1)^j d_j A(t)\big) \cdot H^{-1}(t) \ \ \text{for} \ \ t \in [a, b]. \tag{2.1}$$

Lemma 2.1 is proved in [2].

Lemma 2.2. *Let a continuous from the left matrix-function* $A_0 \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *be such that*

$$\det\big(I_n + d_2 A_0(t)\big) \neq 0 \ \ \text{for} \ \ t \in \mathbb{R}_+.$$

Let, moreover, there exist $t_0 \in \mathbb{R}$ *such that the following conditions hold:*

a) *the Cauchy matrix* C_0 *of the system*

$$dx(t) = dA_0(t) \cdot x(t) \tag{2.2}$$

satisfies the inequality

$$\big|C_0(t, t_0)\big| \leq \exp\big(-\xi(t) + \xi(t_0)\big) \cdot \Omega \ \ \text{for} \ \ t \geq t_0,$$

where $\Omega \in \mathbb{R}_+^{n\times n}$ *and* ξ *is a function from* $\mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *satisfying the condition*

$$\lim_{t\to+\infty} \xi(t) = +\infty;$$

b) *there exists a matrix* $Q \in \mathbb{R}_+^{n\times n}$ *such that*

$$r(Q) < 1 \tag{2.3}$$

and

$$\int\limits_{t_0}^t \exp\big(\xi(t) - \xi(\tau)\big) \cdot \big|C_0(t, \tau)\big| \, dV\big(\mathcal{A}(A_0, A^* - A_0)\big)(\tau) \leq Q \ \ \text{for} \ \ t \geq t_0,$$

where $A^* \in \mathrm{BV}_{loc}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *is a matrix-function satisfying the condition*

$$\det\big(I_n + d_2 A^*(t)\big) \neq 0 \ \ \text{for} \ \ t \in \mathbb{R}_+.$$

Then an arbitrary solution y of the system

$$dy(t) = dA^*(t) \cdot y(t) \tag{2.4}$$

admits the estimate

$$|y(t)| \le \exp\big(-\xi(t) + \xi(t_0)\big) \cdot R|y(t_0)| \ \ \textit{for} \ \ t \ge t_0,$$

where $R = (I_n - Q)^{-1} \cdot \Omega$.

Lemma 2.2 is proved in [7].

Proof of Theorem 1.1. First, we show the sufficiency. According to Lemma 2.1 the mapping

$$x \to y = Hx$$

establishes a one-to-one correspondence between the solutions of systems (1.1_0) and (2.4), respectively, where

$$A^*(t) \equiv \mathcal{L}(H, A)(t).$$

On the other hand, by (1.4)

$$\det H(t+) \ne 0 \ \ \text{for} \quad t \in \mathbb{R}_+.$$

Therefore, by (2.1) we have

$$\det\big(I_n + d_2 A^*(t)\big) \ne 0 \ \ \text{for} \ \ t \in \mathbb{R}_+.$$

Let X $(X(0) = I_n)$ and Y $(Y(0) = I_n)$ be the fundamental matrices of the systems (1.1_0) and (2.4), respectively. Then

$$X(t) = H^{-1}(t)Y(t) = H^{-1}(t)\bigg(I_n + \int_0^t dA^*(\tau) \cdot Y(\tau)\bigg) =$$
$$= H^{-1}(t)\bigg(I_n + \int_0^t d\big(H(\tau) + \mathcal{B}(H, A)\big)(\tau) \cdot X(\tau)\bigg) \ \ \text{for} \ \ t \in \mathbb{R}_+.$$

Hence by virtue of (1.4) we have

$$u(t) \le r + \int_0^t u(\tau) da(\tau) \ \ \text{for } t \in \mathbb{R}_+, \tag{2.5}$$

where

$$u(t) \equiv \|X(t)\|, \ \ a(t) \equiv r\big\|V(H + \mathcal{B}(H, A))(t)\big\|$$

and

$$r = \sup\big\{\|H^{-1}(t)\| : \ t \in \mathbb{R}_+\big\}.$$

It is evident that $a(t)$ $(t \in \mathbb{R}_+)$ is a nondecreasing continuous from the left function. Therefore, from (2.5) according to Gronwall's inequality (see [9, Theorem I.4.30]) we get

$$u(t) \le r \exp\big(2 \bigvee_{t^*}^{t}(b)\big) \le r \exp\Big(\bigvee_0^{+\infty}\big(H + \mathcal{B}(H,A)\big)\Big) \ \text{ for } \ t \ge 0.$$

Hence, by (1.5)

$$\sup\big\{\|X(t)\| : \ t \in \mathbb{R}_+\big\} < +\infty.$$

Thus the stability of the matrix-function A is proved.

Let us show the necessity. Let the matrix-function A be stable. Then there exists $r > 0$ such that

$$\|X(t)\| < r \ \text{ for } \ t \in \mathbb{R}_+,$$

where X $(X(0) = I_n)$ is the fundamental matrix of the system (1.1_0).

If we assume $H(t) \equiv X^{-1}(t)$, then by (1.3) we conclude

$$H(t) + \mathcal{B}(H,A)(t) =$$
$$= X^{-1}(t) + \mathcal{B}(X^{-1},A)(t) = X^{-1}(t) + I_n - X^{-1}(t) = I_n \ \text{ for } \ t \in \mathbb{R}_+.$$

Therefore the estimates (1.4) and (1.5) hold. The theorem is proved. □

Proof of Theorem 1.2. Let us show the sufficiency. Let C and C^* be the Cauchy matrices of the systems (1.1_0) and (2.2), respectively. Then by Lemma 2.1, for every fixed $s \in \mathbb{R}_+$ we have

$$C(t,s) = H^{-1}(t)C^*(t,s)H(s) =$$
$$= H^{-1}(t)\bigg(I_n + \int_s^t d\mathcal{L}(H,A)(\tau)\cdot C^*(\tau,s)\bigg)H(s) =$$
$$= H^{-1}(t)H(s) + H^{-1}(t)\int_s^t d\big(H(\tau) + \mathcal{B}(H,A)(\tau)\big)\cdot H^{-1}(\tau)C^*(\tau,s)H(s)$$
$$\text{for } \ t \in \mathbb{R}_+.$$

Therefore,

$$C(t,s) = H^{-1}(t)H(s) + H^{-1}(t)\int_s^t d\big(H(\tau) + \mathcal{B}(H,A)(\tau)\big)\cdot C(\tau,s)$$
$$\text{for } \ t \ge s.$$

Hence by (1.6) we find

$$\big\|C(t,s)\big\| \le r + \int_s^t \big\|C(\tau,s)\big\|\, da(\tau) \ \text{ for } \ t \ge s,$$

where

$$a(t) \equiv r\big\|V(H + \mathcal{B}(H, A))(t)\big\|$$

and

$$r = \sup\big\{\|H^{-1}(t)H(s)\| : \ t \geq s \geq 0\big\}.$$

Analogously, as in the proof of Theorem 1.1 we get

$$\big\|C(t,s)\big\| \leq 2r\exp\big(2\mathop{\vee}_{s}^{t}(b)\big) \leq 2r\exp\Big(2\mathop{\vee}_{0}^{+\infty}\big(H + \mathcal{B}(H, A)\big)\Big) \ \text{ for } \ t \geq s \geq 0.$$

Thus,

$$\sup\big\{\|C(t,s)\| : \ t \geq s \geq 0\big\} < +\infty.$$

Therefore, the matrix-function A is uniformly stable.

The proof of the necessity is analogous to that for Theorem 1.1. □

Proof of Theorem 1.3. Let $\varepsilon > 0$ be an arbitrary positive number. According to (1.7) there exists $t^* \in \mathbb{R}_+$ such that

$$\|H^{-1}(t)\| < \varepsilon \ \text{ for } \ t \geq t^*.$$

From the last estimate, due to Theorem 1.1, it follows that the matrix-function A is stable. Therefore, there exists $r > 0$ such that

$$\|X(t)\| < r \ \text{ for } \ t \in \mathbb{R}_+,$$

where X $(X(0) = I_n)$ is the fundamental matrix of the system (1.1_0).

As in the proof of Theorem 1.1, we obtain

$$u(t) \leq \varepsilon r + \int_{t^*}^{t} u(\tau)\,da_\varepsilon(\tau) \ \text{ for } \ t \geq t^*,$$

where

$$u(t) \equiv \|X(t)\|$$

and

$$a_\varepsilon(t) \equiv \varepsilon\big\|V\big(H + \mathcal{B}(H, A)\big)(t)\big\|.$$

The function a_ε is continuous from the left. Therefore, as above, by Gronwall's inequality we have

$$\|X(t)\| \leq \varepsilon\exp\Big(\varepsilon\mathop{\vee}_{0}^{+\infty}\big(H + \mathcal{B}(H, A)\big)\Big) \ \text{ for } \ t \geq t^*.$$

Consequently, with regard to (1.5) we have

$$\lim_{t \to +\infty}\|X(t)\| = 0.$$

Hence, the matrix-function A is asymptotically stable.

The proof of the necessity is analogous to that for Theorem 1.1. □

Theorem 1.4 is proved in [7]. □

Proof of Corollary 1.1. The cases of stability, uniform stability and asymptotical stability of the matrix-function A follow from Theorems 1.1–1.3, respectively, if we assume that $H(t) \equiv Y^{-1}(t)$ in those theorems. Indeed, by definition of the operator $\mathcal{B}$ as well as by (1.3) and (1.10), it is easy to verify that

$$Y^{-1}(t)+\mathcal{B}(Y^{-1},A)(t)=Y^{-1}(t)+\mathcal{B}(Y^{-1},A-Q)(t)+\mathcal{B}(Y^{-1},Q)(t)= \\ =\mathcal{B}(Y^{-1},A-Q)(t)+I_n \ \text{ for } \ t\in\mathbb{R}_+$$

and

$$\bigvee_0^{+\infty}\big(H+\mathcal{B}(H,A)\big)=\bigvee_0^{+\infty}\mathcal{B}(Y^{-1},A-Q)<+\infty.$$

Let now the matrix-function Q be ξ-exponentially asymptotically stable. Then there exist the positive numbers η and ρ such that

$$\big\|Y(t)Y^{-1}(s)\big\|\le\rho\exp\big(-\eta(\xi(t)-\xi(s))\big) \ \text{ for } \ t\ge s\ge 0.$$

Therefore, the estimate (1.8) is valid, where $H(t)\equiv Y^{-1}(t)$. On the other hand, by (1.3)

$$Y^{-1}(t)=I_n+\mathcal{B}(Y^{-1},-Q)(t) \ \text{ for } \ t\in\mathbb{R}_+.$$

Then

$$\mathcal{B}_\eta(H,A)(t)=\int_0^t\exp\big(-\eta\xi(\tau)\big)\,d\mathcal{B}(Y^{-1},A-Q)(\tau) \ \text{ for } \ t\in\mathbb{R}_+,$$

where $\mathcal{B}_\eta(H,A)$ is the matrix-function appearing in Theorem 1.4. Hence by (1.10) we conclude that the condition (1.9) holds.

Hence, due to Theorem 1.4 the matrix-function A is ξ-exponentially asymptotically stable as well. The theorem is proved. □

Proof of Theorem 1.5. According to Lemma 2.1, the mapping

$$x\to y=Hx$$

establishes a one-to-one correspondence between the solutions of the systems (1.1_0) and (2.4), respectively, where

$$A^*(t)\equiv\mathcal{L}(H,A)(t).$$

On the other hand, by uniform stability of the matrix-function A_0 there exist a constant matrix $\Omega\in\mathbb{R}_+^{n\times n}$ such that the Cauchy matrix C_0 of the system (2.2) admits the estimate

$$|C_0(t,t_0)|\le\Omega \ \text{ for } \ t\ge t_0\ge 0.$$

Taking this estimate into account, we conclude

$$\int_{t_0}^t|C_0(t,\tau)|\,dV\big(\mathcal{A}(A_0,A^*-A_0)\big)(\tau)\le\Omega\int_{t_0}^t dV\big(\mathcal{A}(A_0,A^*-A_0)\big)(\tau)=$$

$$= \Omega\Big(V\big(\mathcal{A}(A_0, A^* - A_0)\big)(t) - V\big(\mathcal{A}(A_0, A^* - A_0)\big)(t_0)\Big) \ \text{ for } \ t \geq t_0 \geq 0. \quad (2.6)$$

Moreover, by the inequality (1.11) the constant matrix

$$Q = \Omega \overset{+\infty}{\underset{t^*}{\vee}} \big(\mathcal{A}(A_0, A^* - A_0\big) \quad (2.7)$$

admits the estimate (2.3) for some sufficiently large $t^* \in \mathbb{R}_+$.

According to (2.6) and (2.7),

$$\int_{t_0}^{t} |C_0(t,\tau)| \, dV\big(\mathcal{A}(A_0, A^* - A_0)\big)(\tau) \leq Q \ \text{ for } \ t \geq t_0 \geq t^*.$$

Therefore, by Lemma 2.2 every solution y of the system (2.4) admits the estimate

$$\|y(t)\| \leq \rho \|y(t_0)\| \ \text{ for } \ t \geq t_0 \geq t^*,$$

where $\rho > 0$ is a number independent of t_0. The last estimate guarantees the uniform stability of the matrix-function A^*. Hence, there exist a positive number ρ_1 such that

$$\|C^*(t,t_0)\| \leq \rho_1 \ \text{ for } \ t \geq t_0 \geq t^*, \quad (2.8)$$

where C^* is the Cauchy matrix of the system (2.4).

Let now C be the Cauchy matrix of the system (1.1_0). Then, according to Lemma 2.1

$$C(t,t_0) = H^{-1}(t) C^*(t,t_0) H(t_0) \ \text{ for } \ t \geq t_0 \geq 0.$$

From this, by (1.6) and (2.8) we get

$$\|C(t,t_0)\| \leq \rho_1 \rho_2 \ \text{ for } \ t \geq t_0 \geq t^*,$$

where

$$\rho_2 = \sup\Big\{\big\|H^{-1}(t)H(\tau)\big\| : \ t \geq \tau \geq 0\Big\}.$$

Consequently, the matrix-function A is uniformly stable as well. The theorem is proved. □

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