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ON THE EXISTENCE OF BOUNDED SOLUTIONS FOR SYSTEMS OF NONLINEAR IMPULSIVE EQUATIONS

Abstract. Necessary conditions are given for the existence of bounded solutions for systems of nonlinear impulsive equations.

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In this paper we investigate the question of the existence of solutions for the system of impulsive equations

$$\frac{dx}{dt} = f(t,x) \ \text{ for almost all } \ t \in \mathbb{R}_+ \setminus \{\tau_1, \tau_2, \dots\}, \tag{1}$$

$$x(\tau_k+) - x(\tau_k-) = I_k(x(\tau_k-)) \ \ (k = 1, 2, \dots) \tag{2}$$

satisfying the condition

$$\sup\Big\{ \sum_{i=1}^{n} |x_i(t)| : \ t \in \mathbb{R}_+ \Big\} < \infty, \tag{3}$$

where $x = (x_i)_{i=1}^n$, $0 < \tau_1 < \tau_2 < \cdots$, $\tau_k \to \infty$ $(k \to \infty)$ (we will assume $\tau_0 = 0$ if necessary), $f = (f_i)_{i=1}^n \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, and $I_k = (I_{ki})_{i=1}^n : \mathbb{R}^n \to \mathbb{R}^n$ $(k = 1, 2, \dots)$ are continuous operators.

Sufficient conditions are given for the existence of solutions of the boundary value problem (1), (2); (3). Analogous results are contained in [5]–[10] for systems of ordinary differential and functional differential equations.

Throughout the paper the following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ $(a, b \in \mathbb{R})$ is a closed segment.

$\mathbb{R}^{n\times m}$ is the set of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the set of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

$\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$.

$\bigvee\limits_a^b(X)$ is the total variation of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$, i.e., the sum of total variations of the latter's components.

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$X(t-)$ and $X(t+)$ are the left and the right limit of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \le a$ and $X(t) = X(b)$ for $t \ge b$, if necessary);

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t).$$

A matrix-function is said to be measurable, integrable, nondecreasing etc. if each of its components is such.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X) < +\infty$).

$\mathrm{BV}_{loc}(\mathbb{R}_+,\mathbb{R}^{n\times m})$ is the set of all matrix-functions of $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{R}_+$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$.

$\widetilde{C}([a,b],D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$.

$\widetilde{C}_{loc}(\mathbb{R}_+ \setminus \{\tau_1,\tau_2,\dots\},D)$ is the set of all matrix-functions $X : \mathbb{R}_+ \to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{R}_+ \setminus \{\tau_1,\tau_2,\dots\}$ belong to $\widetilde{C}([a,b],D)$.

$L([a,b],D)$ is the set of all matrix-functions $X : [a,b] \to D$, measurable and integrable.

$L_{loc}(\mathbb{R}_+ \setminus \{\tau_1,\tau_2,\dots\},D)$ is the set of all matrix-functions $X : \mathbb{R}_+ \to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{R}_+ \setminus \{\tau_1,\tau_2,\dots\}$ belong to $\widetilde{C}([a,b],D)$.

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^{n\times m}$, then $K([a,b]\times D_1,D_2)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$: a) the function $f_{kj}(\cdot,x) : [a,b] \to D_2$ is measurable for every $x \in D_1$; b) the function $f_{kj}(t,\cdot) : D_1 \to D_2$ is continuous for almost all $t \in [a,b]$, and $\sup\big\{|f_{kj}(\cdot,x)| : \ x \in D_0\big\} \in L([a,b],\mathbb{R};g_{ik})$ for every compact $D_0 \subset D_1$.

$K_{loc}(\mathbb{R}_+\times D_1,D_2)$ is the set of all mappings $F : \mathbb{R}_+\times D_1 \to D_2$ whose restrictions to an arbitrary closed interval $[a,b]$ from $\mathbb{R}_+ \setminus \{\tau_1,\tau_2,\dots\}$ belong to $K([a,b]\times D_1,D_2)$.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x \in \widetilde{C}_{loc}(\mathbb{R}_+ \setminus \{\tau_1,\tau_2,\dots\}) \cap \mathrm{BV}_{loc}\, s(\mathbb{R}_+,\mathbb{R}^n)$ satisfying both the system (1) for almost all $t \in \mathbb{R}_+ \setminus \{\tau_k\}_{k=1}^{m_0}$ and the relation (2) for every $k \in \{1,2,\dots\}$.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for survey of the results on impulsive systems see, e.g.,[1]–[3], [11]–[15], and references therein). But the above-mentioned works do not contain the results analogous to those obtained in [6] for ordinary differential equations.

Using the theory of so called generalized ordinary differential equations (see, e.g., [1], [4] and references therein), we extend these results to systems of impulsive equations.

To establish the results dealing with the boundary value problems for the impulsive system (1), (2), we use the following concept.

It is easy to show that the vector-function x is a solution of the impulsive system (1), (2) if and only if it is a solution of the following system of generalized ordinary differential equations (see, e.g., [1], [4] and references therein)

$$dx(t) = dA(t) \cdot f(t, x(t)),$$

where

$$A(t) \equiv \operatorname{diag}(a_{11}(t), \dots, a_{nn}(t)),$$

$$a_{ii}(t) = \begin{cases} t & \text{for } \ 0 \le t \le \tau_1, \\ t+k & \text{for } \ \tau_k < t \le \tau_{k+1} \ \ (k = 1, 2, \dots); \end{cases} \qquad (i = 1, \dots, n)$$

$$f(\tau_k, x) \equiv I_k(x) \quad (k = 1, 2, \dots).$$

It is evident that the matrix-function A is continuous from the left, $d_2 A(t) = 0$ if $t \in \mathbb{R}_+ \setminus \{\tau_1, \tau_2, \dots\}$ and $d_2 A(\tau_k) = 1$ $(k = 1, 2, \dots)$.

We will assume that $f = (f_i)_{i=1}^n \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$.

The following theorem follows from the corresponding theorem for the system of generalized ordinary differential equations (see [4]).

Theorem 1. *Let there exist numbers* $\sigma_i \in \{-1, 1\}$ $(i=1, \dots, n)$*, continuous from the left vector-functions* $\alpha_m = (\alpha_{mi})_{i=1}^n \in \widetilde{C}_{loc}(\mathbb{R}_+ \setminus \{\tau_1, \tau_2, \dots\}, \mathbb{R}^n) \cap \mathrm{BV}_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ $(m = 1, 2)$ *such that the conditions*

$$\alpha_1(t) \le \alpha_2(t) \quad \textit{for } \ t \in \mathbb{R}_+,$$

$$(-1)^j \sigma_i \Big(f_i\big(t, x_1, \dots, x_{i-1}, \alpha_{ji}(t), x_{i+1}, \dots, x_n\big) - \alpha'_{ji}(t) \Big) \le 0$$

$$\textit{for almost all } \ t \in \mathbb{R}_+ \setminus \{\tau_1, \tau_2, \dots\},$$

$$\alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t) \ \ (j = 1, 2; \ \ i = 1, \dots, n),$$

$$(-1)^m \Big(x_i - I_{ki}(x_1, \dots, x_n) - \alpha_{mi}(\tau_k +) \Big) \le 0$$

$$\textit{for } \ \alpha_1(\tau_k) \le (x_l)_{l=1}^n \le \alpha_2(\tau_k) \ \ (m = 1, 2; \ \ i = 1, \dots, n; \ \ k = 1, 2, \dots)$$

and

$$\sup \big\{ |\alpha_{mi}(t)| : \ t \in \mathbb{R}_+ \big\} < \infty \ \ (m = 1, 2; \ \ i = 1, \dots, n) \tag{4}$$

hold. Then the problem (1), (2); (3) *is solvable.*

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