

Memoirs on Differential Equations and Mathematical Physics

VOLUME 44, 2008, 105–132

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**ON FAEDO–GALERKIN METHOD
FOR EVOLUTION INCLUSIONS
WITH W_{λ_0} -PSEUDOMONOTONE MAPS**

1. INTRODUCTION

One of the most effective approaches to investigation of nonlinear problems represented by partial differential equations, inclusions and inequalities with boundary values, consists in the reduction of them to equations in Banach spaces governed by nonlinear operators. In order to study these objects, modern methods of nonlinear analysis have been used [9], [11], [21]. In [26], by using a special basis, the Cauchy problem for a class of equations with operators of Volterra type has been studied. The important periodic problem for equations with monotone differential operators of Volterra type has been studied in [11]. Periodic solutions for pseudomonotone operators have been considered in [21], while for w_{λ_0} -pseudomonotone single-valued operators in [16].

Convergence of approximate solutions to an exact solution of a differential-operator equation or inclusion is frequently proved on the basis of monotonicity or pseudomonotonicity of the corresponding operator. If the given property of the initial operator takes place, then it is possible to prove convergence of the approximate solutions within weaker a priori estimates than it is demanded when using embedding theorems. The monotonicity concept has been introduced in papers of Weinberg, Kachurovsky, Minty, Sarantonello and others. Significant generalization to monotonicity was given by H. Brezis [3]. Namely, Brezis calls an operator $A : X \rightarrow X^*$ *pseudomonotone* if

- a) the operator A is bounded;
- b) from $u_n \rightarrow u$ weakly in X and from

$$\overline{\lim}_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle_X \leq 0$$

it follows that

$$\underline{\lim}_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle_X \geq \langle A(u), u - v \rangle_X \quad \forall v \in X.$$

In applications, as a pseudomonotone operator the sum of radially continuous monotone bounded operator and strongly continuous operator was considered [11]. Concrete examples of pseudomonotone operators were obtained by extension of elliptic differential operators when only their summands with highest derivatives satisfied the monotonicity property [21]. In papers of J.-L. Lions, H. Gaevsky, K. Greger, K. Zaharias [11], [21] the main results of solvability theory for abstract operator equations and differential operator equations that are monotone or pseudomonotone in the Brezis sense are set out. Also the application of the proved theorems are given to concrete equations of mathematical physics, and in particular, to free boundary problems.

The theory of monotone operators in reflexive Banach spaces is one of the major areas of nonlinear functional analysis. Its basis is formed by so-called variational methods, and since the 60-ies of the last century the theory has been intensively developing in tight interaction with the theory

of convex functions and the theory of partial differential equations. The papers of F. Browder and P. Hess [4], [5] became classical in the given direction of investigations. In particular, in the work [5] of F. Browder and P. Hess a class of generalized pseudomonotone operators enveloping the class of monotone mappings was introduced. Let W be some normed space continuously embedded in the normed space Y . A multi-valued map $A : Y \rightarrow 2^{Y^*}$ is said to be *generalized pseudomonotone on W* if for each pair of sequences $\{y_n\}_{n \geq 1} \subset W$ and $\{d_n\}_{n \geq 1} \subset Y^*$ such that $d_n \in A(y_n)$, $y_n \rightarrow y$ weakly in W , $d_n \rightarrow d$ *-weakly in Y^* , from the inequality

$$\overline{\lim}_{n \rightarrow \infty} \langle d_n, y_n \rangle_Y \leq \langle d, y \rangle_Y$$

it follows that $d \in A(y)$ and $\langle d_n, y_n \rangle_Y \rightarrow \langle d, y \rangle_Y$.

A grave disadvantage of the given theory is the fact that in the general case it is impossible to prove that the set of pseudomonotone (in the classical sense) maps is closed with respect to summation (the given statement is problematic). This disadvantage becomes more substantial when investigating differential-operator inclusions and evolutionary variational inequalities when we necessarily consider the sum of the classical pseudomonotone mapping and the subdifferential (in Gateaux or Clarke sense) for multi-valued map which is generalized pseudomonotone. I. V. Skripnik's idea of passing to subsequences in classical definitions [33], realized for stationary inclusions in the papers of M. Z. Zgurovsky and V. S. Mel'nik [22], [25] enabled one to consider the essentially wider class of λ_0 -pseudomonotone maps, closed with respect to summation of maps, which for classical definitions appeared problematic. In the papers of V. S. Mel'nik and P. O. Kasyanov [16] there was introduced the class of w_{λ_0} -pseudomonotone maps which includes, in particular, the class of generalized pseudomonotone multi-valued operators and also it is *closed with respect to summation of maps*. A multi-valued map $A : Y \rightarrow 2^{Y^*}$ with the nonempty, convex, bounded, closed values is called w_{λ_0} -pseudomonotone (λ_0 -pseudomonotone on W) if for any sequence $\{y_n\}_{n \geq 0} \subset W$ such that $y_n \rightarrow y_0$ weakly in W , $d_n \rightarrow d_0$ *-weakly in Y^* as $n \rightarrow +\infty$, where $d_n \in A(y_n) \forall n \geq 1$, from the inequalities

$$\overline{\lim}_{n \rightarrow \infty} \langle d_n, y_n - y_0 \rangle_Y \leq 0$$

it follows the existence of such subsequences $\{y_{n_k}\}_{k \geq 1}$ of $\{y_n\}_{n \geq 1}$ and $\{d_{n_k}\}_{k \geq 1}$ of $\{d_n\}_{n \geq 1}$ for which

$$\underline{\lim}_{k \rightarrow \infty} \langle d_{n_k}, y_{n_k} - w \rangle_Y \geq [A(y_0), y_0 - w]_- \quad \forall w \in Y.$$

Now we have to prove solvability for differential-operator inclusions with λ_0 -pseudomonotone on $D(L)$ multi-valued maps in Banach spaces:

$$Lu + A(u) + B(u) \ni f, \quad u \in D(L), \quad (1)$$

where $A : X_1 \rightarrow 2^{X_1^*}$, $B : X_2 \rightarrow 2^{X_2^*}$ are multi-valued maps of $D(L)_{\lambda_0}$ -pseudomonotone type with nonempty, convex, closed, bounded values, X_1 ,

X_2 are Banach spaces continuously embedded in some Hausdorff linear topological space, $X = X_1 \cap X_2$, $L : D(L) \subset X \rightarrow X^*$ is linear, monotone, closed, densely defined operator with the linear domain $D(L)$.

Let us remark that any multi-valued map $A : Y \rightarrow 2^{Y^*}$ defined on a Banach space Y naturally generates respectively *upper* and *lower forms*:

$$[A(y), \omega]_+ = \sup_{d \in A(y)} \langle d, \omega \rangle_Y, \quad [A(y), \omega]_- = \inf_{d \in A(y)} \langle d, \omega \rangle_Y, \quad y, \omega \in X.$$

Properties of these objects have been investigated by M. Z. Zgurovsky and V. S. Mel'nik. Thus, together with the classical coercivity condition for the operator A ,

$$\frac{\langle A(y), y \rangle_Y}{\|y\|_Y} \rightarrow +\infty \text{ as } \|y\|_Y \rightarrow +\infty,$$

which ensures important a priori estimates, arises $+$ -coercivity (and, respectively, $-$ -coercivity):

$$\frac{[A(y), y]_{+(-)}}{\|y\|_Y} \rightarrow +\infty \text{ as } \|y\|_Y \rightarrow +\infty.$$

$+$ -coercivity is a much weaker condition than $-$ -coercivity.

When investigating multi-valued maps of w_{λ_0} -pseudomonotone type, it was found out that even for subdifferentials of convex lower semicontinuous functionals the boundedness condition is not natural [15]. Thus it was necessary to introduce an adequate relaxation of the boundedness condition which would envelope at least the class of monotone multi-valued maps. In the paper [13], the following definition was introduced: a multi-valued map $A : Y \rightarrow 2^{Y^*}$ satisfies *the property* (II) if for any bounded set $B \subset Y$, any $y_0 \in Y$ and for some $k > 0$, $d \in A$ for which

$$\langle d(y), y - y_0 \rangle_Y \leq k \text{ for all } y \in B,$$

there exists $C > 0$ such that

$$\|d(y)\|_{Y^*} \leq C \text{ for all } y \in B.$$

Recent development of the monotonicity method in the theory of differential-operator inclusions and evolutionary variational inequalities ensures resolvability of the given objects under the conditions of $-$ -coercivity, boundedness and the generalized pseudomonotonicity (it is necessary to notice that the proof is not constructive). With relation to applications, it would be topical to relax some conditions on multi-valued maps in the problem (1) replacing $-$ -coercivity by $+$ -coercivity, boundedness by the condition (II) and pseudomonotonicity in classical sense or generalized pseudomonotonicity by w_{λ_0} -pseudomonotonicity.

At present the operator and differential-operator equations and inclusions as well as evolutionary variational inequalities are studied intensively enough by many authors: J.-P. Aubin, V. Barbu, Yu. G. Borisovich, S. Carl, H. Frankowska, B. D. Gelman, M. F. Gorodnij, S. Hu, M. I. Kamenskii,

P. I. Kogut, O. A. Kovalevsky, A. D. Mishkis, D. Motreanu, V. V. Obuchovskii, N. S. Papageorgiou, A. M. Samoilenko, N. A. Perestyuk, V. A. Plotnikov, N. V. Skripnik, V. Yu. Slusarchuk, O. M. Solonucha, A. N. Vakulenko, M. Z. Zgurovsky and others [1]–[34].

By analogy with the differential-operator equations, at least four approaches are well-known: Faedo–Galerkin method, elliptic regularization, the theory of semigroups, difference approximations. Extension of these approaches on evolutionary inclusions encounters a series of basic difficulties. For differential-operator inclusions, the method of semigroups is realized in the works of A. A. Tolstonogov, Yu. I. Umansky [34] and V. Barbu [2]. The method of finite differences first time was extended on evolutionary inclusions and variational inequalities by P. O. Kasyanov, V. S. Melnik and L. Toskano. The method of singular perturbations (H. Brezis [3] and Yu. A. Dubinsky [8]) and the Faedo–Galerkin method with differential-operator inclusions for w_{λ_0} -pseudomonotone multi-valued maps have not been systematically investigated as yet. It is required to prove the singular perturbations method and the Faedo–Galerkin method for differential-operator inclusions with w_{λ_0} -pseudomonotone multi-valued maps in Banach spaces.

In the present paper, we introduce a new construction to prove the existence of periodic solutions for differential-operator inequalities by the Faedo–Galerkin (FG) method for w_{λ_0} -pseudomonotone multi-valued operators. From the point of view of the applications, we have essentially widened the class of the operators considered by other authors (see [13], [26]).

2. PROBLEM DEFINITION

Let $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ be some reflexive separable Banach spaces continuously embedded in the Hilbert space $(H, (\cdot, \cdot))$ and such that

$$V := V_1 \cap V_2 \text{ is dense in spaces } V_1, V_2 \text{ and } H. \quad (2)$$

After the identification $H \equiv H^*$, we get

$$V_1 \subset H \subset V_1^*, \quad V_2 \subset H \subset V_2^* \quad (3)$$

with continuous and dense embeddings [11], where $(V_i^*, \|\cdot\|_{V_i^*})$ is the space topologically conjugate to V_i with respect to the canonical bilinear form

$$\langle \cdot, \cdot \rangle_{V_i} : V_i^* \times V_i \rightarrow \mathbb{R}$$

($i = 1, 2$) which coincides on H with the inner product (\cdot, \cdot) of H . Let us consider the functional spaces

$$X_i = L_{r_i}(S; H) \cap L_{p_i}(S; V_i),$$

where $S = [0, T]$, $0 < T < +\infty$, $1 < p_i \leq r_i < +\infty$ ($i = 1, 2$). The spaces X_i are Banach spaces with the norms $\|y\|_{X_i} = \|y\|_{L_{p_i}(S; V_i)} + \|y\|_{L_{r_i}(S; H)}$. Moreover, X_i is a reflexive space.

Let us also consider the Banach space $X = X_1 \cap X_2$ with the norm $\|y\|_X = \|y\|_{X_1} + \|y\|_{X_2}$. Since the spaces $L_{q_i}(S; V_i^*) + L_{r'_i}(S; H)$ and X_i^*

are isometrically isomorphic, we identify them. Analogously,

$$X^* = X_1^* + X_2^* = L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) + L_{r'_1}(S; H) + L_{r'_2}(S; H),$$

where $r_i^{-1} + r'_i{}^{-1} = p_i^{-1} + q_i^{-1} = 1$.

Let us define the duality form on $X^* \times X$:

$$\begin{aligned} \langle f, y \rangle &= \int_S (f_{11}(\tau), y(\tau))_H d\tau + \int_S (f_{12}(\tau), y(\tau))_H d\tau + \\ &+ \int_S \langle f_{21}(\tau), y(\tau) \rangle_{V_1} d\tau + \int_S \langle f_{22}(\tau), y(\tau) \rangle_{V_2} d\tau = \\ &= \int_S (f(\tau), y(\tau)) d\tau, \end{aligned}$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r'_i}(S; H)$, $f_{2i} \in L_{q_i}(S; V_i^*)$. Note that for each $f \in X^*$

$$\begin{aligned} \|f\|_{X^*} = & \inf_{\substack{f=f_{11}+f_{12}+f_{21}+f_{22}: \\ f_{1i} \in L_{r'_i}(S; H), f_{2i} \in L_{q_i}(S; V_i^*) \ (i=1,2)}} \max \left\{ \|f_{11}\|_{L_{r'_1}(S; H)}; \right. \\ & \left. \|f_{12}\|_{L_{r'_2}(S; H)}; \|f_{21}\|_{L_{q_1}(S; V_1^*)}; \|f_{22}\|_{L_{q_2}(S; V_2^*)} \right\}. \end{aligned}$$

Let $A : X_1 \rightrightarrows X_1^*$ and $B : X_2 \rightrightarrows X_2^*$ be multi-valued maps with nonempty closed convex values, $L : D(L) \subset X \rightarrow X^*$ be a linear closed densely defined operator. We consider the problem

$$\begin{cases} Ly + A(y) + B(y) \ni f, \\ y \in D(L), \end{cases} \quad (4)$$

where $f \in X^*$ is arbitrarily fixed.

3. CLASSES OF MAPS

Let Y be some reflexive Banach space, Y^* be its topologically conjugate,

$$\langle \cdot, \cdot \rangle_Y : Y^* \times Y \rightarrow \mathbb{R}$$

be the duality form on Y . For each nonempty subset $B \subset Y^*$, let us consider its weak closed convex hull $\overline{\text{co}}(B) := \text{cl}_{X_w^*}(\text{co}(B))$. Further, by $C_w(Y^*)$ we will denote the class of all nonempty convex weakly compact subsets of Y^* .

For each multi-valued map $A : Y \rightrightarrows Y^*$, we can consider its *upper* and *lower function of support*:

$$[A(y), \omega]_+ = \sup_{d \in A(y)} \langle d, \omega \rangle_X, \quad [A(y), \omega]_- = \inf_{d \in A(y)} \langle d, \omega \rangle_X,$$

where $y, \omega \in X$. We also consider its *upper* and *lower norms*:

$$\|A(y)\|_+ = \sup_{d \in A(y)} \|d\|_{X^*}, \quad \|A(y)\|_- = \inf_{d \in A(y)} \|d\|_{X^*}.$$

The main properties of the given maps are considered in [18]. In particular, from [29] the following properties of the introduced symbols with brackets and norms take place.

Let $\mathcal{A}, \mathcal{B} : Y \rightarrow C_v(Y^*)$; then for arbitrary $y, v, v_1, v_2 \in Y$

- 1) the functional $Y \ni v \rightarrow [\mathcal{A}(y), v]_+$ is convex positively homogeneous and lower semicontinuous;
- 2) $[\mathcal{A}(y), v_1 + v_2]_+ \geq [\mathcal{A}(y), v_1]_+ + [\mathcal{A}(y), v_2]_+$,
 $[\mathcal{A}(y), v_1 + v_2]_- \leq [\mathcal{A}(y), v_1]_- + [\mathcal{A}(y), v_2]_-$;
- 3) $[\mathcal{A}(y) + \mathcal{B}(y), v]_+ = [\mathcal{A}(y), v]_+ + [\mathcal{B}(y), v]_+$,
 $[\mathcal{A}(y) + \mathcal{B}(y), v]_- = [\mathcal{A}(y), v]_- + [\mathcal{B}(y), v]_-$;
- 4) $[\mathcal{A}(y), v]_+ \leq \|\mathcal{A}(y)\|_+ \|v\|_Y$, $[\mathcal{A}(y), v]_- \leq \|\mathcal{A}(y)\|_- \|v\|_Y$;
- 5) the functional $\|\cdot\|_+ : C_v(Y^*) \rightarrow \mathbb{R}_+$ defines a norm on $C_v(Y^*)$;
- 6) the functional $\|\cdot\|_- : C_v(Y^*) \rightarrow \mathbb{R}_+$ satisfies the conditions:
 - a) $\bar{0} \in \mathcal{A}(y) \iff \|\mathcal{A}(y)\|_- = 0$,
 - b) $\|\alpha \mathcal{A}(y)\|_- = |\alpha| \|\mathcal{A}(y)\|_- \quad \forall \alpha \in \mathbb{R}, y \in Y$,
 - c) $\|\mathcal{A}(y) + \mathcal{B}(y)\|_- \leq \|\mathcal{A}(y)\|_- + \|\mathcal{B}(y)\|_-$;
- 7) $\|\mathcal{A}(y) - \mathcal{B}(y)\|_+ \geq \left| \|\mathcal{A}(y)\|_+ - \|\mathcal{B}(y)\|_+ \right|$,
 $\|\mathcal{A}(y) - \mathcal{B}(y)\|_- \geq \left| \|\mathcal{A}(y)\|_- - \|\mathcal{B}(y)\|_- \right|$,
 $d_H(\mathcal{A}(y), \mathcal{B}(y)) \geq \left| \|\mathcal{A}(y)\|_{+(-)} - \|\mathcal{B}(y)\|_{+(-)} \right|$,
 where $d_H(\cdot, \cdot)$ is the Hausdorff metric;
- 8) $d \in \mathcal{A}(y) \iff \forall \omega \in Y [\mathcal{A}(y), \omega]_+ \geq \langle d, \omega \rangle_Y$.

Now we consider the main classes of maps of w_{λ_0} -pseudomonotone type. In what follows, $y_n \rightharpoonup y$ in Y will mean that y_n weakly converges to y in the reflexive Banach space Y .

Definition 1. The multi-valued map $A : Y \rightarrow C_v(Y^*)$ is called:

- *+(-)-coercive* if $\|y\|_Y^{-1} [A(y), y]_{+(-)} \rightarrow +\infty$ as $\|y\|_Y \rightarrow +\infty$;
- *weakly +(-)-coercive* if for each $f \in Y^*$ there exists $R > 0$ such that

$$[A(y) - f, y]_{+(-)} \geq 0 \quad \text{as } \|y\|_Y = R, \quad y \in Y;$$

- *bounded* if for any $L > 0$ there exists $l > 0$ such that

$$\|A(y)\|_+ \leq l \quad \forall y \in Y : \|y\|_Y \leq L;$$

- *locally bounded* if for any fixed $y \in Y$ there exist constants $m > 0$ and $M > 0$ such that $\|A(\xi)\|_+ \leq M$ when $\|y - \xi\|_Y \leq m$, $\xi \in Y$;
- *finite-dimensionally locally bounded* if for each finite-dimensional subspace $F \subset Y$ $A|_F$ is locally bounded on $(F, \|\cdot\|_Y)$.

Let W be a normed space with the norm $\|\cdot\|_W$. We consider $W \subset Y$ with continuous embedding, $C(r_1; \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous function for each $r_1 \geq 0$ and such that $\tau^{-1}C(r_1; \tau r_2) \rightarrow 0$ as $\tau \rightarrow +0 \quad \forall r_1, r_2 \geq 0$, and

$\|\cdot\|'_W$ is a (semi)-norm on Y that is compact with respect to $\|\cdot\|_W$ on W and continuous with respect to $\|\cdot\|_Y$ on Y .

Definition 2. The multi-valued map $A : Y \rightarrow C_v(Y^*)$ is called:

- *radially semi-continuous* if $\forall x, h \in Y$ the following inequality takes place

$$\overline{\lim}_{t \rightarrow +0} [A(x + th), h]_- \leq [A(x), h]_+;$$

- *an operator with semi-bounded variation on W (with (Y, W) -s.b.v.)* if $\forall R > 0$ and $\forall y_1, y_2 \in Y$ with $\|y_1\|_Y \leq R, \|y_2\|_Y \leq R$, the inequality

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C(R; \|y_1 - y_2\|'_W)$$

is fulfilled;

- *λ -pseudomonotone on W (w_λ -pseudomonotone)* if for every sequence $\{y_n\}_{n \geq 1} \subset W$ such that $y_n \rightharpoonup y_0$ in W with $y_0 \in W$, from the inequality

$$\overline{\lim}_{n \rightarrow \infty} \langle d_n, y_n - y_0 \rangle_Y \leq 0, \tag{5}$$

where $d_n \in A(y_n), n \geq 1$, it follows the existence of subsequences

$$\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1} \quad \text{and} \quad \{d_{n_k}\}_{k \geq 1} \subset \{d_n\}_{n \geq 1},$$

such that

$$\underline{\lim}_{k \rightarrow \infty} \langle d_{n_k}, y_{n_k} - w \rangle_Y \geq [A(y), y_0 - w]_- \quad \forall w \in Y; \tag{6}$$

- *λ_0 -pseudomonotone on W (w_{λ_0} -pseudomonotone)* if for each sequence $\{y_n\}_{n \geq 1} \subset W$ such that

$$y_n \rightharpoonup y_0 \text{ in } W, \quad A(y_n) \ni d_n \rightharpoonup d_0 \text{ in } Y^* \text{ with } y_0 \in Y, \quad d_0 \in Y^*,$$

from the inequality (5) it follows the existence of

$$\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1} \quad \text{and} \quad \{d_{n_k}\}_{k \geq 1} \subset \{d_n\}_{n \geq 1}$$

such that (6) is true.

The above multi-valued map satisfies:

- *the property (κ)* if for each bounded set D in Y there exists $c \in \mathbb{R}$ such that

$$[A(v), v]_+ \geq -c\|v\|_Y \quad \forall v \in D.$$

- *the property (II)* if for each nonempty bounded subset $B \subset Y$, for each $k > 0$ and for each selector $d \in A$ such that

$$\langle d(y), y \rangle_Y \leq k \quad \text{for each } y \in B,$$

it follows that there exists $K > 0$ such that

$$\|d(y)\|_{Y^*} \leq K \quad \text{for each } y \in B.$$

Remark 1. The idea of the passage to subsequences in the latter definition was adopted by us from the work of I. V. Skripnik [33].

Let $Y = Y_1 \cap Y_2$, where $(Y_1, \|\cdot\|_{Y_1})$ and $(Y_2, \|\cdot\|_{Y_2})$ are some reflexive Banach spaces.

Definition 3. The pair of maps $A : Y_1 \rightarrow C_v(Y_1^*)$ and $B : Y_2 \rightarrow C_v(Y_2^*)$ is called *s-mutually bounded*, if for each $M > 0$, for each bounded set $D \subset Y$ and for each selectors $d_A \in A$ and $d_B \in B$ there exists $K > 0$ such that from

$$y \in D \text{ and } \langle d_A(y), y \rangle_{Y_1} + \langle d_B(y), y \rangle_{Y_2} \leq M$$

we have

$$\text{or } \|d_A(y)\|_{Y_1^*} \leq K \text{ or } \|d_B(y)\|_{Y_2^*} \leq K.$$

Remark 2. A bounded map $A : Y \rightarrow Y^*$ satisfies the property (II); a map $A : Y \rightarrow Y^*$ that satisfies the property (II) satisfies the property (κ); a λ -pseudomonotone on W map is λ_0 -pseudomonotone on W . The converse statement is correct for bounded multi-valued maps.

If one of the operators of the pair $(A; B)$ is bounded, then the pair $(A; B)$ is *s-mutually bounded*. Moreover, if each of maps satisfies the condition (II), then their sum also satisfies the condition (II) and the pair $(A; B)$ is *s-mutually bounded*.

Now let $W = W_1 \cap W_2$, where $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ are Banach spaces such that $W_i \subset Y_i$ with continuous embedding.

Lemma 1 ([18]). *Let $A : Y_1 \rightarrow C_v(Y_1^*)$ and $B : Y_2 \rightarrow C_v(Y_2^*)$ be s-mutually bounded λ_0 -pseudomonotone on W_1 and respectively on W_2 multivalued maps. Then $C := A + B : Y \rightarrow C_v(Y^*)$ is a λ_0 -pseudomonotone on W map.*

Remark 3. If the pair $(A; B)$ is not *s-mutually bounded*, then the given proposition takes place only for λ -pseudomonotone (respectively on W_1 and on W_2) maps.

Lemma 2 ([18]). *Let $A : Y_1 \rightrightarrows Y_1^*$, $B : Y_2 \rightrightarrows Y_2^*$ be +-coercive maps which satisfy the condition (κ). Then the map $C := A + B : Y \rightrightarrows Y^*$ is +-coercive.*

Remark 4. Under the conditions of the last lemma it follows that the operator $C = A + B : Y \rightarrow C_v(Y^*)$ is weakly +-coercive.

Proposition 1 ([18]). *Let $A : X \rightrightarrows X^*$ be a λ_0 -pseudomonotone operator on W , the embedding of W in the Banach space Y be compact and dense, the embedding of X in Y be continuous and dense, and let $\overline{c\sigma}^* B : Y \rightrightarrows Y^*$ be a locally bounded map such that the graph of $\overline{c\sigma}^* B$ is closed in $Y \times Y_w^*$ (i.e. with respect to the strong topology of Y and the weakly star topology in Y^*). Then $C = A + B$ is a λ_0 -pseudomonotone on W map.*

Now we consider a functional $\varphi : X \mapsto \mathbb{R}$.

Definition 4. The functional φ is said to be *locally Lipschitz*, if for any $x_0 \in X$ there are $r, c > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq c\|x - y\|_X \quad \forall x, y \in B_r(x_0) = \{x \in X \mid \|x - x_0\|_X < r\}.$$

For a locally Lipschitz functional φ defined on a Banach space X , we consider the upper Clarke’s derivative [7]

$$\varphi_{Cl}^\uparrow(x, h) = \overline{\lim}_{v \rightarrow x, \alpha \searrow 0^+} \frac{1}{\alpha} (\varphi(v + \alpha h) - \varphi(v)) \in \mathbb{R}, \quad x, h \in X$$

and Clarke’s generalized gradient

$$\partial_{Cl}\varphi(x) = \left\{ p \in X^* \mid \langle p, v - x \rangle_X \leq \varphi_{Cl}^\uparrow(x, v - x) \forall v \in X \right\}, \quad x \in X.$$

Proposition 2 ([18]). *Let W be a Banach space compactly embedded in some Banach space Y , $\varphi : Y \mapsto \mathbb{R}$ be a locally Lipschitz functional. Then Clarke’s generalized gradient $\partial_{Cl}\varphi : Y \rightrightarrows Y^*$ is λ_0 -pseudomonotone on W .*

Definition 5. The operator $L : D(L) \subset Y \rightarrow Y^*$ is called

- *monotone*, if for each $y_1, y_2 \in D(L)$ $\langle Ly_1 - Ly_2, y_1 - y_2 \rangle_Y \geq 0$;
- *maximal monotone*, if it is monotone and from $\langle w - Lu, v - u \rangle_Y \geq 0$ for each $u \in D(L)$ it follows that $v \in D(L)$ and $Lv = w$.

Remark 5. If a reflexive Banach space Y is strictly convex with its conjugate, then [21, Lemma 3.1.1] a linear operator $L : D(L) \subset Y \rightarrow Y^*$ is maximal monotone and dense defined if and only if L is a closed non-bounded operator such that

$$\langle Ly, y \rangle_Y \geq 0 \quad \forall y \in D(L) \quad \text{and} \quad \langle L^*y, y \rangle_Y \geq 0 \quad \forall y \in D(L^*),$$

where $L^* : D(L^*) \subset Y \rightarrow Y^*$ is the operator conjugate to L in the sense of non-bounded operators theory (see [12]).

4. AUXILIARY STATEMENTS

From (2) and (3) $V = V_1 \cap V_2 \subset H$ with the continuous and dense embedding. Since V is a separable Banach space, there exists a complete in V and consequently in H countable system of vectors $\{h_i\}_{i \geq 1} \subset V$.

Let for each $n \geq 1$ $H_n = \text{span}\{h_i\}_{i=1}^n$, on which we consider the inner product induced from H that we again denote by (\cdot, \cdot) ; $P_n : H \rightarrow H_n \subset H$ be the operator of orthogonal projection from H on H_n , i.e.,

$$\forall h \in H \quad P_n h = \underset{h_n \in H_n}{\operatorname{argmin}} \|h - h_n\|_H.$$

Definition 6. We say that the triple $(\{h_i\}_{i \geq 1}; V; H)$ satisfies the condition (γ) if $\sup_{n \geq 1} \|P_n\|_{\mathcal{L}(V, V)} < +\infty$, i.e., there exists $C \geq 1$ such that

$$\forall v \in V, \quad \forall n \geq 1 \quad \|P_n v\|_V \leq C \cdot \|v\|_V.$$

Let us remark that a construction of the basis which satisfies (or not) the above condition was introduced in the papers [10], [16], [17], [35].

Remark 6. When the system of vectors $\{h_i\}_{i \geq 1} \subset V$ is orthogonal in H , the condition (γ) means that the given system is a Schauder basis in the Banach space V [35].

Remark 7. Since $P_n \in \mathcal{L}(V, V)$, its conjugate operator $P_n^* \in \mathcal{L}(V^*, V^*)$ and $\|P_n\|_{\mathcal{L}(V, V)} = \|P_n^*\|_{\mathcal{L}(V^*, V^*)}$. It is clear that for each $h \in H$, $P_n h = P_n^* h$. Hence, we identify P_n with P_n^* . Then the condition (γ) means that for each $v \in V$ and $n \geq 1$ $\|P_n v\|_{V^*} \leq C \cdot \|v\|_{V^*}$.

Due to the equivalence of H^* and H , it follows that $H_n^* \equiv H_n$. For each $n \geq 1$, we consider the Banach space $X_n = L_{p_0}(S; H_n) \subset X$, where $p_0 := \max\{r_1, r_2\}$, with the norm $\|\cdot\|_{X_n}$ induced from the space X . This norm is equivalent to the natural norm in $L_{p_0}(S; H_n)$ [11].

The space $L_{q_0}(S; H_n)$ ($q_0^{-1} + p_0^{-1} = 1$) with the norm

$$\|f\|_{X_n^*} := \sup_{x \in X_n \setminus \{0\}} \frac{|\langle f, x \rangle|}{\|x\|_X} = \sup_{x \in X_n \setminus \{0\}} \frac{|\langle f, x \rangle_{X_n}|}{\|x\|_{X_n}}$$

is isometrically isomorphic to the conjugate space X_n^* of X_n (further the given spaces will be identified); moreover, the map

$$X_n^* \times X_n \ni f, x \rightarrow \int_S (f(\tau), x(\tau))_{H_n} d\tau = \int_S (f(\tau), x(\tau)) d\tau = \langle f, x \rangle_{X_n}$$

is the duality form on $X_n^* \times X_n$. This statement is correct since

$$\begin{aligned} X_n^* = L_{q_0}(S; H_n) \subset L_{q_0}(S; H) \subset L_{r_1'}(S; H) + \\ + L_{r_2'}(S; H) + L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) = X^* \end{aligned}$$

(see [11]). Let us remark that $\langle \cdot, \cdot \rangle_{X_n^* \times X_n} = \langle \cdot, \cdot \rangle_{X_n}$.

Proposition 3 ([16, Proposition 1]). *For each $n \geq 1$ $X_n = P_n X$, i.e., $X_n = \{P_n y(\cdot) \mid y(\cdot) \in X\}$, and we have*

$$\langle f, P_n y \rangle = \langle f, y \rangle \quad \forall y \in X \quad \text{and} \quad f \in X_n^*.$$

Moreover, if the triples $(\{h_j\}_{j \geq 1}; V_i; H)$ for $i = 1, 2$ satisfy the condition (γ) with $C = C_i$, then

$$\|P_n y\|_X \leq \max\{C_1, C_2\} \cdot \|y\|_X \quad \forall y \in X \quad \text{and} \quad n \geq 1.$$

For each $n \geq 1$ we denote by I_n the canonical embedding of X_n in X ($\forall x \in X_n$ $I_n x = x$), and by $I_n^* : X^* \rightarrow X_n^*$ its conjugate operator. We point out that

$$\|I_n\|_{\mathcal{L}((X_n, \|\cdot\|_X); (X, \|\cdot\|_X))} = \|I_n^*\|_{\mathcal{L}((X^*, \|\cdot\|_{X^*}); (X_n^*, \|\cdot\|_{X_n^*}))} = 1.$$

Proposition 4 ([16, Proposition 2]). *For each $n \geq 1$ and $f \in X^*$ $(I_n^* f)(t) = P_n f(t)$ for a.a. $t \in S$. Moreover, if the triples $(\{h_j\}_{j \geq 1}; V_i; H)$ for $i = 1, 2$ satisfy the condition (γ) with $C = C_i$, then*

$$\text{for each } f \in X^* \text{ and } n \geq 1 \quad \|I_n^* f\|_{X_n^*} \leq \max\{C_1, C_2\} \cdot \|f\|_{X^*},$$

i.e., $\sup_{n \geq 1} \|I_n^\|_{\mathcal{L}(X^*; X_n^*)} \leq \max\{C_1, C_2\}$.*

From the last two propositions and the properties of I_n^* it immediately follows the following

Corollary 1. For each $n \geq 1$ $X_n^* = P_n X^* = I_n^* X$, i.e.,

$$X_n^* = \{P_n f(\cdot) \mid f(\cdot) \in X^*\} = \{I_n^* f \mid f \in X^*\}.$$

Proposition 5 ([16, Proposition 3]). The set $\bigcup_{n \geq 1} X_n$ is dense in $(X, \|\cdot\|_X)$.

For some linear densely defined operator $L : D(L) \subset Y \rightarrow Y^*$, we consider the normed space $D(L)$ with the graph norm

$$\|y\|_{D(L)} = \|y\|_Y + \|Ly\|_{Y^*} \quad \forall y \in D(L). \quad (7)$$

Proposition 6. Let Y be a reflexive Banach space, $L : D(L) \subset Y \rightarrow Y^*$ be a linear maximal monotone operator. Then every bounded sequence of the space $D(L)$ with the graph norm (7) has a weakly convergent in $D(L)$ subsequence.

5. THE FAEDO–GALERKIN METHOD

For each $n \geq 1$ let us set

$$L_n := I_n^* L I_n : D(L_n) = D(L) \cap X_n \subset X_n \rightarrow X_n^*, \quad f_n := I_n^* f \in X_n^*,$$

$$A_n := I_n^* A I_n : X_n \rightarrow C_v(X_n^*), \quad B_n := I_n^* B I_n : X_n \rightarrow C_v(X_n^*).$$

Remark 8. We will denote by I_n^* also the conjugate operators of the canonical embeddings of X_n in X_1 and of X_n in X_2 , because these operators coincide with I_n^* on $X_1^* \cap X_2^*$ which is dense in X_1^*, X_2^*, X^* .

By analogy with Proposition 6, we consider the normed space $D(L)$ with the graph norm (7). We note that if the linear operator L is closed and densely defined, then $(D(L), \|\cdot\|_{D(L)})$ is a Banach space continuously embedded in X .

In addition to the problem (4), we consider the following class of problems:

$$\begin{cases} L_n y_n + A_n(y_n) + B_n(y_n) \ni f_n, \\ y_n \in D(L_n). \end{cases} \quad (8)$$

Remark 9. We consider on $D(L_n)$ the graph norm

$$\|y_n\|_{D(L_n)} = \|y_n\|_{X_n} + \|L_n y_n\|_{X_n^*} \quad \text{for each } y_n \in D(L_n).$$

Definition 7. We say that the solution $y \in D(L)$ of (4) is obtained by the Faedo–Galerkin method, if y is the weak limit of a subsequence $\{y_{n_k}\}_{k \geq 1}$ from $\{y_n\}_{n \geq 1}$ in $D(L)$, where for each $n \geq 1$ y_n is a solution of the problem (8).

6. THE MAIN SOLVABILITY THEOREM

Theorem 1. Let $L : D(L) \subset X \rightarrow X^*$ be a linear operator, $A : X_1 \rightarrow C_v(X_1^*)$ and $B : X_2 \rightarrow C_v(X_2^*)$ be multi-valued maps such that

- 1) L is maximal monotone on $D(L)$ and satisfies
 - the condition L_1 : for each $n \geq 1$ and $x_n \in D(L_n)$ $Lx_n \in X_n^*$;

- the condition L_2 : for each $n \geq 1$ the set $D(L_n)$ is dense in X_n ;
 - the condition L_3 : for each $n \geq 1$ L_n is maximal monotone on $D(L)$;
- 2) there exist Banach spaces W_1 and W_2 such that $W_1 \subset X_1$, $W_2 \subset X_2$ and $D(L) \subset W_1 \cap W_2$ with continuous embedding;
- 3) A is λ_0 -pseudomonotone on W_1 and satisfies the condition (II);
- 4) B is λ_0 -pseudomonotone on W_2 and satisfies the condition (II);
- 5) the pair $(A; B)$ is s -mutually bounded and the sum $C = A + B : X \rightrightarrows X^*$ is finite-dimensionally locally bounded and weakly $+$ -coercive.

Furthermore, let $\{h_j\}_{j \geq 1} \subset V$ be a complete system of vectors in V_1, V_2 , H such that $\forall i = 1, 2$ the triple $(\{h_j\}_{j \geq 1}; V_i; H)$ satisfies the condition (γ) . Then for each $f \in X^*$ the set

$$K_H(f) := \left\{ y \in D(L) \mid y \text{ is a solution of (4),} \right. \\ \left. \text{obtained by the Faedo-Galerkin method} \right\}$$

is non-empty and the representation

$$K_H(f) = \bigcap_{n \geq 1} \left[\bigcup_{m \geq n} K_m(f_m) \right]_{X_w} \quad (9)$$

is true, where for each $n \geq 1$

$$K_n(f_n) = \left\{ y_n \in D(L_n) \mid y_n \text{ is a solution of (8)} \right\}$$

and $[\cdot]_{X_w}$ is the closure operator in the space X with respect to the weak topology.

Moreover, if the operator $A + B : X \rightrightarrows X^*$ is $--$ coercive, then $K_H(f)$ is weakly compact in X and in $D(L)$ with respect to the graph norm (7).

Remark 10. The sufficient condition for the weak $+$ -coercivity of $A + B$ is as follows: A is $+$ -coercive and it satisfies the condition (κ) on X_1 , B is $+$ -coercive and it satisfies the condition (κ) on X_2 (see Lemma 2).

Remark 11. From the condition L_2 on the operator L and from the Proposition 5 it follows that L is densely defined.

Proof. By Lemma 1 and Remark 2 we consider the λ_0 -pseudomonotone on $W_1 \cap W_2$ (and hence on $D(L)$), finite-dimensionally locally bounded, weakly $+$ -coercive map

$$X \ni y \rightarrow C(y) := A(y) + B(y) \in C_v(X^*),$$

which satisfies the condition (II).

Let $f \in X^*$ be fixed. Now let us use the weak $+$ -coercivity condition for C . There exists $R > 0$ such that

$$[C(y) - f, y]_+ \geq 0 \quad \forall y \in X : \|y\|_X = R. \quad (10)$$

6.1. Solvability of approximating problems.

Lemma 3 ([18]). *For each $n \geq 1$ there exists a solution of the problem (8) $y_n \in D(L_n)$ such that $\|y_n\|_X \leq R$.*

6.2. Passing to limit. Due to Lemma 3, we have a sequence of Galerkin approximate solutions $\{y_n\}_{n \geq 1}$ that satisfies the conditions

$$\forall n \geq 1 \quad \|y_n\|_X \leq R; \tag{11}$$

$$\forall n \geq 1 \quad y_n \in D(L_n) \subset D(L), \quad L_n y_n + d_n(y_n) = f_n, \tag{12}$$

where $d_n(y_n) = I_n^* d(y_n)$, $d(y_n) \in C(y_n)$ is a selector.

In order to prove the given theorem, we need to obtain the following

Lemma 4 ([18]). *Let for some subsequence $\{n_k\}_{k \geq 1}$ of the natural scale the sequence $\{y_{n_k}\}_{k \geq 1}$ satisfy the following conditions:*

- $\forall k \geq 1 \quad y_{n_k} \in D(L_{n_k}) = D(L) \cap X_{n_k}$;
- $\forall k \geq 1 \quad L_{n_k} y_{n_k} + d_{n_k}(y_{n_k}) = f_{n_k}$, $d_{n_k}(y_{n_k}) = I_{n_k}^* d(y_{n_k})$, $d(y_{n_k}) \in C(y_{n_k})$;
- $y_{n_k} \rightharpoonup y$ in X as $k \rightarrow +\infty$ for some $y \in X$.

Then $y \in K_H(f)$.

By (11), (12), Lemma 4, the Banach-Alaoglu theorem and the topological property of the upper limit [20, Property 2.29.IV.8] it follows that

$$\emptyset \neq \bigcap_{n \geq 1} \left[\bigcup_{m \geq n} K_m(f_m) \right]_{X_w} \subset K_H(f).$$

The converse inclusion is obvious; it follows from the same topological property of the upper limit and from $D(L) \subset X$ with continuous embedding.

Now let us prove that $K_H(f)$ is weakly compact in X and in $D(L)$ under the $--$ -coercivity condition on the operator $C = A + B : X \rightarrow C_v(X^*)$. Since (9) holds and $D(L) \subset X$ with continuous embedding, it suffices to show that the given set is bounded in $D(L)$. Let $\{y_n\}_{n \geq 1} \subset K_H(f)$ be an arbitrary sequence. Then for some $d_n \in C(y_n)$

$$L y_n + d(y_n) = f.$$

If $\{y_n\}_{n \geq 1}$ is such that

$$\|y_n\|_X \rightarrow +\infty \text{ as } n \rightarrow \infty,$$

then we obtain the contradiction

$$\begin{aligned} +\infty &\leftarrow \frac{1}{\|y_n\|_X} [C(y_n), y_n]_- \leq \frac{1}{\|y_n\|_X} \langle d(y_n), y_n \rangle \leq \\ &\leq \frac{1}{\|y_n\|_X} \langle L y_n + d(y_n), y_n \rangle = \frac{1}{\|y_n\|_X} \langle f, y_n \rangle \leq \|f\|_{X^*} < +\infty. \end{aligned} \tag{13}$$

Hence, for some $k > 0$

$$\|y_n\|_X \leq k \quad \forall n \geq 1. \tag{14}$$

Due to the condition (II) for C , from (13)–(14) it follows the existence of $K > 0$ such that:

$$\|d_n\|_{X^*} \leq K.$$

Hence,

$$\|Ly_n\|_{X^*} \leq K + \|f\|_{X^*} \quad \text{and} \quad \|y_n\|_{D(L)} \leq k + K + \|f\|_{X^*}.$$

The theorem is proved. \square

7. AN APPLICATION

7.1. On the solvability of an initial time problem by FG method.

Let $A : X_1 \rightarrow C_v(X_1^*)$ and $B : X_2 \rightarrow C_v(X_2^*)$ be multi-valued maps. We consider the problem:

$$\begin{cases} y' + A(y) + B(y) \ni f, \\ y(0) = \bar{0} \end{cases} \quad (15)$$

in order to find the solutions by FG method in the class

$$W = \{y \in X \mid y' \in X^*\},$$

where the derivative y' of an element $y \in X$ is considered in the sense of $\mathcal{D}^*(S; V^*)$. We consider the norm on W

$$\|y\|_W = \|y\|_X + \|y'\|_{X^*} \quad \text{for each } y \in W.$$

We also consider the spaces $W_i = \{y \in X_i \mid y' \in X^*\}$, $i = 1, 2$.

Remark 12. The space W is continuously embedded in $C(S; H)$. Hence, the initial condition in (15) has sense.

In parallel with the problem (15), we consider the following class of problems in order to search the solutions in $W_n = \{y \in X_n \mid y' \in X_n^*\}$:

$$\begin{cases} y'_n + A_n(y_n) + B_n(y_n) \ni f_n, \\ y_n(0) = \bar{0}, \end{cases} \quad (16)$$

where the maps A_n , B_n , f_n were introduced in Section 5, the derivative y'_n of an element $y_n \in X_n$ is considered in the sense of $\mathcal{D}^*(S; H_n)$.

Let $W_{\bar{0}} := \{y \in W \mid y(0) = \bar{0}\}$ and introduce the map

$$L : D(L) = W_{\bar{0}} \subset X \rightarrow X^*$$

by $Ly = y'$ for each $y \in W_{\bar{0}}$.

From the main solvability theorem it follows the following

Corollary 2. *Let $A : X_1 \rightarrow C_v(X_1^*)$ and $B : X_2 \rightarrow C_v(X_2^*)$ be multi-valued maps such that*

- 1) A is λ_0 -pseudomonotone on W_1 and it satisfies the condition (II);
- 2) B is λ_0 -pseudomonotone on W_2 and it satisfies the condition (II);
- 3) the pair $(A; B)$ is s -mutually bounded and the sum $C = A + B : X \rightrightarrows X^*$ is finite-dimensionally locally bounded and weakly $+$ -coercive.

Furthermore, let $\{h_j\}_{j \geq 1} \subset V$ be a complete system of vectors in V_1, V_2, H such that for $i = 1, 2$ the triple $(\{h_j\}_{j \geq 1}; V_i; H)$ satisfies the condition (γ) .

Then for each $f \in X^*$ the set

$$K_H^{\bar{0}}(f) := \left\{ y \in W \mid y \text{ is the solution of (15),} \right. \\ \left. \text{obtained by the Faedo–Galerkin method} \right\}$$

is non-empty and the representation

$$K_H^{\bar{0}}(f) = \bigcap_{n \geq 1} \left[\bigcup_{m \geq n} K_m^{\bar{0}}(f_m) \right]_{X_w}$$

is true, where for each $n \geq 1$

$$K_n^{\bar{0}}(f_n) = \left\{ y_n \in W_n \mid y_n \text{ is the solution of (16)} \right\}.$$

Moreover, if the operator $A + B : X \rightrightarrows X^*$ is $--$ -coercive, then $K_H^{\bar{0}}(f)$ is weakly compact in X and in W .

Proof. First let us prove the maximal monotonicity of L on $W_{\bar{0}}$. For $v \in X, w \in X^*$ such that for each $u \in W_{\bar{0}}$ $\langle w - Lu, v - u \rangle \geq 0$ is true, let us prove that $v \in W_{\bar{0}}$ and $v' = w$. If we take $u = h\varphi x \in W_{\bar{0}}$ with $\varphi \in \mathcal{D}(S), x \in V$ and $h > 0$, we get

$$\begin{aligned} 0 &\leq \langle w - \varphi' hx, v - \varphi hx \rangle = \\ &= \langle w, v \rangle - \left\langle \int_S (\varphi'(s)v(s) + \varphi(s)w(s)) ds, hx \right\rangle + \langle \varphi' hx, \varphi hx \rangle = \\ &= \langle w, v \rangle + h \langle v'(\varphi) - w(\varphi), x \rangle, \end{aligned}$$

where $v'(\varphi)$ and $w(\varphi)$ are the values of the distributions v' and w on $\varphi \in \mathcal{D}(S)$. So, for each $\varphi \in \mathcal{D}(S)$ and $x \in V$ $\langle v'(\varphi) - w(\varphi), x \rangle \geq 0$ is true. Thus we obtain $v'(\varphi) = w(\varphi)$ for all $\varphi \in \mathcal{D}(S)$. This means that $v' = w \in X^*$.

Now we prove $v(0) = \bar{0}$. If we use [11, Theorem IV.1.17] with $u(t) = v(T)\frac{t}{T} \in W_{\bar{0}}$, we obtain that

$$\begin{aligned} 0 &\leq \langle v' - Lu, v - u \rangle = \langle v' - u', v - u \rangle = \\ &= \frac{1}{2} \left(\|v(T) - v(T)\|_H^2 - \|v(0)\|_H^2 \right) = -\frac{1}{2} \|v(0)\|_H^2 \leq 0, \end{aligned}$$

and then $v(0) = \bar{0}$.

In order to prove the given statement, it is enough to show that L satisfies the conditions L_1 – L_3 . The condition L_1 follows from the following

Proposition 7 ([16, Proposition 6]). *For each $y \in X, n \geq 1$ we have $P_n y' = (P_n y)'$, where the derivative of an element $x \in X$ is to be considered in the sense of $\mathcal{D}^*(S; V^*)$.*

The condition L_2 follows from [11, Lemma VI.1.5] and from the fact that the set $C^1(S; H_n)$ is dense in $L_{p_0}(S; H_n) = X_n$. The condition L_3 follows from the previous conclusions with $V = H = H_n$ and $X = X_n$. \square

Remark 13. In the latter Corollary we may relinquish the condition (γ) in the following way:

Following by [21], we may assume that there is a separable Hilbert space V_σ such that $V_\sigma \subset V_1$, $V_\sigma \subset V_2$ with continuous and dense embedding, $V_\sigma \subset H$ with compact and dense embedding. Then

$$V_\sigma \subset V_1 \subset H \subset V_1^* \subset V_\sigma^*, \quad V_\sigma \subset V_2 \subset H \subset V_2^* \subset V_\sigma^*$$

with continuous and dense embedding. For $i = 1, 2$, let us set

$$\begin{aligned} X_{i,\sigma} &= L_{r_i}(S; H) \cap L_{p_i}(S; V_\sigma), & X_\sigma &= X_{1,\sigma} \cap X_{2,\sigma}, \\ X_{i,\sigma}^* &= L_{r'_i}(S; H) + L_{q_i}(S; V_\sigma^*), & X_\sigma^* &= X_{1,\sigma}^* + X_{2,\sigma}^*, \\ W_{i,\sigma} &= \{y \in X_i \mid y' \in X_{i,\sigma}^*\}, & W_\sigma &= W_{1,\sigma} \cap W_{2,\sigma}. \end{aligned}$$

As a complete system of vectors $\{h_i\}_{i \geq 1} \subset V_\sigma$, let us take a special basis, i.e.,

- (i) $\{h_i\}_{i \geq 1}$ orthonormal in H ;
- (ii) $\{h_i\}_{i \geq 1}$ orthogonal in V_σ ;
- (iii) $\forall i \geq 1$ $(h_i, v)_{V_\sigma} = \lambda_i(h_i, v) \forall v \in V_\sigma$, where $0 \leq \lambda_1 \leq \lambda_2, \dots, \lambda_j \rightarrow \infty$ as $j \rightarrow \infty$, $(\cdot, \cdot)_{V_\sigma}$ is the natural inner product in V_σ .

Then

$$\sup_{n \geq 1} \|I_n^*\|_{\mathcal{L}(X_\sigma^*; X_\sigma^*)} = 1. \quad (17)$$

To use this construction, we need to consider some stronger condition for A, B :

A is λ_0 -pseudomonotone on $W_{1,\sigma}$;

B is λ_0 -pseudomonotone on $W_{2,\sigma}$.

So, in the proof of Theorem 1 we need to modify only "Passing to limit".

Due to Lemma 3, we have a sequence of Galerkin approximate solutions $\{y_n\}_{n \geq 1}$, that satisfies the following conditions:

$$\text{a) } \forall n \geq 1: \|y_n\|_X \leq R; \quad (18)$$

$$\text{b) } \forall n \geq 1: y_n \in W_n \subset W, \quad y'_n + C_n(y_n) \ni f_n; \quad (19)$$

$$\text{c) } \forall n \geq 1: y_n(0) = \bar{0}. \quad (20)$$

From the inclusion (19) we have that

$$\forall n \geq 1 \exists d_n \in C(y_n): I_n^* d_n =: d_n^1 = f_n - y'_n \in C_n(y_n) = I_n^* C(y_n). \quad (21)$$

Lemma 5. *From the sequences $\{y_n\}_{n \geq 1}$, $\{d_n\}_{n \geq 1}$ satisfying (18)–(21), subsequences $\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1}$ and $\{d_{n_k}\}_{k \geq 1} \subset \{d_n\}_{n \geq 1}$ can be selected in such a way that for some $y \in W_0$, $d \in X^*$, $z \in H$ the following types of convergence will take place:*

$$1) \quad y_{n_k} \rightharpoonup y \text{ in } X \text{ as } k \rightarrow \infty; \quad (22)$$

$$2) \quad y'_{n_k} \rightharpoonup y' \text{ in } X_\sigma^* \text{ as } k \rightarrow \infty; \quad (23)$$

$$3) \quad d_{n_k} \rightharpoonup d \text{ in } X^* \text{ as } k \rightarrow \infty; \quad (24)$$

$$4) \quad y_{n_k}(T) \rightharpoonup z \text{ in } H \text{ as } k \rightarrow \infty. \quad (25)$$

Moreover, in (25)

$$z = y(T). \quad (26)$$

Proof. 1°. The boundedness of $\{d_n\}_{n \geq 1}$ in X^* is clear. So,

$$\exists c_1 > 0 : \quad \forall n \geq 1 \quad \|d_n\|_{X^*} \leq c_1. \quad (27)$$

2°. Let us prove the boundedness of $\{y'_n\}_{n \geq 1}$ in X^* . From (21) it follows that $\forall n \geq 1 \quad y'_n = I_n^*(f - d_n)$, and therefore keeping in mind (17), (18), (19) and (27) we have:

$$\|y'_n\|_{X_\sigma^*} \leq \|y_n\|_{W_\sigma} \leq R + c(\|f\|_{X^*} + c_1) =: c_2 < +\infty, \quad (28)$$

where $c > 0$ is the constant from the inequality

$$\|f\|_{X_\sigma^*} \leq c\|f\|_{X^*} \quad \forall f \in X^*.$$

3°. From (21) and from $W_n \subset C(S; H)$ with continuous embedding, we obtain that for each $t \in S$, $n \geq 1$

$$\begin{aligned} \|y_n(t)\|_H^2 &= 2 \int_0^t (y'_n(s), y_n(s)) \, ds = 2 \int_0^t (f(s) - d_n(s), y_n(s)) \, ds \leq \\ &\leq (\|f\|_{X^*} + c_1)R. \end{aligned}$$

Hence there exists $c_3 > 0$ such that

$$\forall n \geq 1 \quad \text{for all } t \in S \quad \|y_n(t)\|_H \leq c_3 < +\infty.$$

In particular,

$$\forall n \geq 1 \quad \|y_n(T)\|_H \leq c_3. \quad (29)$$

4°. From the estimates (18), (27)–(29), due to the Banach–Alaoglu theorem, it follows the existence of subsequences

$$\{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1}, \quad \{d_{n_k}\}_{k \geq 1} \subset \{d_n\}_{n \geq 1}$$

and of elements $y \in W_\sigma$, $d \in X^*$ and $z \in H$, for which convergence of types (22)–(25) take place.

5°. Let us prove that

$$y' = f - d. \quad (30)$$

Let $\varphi \in D(S)$, $n \in \mathbb{N}$ and $h \in H_n$. Then $\forall k : n_k \geq n$ we have:

$$\begin{aligned} \left(\int_S \varphi(\tau)(y'_{n_k}(\tau) + d_{n_k}(\tau)) \, d\tau, h \right) &= \int_S \left(\varphi(\tau)(y'_{n_k}(\tau) + d_{n_k}(\tau)), h \right) \, d\tau = \\ &= \int_S \left(y'_{n_k}(\tau) + d_{n_k}(\tau), \varphi(\tau)h \right) \, d\tau = \langle y'_{n_k} + d_{n_k}, \psi \rangle, \end{aligned}$$

where $\psi(\tau) = h \cdot \varphi(\tau) \in X_n \subset X$.

Note that we use the property of Bochner's integral here [11, Theorem IV.1.8, c. 153]. Since for $n_k \geq n$ $H_{n_k} \supset H_n$, we have

$$\langle y'_{n_k} + d_{n_k}, \psi \rangle = \langle f_{n_k}, \psi \rangle.$$

So, $\forall k \geq 1$: $n_k \geq n$ it follows that

$$\begin{aligned} \langle f_{n_k}, \psi \rangle &= \langle f, I_{n_k} \psi \rangle = \\ &= \int_S (f(\tau), \varphi(\tau)h) d\tau = \int_S (\varphi(\tau)f(\tau), h) d\tau = \left(\int_S \varphi(\tau)f(\tau) d\tau, h \right). \end{aligned}$$

Therefore, for all k : $n_k \geq n$

$$\begin{aligned} \left(\int_S \varphi(\tau)y'_{n_k}(\tau) d\tau, h \right) &= \left(\int_S \varphi(\tau)(f(\tau) - d_{n_k}(\tau)) d\tau, h \right) = \\ &= \int_S ((f(\tau) - d_{n_k}(\tau)), \varphi(\tau)h) d\tau = \langle f - d_{n_k}, \psi \rangle \rightarrow \langle f - d, \psi \rangle = \\ &= \left(\int_S \varphi(\tau)(f(\tau) - d(\tau)) d\tau, h \right) \text{ as } k \rightarrow \infty. \quad (31) \end{aligned}$$

The latter follows from the weak convergence of d_{n_k} to d in X^* .

From the convergence (23) we have:

$$\begin{aligned} \left(\int_S \varphi(\tau)y'_{n_k}(\tau) d\tau, h \right) &\rightarrow \\ &\rightarrow \left(\int_S \varphi(\tau)y'(\tau) d\tau, h \right) = (y'(\varphi), h) \text{ as } k \rightarrow +\infty, \quad (32) \end{aligned}$$

where

$$\forall \varphi \in \mathcal{D}(S) \quad y'(\varphi) = -y(\varphi') = - \int_S y(\tau)\varphi'(\tau) d\tau$$

is the derivative of the element y considered in the sense of $\mathcal{D}^*(S, V^*)$.

Hence, from (31) and (32) it follows that

$$\forall \varphi \in \mathcal{D}(S) \quad \forall h \in \bigcup_{n \geq 1} H_n \quad (y'(\varphi), h) = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau)) d\tau, h \right).$$

Since $\bigcup_{n \geq 1} H_n$ is dense in V , we have

$$\forall \varphi \in \mathcal{D}(S) \quad y'(\varphi) = \int_S \varphi(\tau)(f(\tau) - d(\tau)) d\tau.$$

So, $y' = f - d \in X^*$ and $y \in W$.

6°. Prove that $y(0) = \bar{0}$. Let $h \in H_n$, $\varphi \in \mathcal{D}(S)$, $n \in \mathbb{N}$, $\psi(\tau) := (T - \tau)h \in X_n$. From (30) it follows:

$$\begin{aligned} \langle y', \psi \rangle &= \int_S (y'(\tau), \psi(\tau)) d\tau = \int_S (f(\tau) - d(\tau), \psi(\tau)) d\tau = \\ &= \lim_{k \rightarrow \infty} \int_S (f(\tau) - d_{n_k}(\tau), \psi(\tau)) d\tau = \lim_{k \rightarrow \infty} \langle f - d_{n_k}, I_{n_k} \psi \rangle = \\ &= \lim_{k \rightarrow \infty} \langle I_{n_k}^* (f - d_{n_k}), \psi \rangle = \lim_{k \rightarrow \infty} \langle (f_{n_k} - d_{n_k}^1), \psi \rangle = \lim_{k \rightarrow \infty} \langle y'_{n_k}, \psi \rangle. \end{aligned}$$

Noting that $\psi'(\tau) = -h$, $\tau \in S$, we obtain:

$$\begin{aligned} \lim_{k \rightarrow \infty} \langle y'_{n_k}, \psi \rangle &= \lim_{k \rightarrow \infty} \left\{ -\langle \psi', y_{n_k} \rangle + (y_{n_k}(T), \psi(T)) \right\} = \\ &= \lim_{k \rightarrow \infty} \left\{ \int_S (y_{n_k}(\tau), h) d\tau \right\} = \int_S (y(\tau), h) d\tau = -\langle \psi', y \rangle. \end{aligned}$$

The latter is true due to $y_{n_k} \rightharpoonup y$ in X . On the other hand,

$$-\langle \psi', y \rangle = \langle y', \psi \rangle - (y(T), \psi(T)) + (y(0), \psi(0)) = \langle y', \psi \rangle + T(y(0), h).$$

Hence, $\forall h \in \bigcup_{n \geq 1} H_n$

$$\langle y', \psi \rangle = \langle y', \psi \rangle + T(y(0), h) \iff (y(0), h) = 0.$$

From the density of $\bigcup_{n \geq 1} H_n$ in H it follows that $y(0) = \bar{0}$ and $y \in W_0^-$.

7°. To complete the proof, we must show that $y(T) = z$. The proof is similar to that of 6°.

Lemma 5 is proved. \square

Now, to prove that y is a solution of the problem (16), it is necessary to show that y satisfies the inclusion from (16). Due to the identity (30), it is sufficient to prove that $d \in C(y)$.

First let us ascertain that

$$\overline{\lim}_{k \rightarrow \infty} \langle d_{n_k}, y_{n_k} - y \rangle \leq 0. \quad (33)$$

Indeed, due to (30), $\forall k \geq 1$ we have:

$$\begin{aligned} \langle d_{n_k}, y_{n_k} - y \rangle &= \langle d_{n_k}, y_{n_k} \rangle - \langle d_{n_k}, y \rangle = \langle d_{n_k}^1, y_{n_k} \rangle - \langle d_{n_k}, y \rangle = \\ &= \langle f_{n_k} - y'_{n_k}, y_{n_k} \rangle - \langle d_{n_k}, y \rangle = \langle f_{n_k}, y_{n_k} \rangle - \langle y'_{n_k}, y_{n_k} \rangle - \langle d_{n_k}, y \rangle = \\ &= \langle f, y_{n_k} \rangle - \langle d_{n_k}, y \rangle - \frac{1}{2} \|y_{n_k}(T)\|_H^2. \end{aligned} \quad (34)$$

Further in left and right sides of the equality (34) we pass to upper limit as $k \rightarrow \infty$. We have:

$$\overline{\lim}_{k \rightarrow \infty} \langle d_{n_k}, y_{n_k} - y \rangle \leq \overline{\lim}_{k \rightarrow \infty} \langle f, y_{n_k} \rangle + \overline{\lim}_{k \rightarrow \infty} \langle d_{n_k}, -y \rangle - \underline{\lim}_{k \rightarrow \infty} \frac{1}{2} \|y_{n_k}(T)\|_H^2 \leq$$

$$\leq \langle f, y \rangle_X - \langle d, y \rangle - \frac{1}{2} \|y(T)\|_H^2 = \langle f - d, y \rangle - \langle y', y \rangle = 0.$$

From the conditions (22), (23), (24), (33) and λ_0 -pseudomonotonicity of C on W_σ (it holds due to Lemma 1) it follows that there exist $\{d_m\} \subset \{d_{n_k}\}_{k \geq 1}$, $\{y_m\} \subset \{y_{n_k}\}_{k \geq 1}$ such that

$$\forall \omega \in X \quad \varliminf_{m \rightarrow \infty} \langle d_m, y_m - \omega \rangle \geq [C(y), y - \omega]_-. \quad (35)$$

If we prove that

$$\langle d, y \rangle \geq \overline{\lim}_{m \rightarrow \infty} \langle d_m, y_m \rangle, \quad (36)$$

then from (35) and from the convergence of (27) we will have:

$$\forall \omega \in X \quad [C(y), y - \omega]_- \leq \langle d, y - \omega \rangle,$$

and we will obtain that this is equivalent to the inclusion $y \in C(y) \in C_v(X^*)$. Therefore, y will be a solution of the problem (16).

Let us prove (36):

$$\begin{aligned} \overline{\lim}_{m \rightarrow \infty} \langle d_m, y_m \rangle &= \overline{\lim}_{m \rightarrow \infty} \langle f_m - y'_m, y_m \rangle \leq \\ &\leq \overline{\lim}_{m \rightarrow \infty} \langle f_m, y_m \rangle + \overline{\lim}_{m \rightarrow \infty} (-\langle y'_m, y_m \rangle) = \\ &= \overline{\lim}_{m \rightarrow \infty} \langle f, y_m \rangle - \frac{1}{2} \overline{\lim}_{m \rightarrow \infty} \|y_m(T)\|_H^2 \leq \\ &\leq \langle f, y \rangle - \frac{1}{2} \|y(T)\|_H^2 = \langle f, y \rangle - \langle y', y \rangle = \langle d, y \rangle. \end{aligned}$$

So, $y \in W$ is a solution of the problem (16).

7.2. On searching the periodic solutions for differential-operator inclusions by FG method. Let $A : X_1 \rightarrow C_v(X_1^*)$ and $B : X_2 \rightarrow C_v(X_2^*)$ be multi-valued maps. We consider the following problem:

$$\begin{cases} y' + A(y) + B(y) \ni f, \\ y(0) = y(T) \end{cases} \quad (37)$$

in order to find the solutions by FG method in the class

$$W = \{y \in X \mid y' \in X^*\},$$

where the derivative y' of an element $y \in X$ is considered in the sense of scalar distributions space $\mathcal{D}^*(S; V^*) = \mathcal{L}(\mathcal{D}(S); V_w^*)$, with $V = V_1 \cap V_2$, V_w^* equals to V^* with the topology $\sigma(V^*, V)$ [32]. We consider the following norm on W

$$\|y\|_W = \|y\|_X + \|y'\|_{X^*} \quad \text{for each } y \in W.$$

We also consider the spaces $W_i = \{y \in X_i \mid y' \in X^*\}$, $i = 1, 2$.

Remark 14. It is clear that the space W is continuously embedded in $C(S; V^*)$. Hence, the condition in (37) has sense.

In parallel with the problem (37), we consider the following class of problems in order to search the solutions in $W_n = \{y \in X_n \mid y' \in X_n^*\}$:

$$\begin{cases} y'_n + A_n(y_n) + B_n(y_n) \ni f_n, \\ y_n(0) = y_n(T), \end{cases} \quad (38)$$

where the maps A_n, B_n, f_n were introduced in Section 5, the derivative y'_n of an element $y_n \in X_n$ is considered in the sense of $\mathcal{D}^*(S; H_n)$.

Let $W_{per} := \{y \in W \mid y(0) = y(T)\}$ and introduce the map

$$L : D(L) = W_{per} \subset X \rightarrow X^*$$

by $Ly = y'$ for each $y \in W_{per}$.

From the main solvability theorem it follows

Corollary 3. *Let $A : X_1 \rightarrow C_v(X_1^*)$ and $B : X_2 \rightarrow C_v(X_2^*)$ be multi-valued maps such that*

- 1) *A is λ_0 -pseudomonotone on W_1 and it satisfies the condition (II);*
- 2) *B is λ_0 -pseudomonotone on W_2 and it satisfies the condition (II);*
- 3) *the pair $(A; B)$ is s -mutually bounded and the sum $C = A + B : X \rightrightarrows X^*$ is finite-dimensionally locally bounded and weakly $+$ -coercive.*

Furthermore, let $\{h_j\}_{j \geq 1} \subset V$ be a complete vector system in V_1, V_2, H such that for $i = 1, 2$ the triple $(\{h_j\}_{j \geq 1}; V_i; H)$ satisfies the condition (γ) .

Then for each $f \in X^$ the set*

$$K_H^{per}(f) := \left\{ y \in W \mid y \text{ is the solution of (37),} \right. \\ \left. \text{obtained by the Faedo–Galerkin method} \right\}$$

is non-empty and the representation

$$K_H^{per}(f) = \bigcap_{n \geq 1} \left[\bigcup_{m \geq n} K_m^{per}(f_m) \right]_{X_w}$$

is true, where for each $n \geq 1$

$$K_n^{per}(f_n) = \left\{ y_n \in W_n \mid y_n \text{ is the solution of (38)} \right\}.$$

Moreover, if the operator $A + B : X \rightrightarrows X^$ is $--$ -coercive, then $K_H^{per}(f)$ is weakly compact in X and in W .*

Proof. First let us prove the maximal monotonicity of L on W_{per} . For $v \in X, w \in X^*$ such that for each $u \in W_{per}$ $\langle w - Lu, v - u \rangle \geq 0$ is true, let us prove that $v \in W_{per}$ and $v' = w$. By analogy with the proof of Corollary 2, we obtain $v' = w \in X^*$. Now we prove $v(0) = v(T)$. If we use [11, Theorem VI.1.17] with $u(t) \equiv v(T) \in W_{per}$, we obtain that

$$\begin{aligned} 0 &\leq \langle v' - Lu, v - u \rangle = \langle v' - u', v - u \rangle = \\ &= \frac{1}{2} \left(\|v(T) - v(T)\|_H^2 - \|v(0) - v(T)\|_H^2 \right) = -\frac{1}{2} \|v(0) - v(T)\|_H^2 \leq 0 \end{aligned}$$

and then $v(0) = v(T)$.

In order to prove the given statement, it is enough to show that L satisfies the conditions L_1 – L_3 . The condition L_1 follows from Proposition 7. The condition L_2 follows from [11, Lemma VI.1.5] and from the fact that the set $C^1(S; H_n)$ is dense in $L_{p_0}(S, H_n) = X_n$. The condition L_3 follows from [11, Lemma VI.1.7] with $V = H = H_n$ and $X = X_n$. \square

Remark 15. In the latter Corollary we may relinquish the condition (γ) in the way introduced in Remark 13. The proof is similar.

7.3. Example. Let us consider a bounded domain $\Omega \subset \mathbb{R}^n$ with rather smooth boundary $\partial\Omega$, $S = [0, T]$, $Q = \Omega \times (0; T)$, $\Gamma_T = \partial\Omega \times (0; T)$. Let, for $i = 1, 2$, $m_i \in \mathbb{N}$, N_1^i (respectively N_2^i) be the number of the derivatives with respect to the variable x of order $\leq m_i - 1$ (respectively m_i) and $\{A_\alpha^i(x, t, \eta, \xi)\}_{|\alpha| \leq m_i}$ be a family of real functions defined in $Q \times \mathbb{R}^{N_1^i} \times \mathbb{R}^{N_2^i}$. Let

$$\begin{aligned} D^k u &= \{D^\beta u, |\beta| = k\} \text{ be the differentiations by } x, \\ \delta_i u &= \{u, Du, \dots, D^{m_i-1} u\}, \\ A_\alpha^i(x, t, \delta_i u, D^{m_i} v) &: x, t \rightarrow A_\alpha^i(x, t, \delta_i u(x, t), D^{m_i} v(x, t)). \end{aligned}$$

Moreover, let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be some locally Lipschitz real function and let its Clarke's generalized gradient $\Phi = \partial_{CI}\psi : \mathbb{R} \rightrightarrows \mathbb{R}$ satisfy the growth condition

$$\exists C > 0 : \|\Phi(t)\|_+ \leq C(1 + |t|), \quad [\Phi(t), t]_+ \geq \frac{1}{C}(t^2 - 1) \quad \forall t \in \mathbb{R}. \quad (39)$$

Let us consider the following problem with Dirichlet boundary conditions:

$$\begin{aligned} &\frac{\partial y(x, t)}{\partial t} + \sum_{|\alpha| \leq m_1} (-1)^{|\alpha|} D^\alpha (A_\alpha^1(x, t, \delta_1 y, D^{m_1} y)) + \\ &+ \sum_{|\alpha| \leq m_2} (-1)^{|\alpha|} D^\alpha (A_\alpha^2(x, t, \delta_2 y, D^{m_2} y)) + \Phi(y(x, t)) \ni f(x, t) \text{ in } Q, \quad (40) \end{aligned}$$

$$D^\alpha y(x, t) = 0 \text{ on } \Gamma_T \text{ as } |\alpha| \leq m_i - 1 \text{ and } i = 1, 2 \quad (41)$$

$$\text{and } y(x, 0) = y(x, T) \text{ in } \Omega, \quad (42)$$

$$\text{or } y(x, 0) = 0 \text{ in } \Omega. \quad (43)$$

Let us assume $H = L_2(\Omega)$ and $V_i = W_0^{m_i, p_i}(\Omega)$ with $p_i > 1$ such that $V_i \subset H$ with continuous embedding. Consider the function $\varphi : L_2(S; H) \rightarrow \mathbb{R}$ defined by

$$\varphi(y) = \int_Q \psi(y(x, t)) \, dx \, dt \quad \forall y \in L_2(S; H).$$

Using the growth condition (39) and Lebesgue's mean value theorem, we note that the function φ is well-defined and Lipschitz continuous on bounded sets in $L_2(S; H)$, thus locally Lipschitz, so that Clarke's generalized gradient

$\partial_{CI}\varphi : L_2(S; H) \rightrightarrows L_2(S; H)$ is well-defined. Moreover, the Aubin–Clarke theorem (see [7, p. 83]) ensures that for each $y \in L_2(S; H)$ we have

$$p \in \partial_{CI}\varphi(y) \implies p \in L_{q_3}(Q) \text{ with } p(x, t) \in \partial_{CI}\psi(y(x, t)) \text{ for a.e. } (x, t) \in Q.$$

Under suitable conditions on the coefficients A_α^i , the given problems can be written as:

$$y' + A_1(y) + A_2(y) + \partial_{CI}\varphi(y) \ni f, \quad y(0) = y(T), \tag{44}$$

or, respectively

$$y' + A_1(y) + A_2(y) + \partial_{CI}\varphi(y) \ni f, \quad y(0) = \bar{0}, \tag{45}$$

where

$$f \in X^* = L_2(S; L_2(\Omega)) + L_{q_1}(S; W^{-m_1, q_1}(\Omega)) + L_{q_2}(S; W^{-m_2, q_2}(\Omega)),$$

$$p_i^{-1} + q_i^{-1} = 1.$$

Each element $y \in W$ that satisfies (44) (or (45)) is called a generalized solution of the problem (7.3), (41), (42) (respectively (7.3), (41), (43)).

Choice of basis. As a complete system of vectors $\{h_j\}_{j \geq 1} \subset W_0^{m_1, p_1}(\Omega) \cap W_0^{m_2, p_2}(\Omega)$ we may consider the spacial basis for $H_0^l(\Omega)$ with $l \in \mathbb{N}$ such that $H_0^l(\Omega) \subset W_0^{m_i, p_i}(\Omega)$ with continuous embedding ($i = 1, 2$), or we may assume that there is a complete system of vectors $\{h_j\}_{j \geq 1} \subset W_0^{m_1, p_1}(\Omega) \cap W_0^{m_2, p_2}(\Omega)$ such that the triples

$$\left(\{h_j\}_{j \geq 1}; W_0^{m_i, p_i}(\Omega); L_2(\Omega) \right), \quad i = 1, 2,$$

satisfy the condition (γ) .

For example, when $n = 1$, as $\{h_j\}_{j \geq 1}$ we may take the “special” basis for the pair $(H_0^{\max\{m_1, m_2\} + \varepsilon}(\Omega); L_2(\Omega))$ with a suitable $\varepsilon \geq 0$ [21], [16]. As it is well-known, the triple $(\{h_j\}_{j \geq 1}; L_p(\Omega); L_2(\Omega))$ satisfies the condition (γ) for $p > 1$. Then, using (for example) the results of [16], [17], we obtain the necessary condition.

Definition of the operators A_i . Let $A_\alpha^i(x, t, \eta, \xi)$, defined in $Q \times \mathbb{R}^{N_1^i} \times \mathbb{R}^{N_2^i}$, satisfy the conditions.

1) for almost all $x, \in Q$ the map $\eta, \xi \rightarrow A_\alpha^i(x, t, \eta, \xi)$ is continuous on $\mathbb{R}^{N_1^i} \times \mathbb{R}^{N_2^i}$;

2) for all η, xi the map $x, t \rightarrow A_\alpha^i(x, t, \eta, \xi)$ is measurable on Q , (46)

3) for all $u, v \in L^{p_i}(0, T; V_i) =: \mathcal{V}_i$ $A_\alpha^i(x, t, \delta_i u, D^{m_i} u) \in L^{q_i}(Q)$. (47)

Then for each $u \in \mathcal{V}_i$ the map

$$w \rightarrow a_i(u, w) = \sum_{|\alpha| \leq m_i} \int_Q A_\alpha^i(x, t, \delta_i u, D^{m_i} u) D^\alpha w \, dx \, dt$$

is continuous on \mathcal{V}_i and then

$$\text{there exists } A_i(u) \in \mathcal{V}_i^* \text{ such that } a_i(u, w) = \langle A_i(u), w \rangle. \tag{48}$$

Conditions on A_i . Similarly to [21, Sections 2.2.5, 2.2.6, 3.2.1] we have

$$A_i(u) = A_i(u, u), \quad A_i(u, v) = A_{i1}(u, v) + A_{i2}(u),$$

where

$$\begin{aligned} \langle A_{i1}(u, v), w \rangle &= \sum_{|\alpha|=m_i} \int_Q A_\alpha^i(x, t, \delta_i u, D^{m_i} v) D^\alpha w \, dx \, dt, \\ \langle A_{i2}(u), w \rangle &= \sum_{|\alpha| \leq m_i - 1} \int_Q A_\alpha^i(x, t, \delta_i u, D^{m_i} u) D^\alpha w \, dx \, dt. \end{aligned}$$

We add the following conditions:

$$\langle A_{i1}(u, u), u - v \rangle - \langle A_{i1}(u, v), u - v \rangle \geq 0 \quad \forall u, v \in \mathcal{V}_i; \quad (49)$$

if $u_j \rightharpoonup u$ in \mathcal{V}_i , $u'_j \rightharpoonup u'$ in \mathcal{V}_i^* and

$$\text{if } \langle A_{i1}(u_j, u_j) - A_{i1}(u_j, u), u_j - u \rangle \rightarrow 0, \quad (50)$$

then $A_\alpha^i(x, t, \delta u_j, D^{m_i} u_j) \rightharpoonup A_\alpha^i(x, t, \delta u, D^{m_i} u)$ in $L^{q_i}(Q)$;

$$\text{“coercivity”}. \quad (51)$$

Remark 16. Similarly to [21, Theorem 2.2.8], the sufficient conditions for (49), (50) are:

$$\sum_{|\alpha|=m_i} A_\alpha^i(x, t, \eta, \xi) \xi_\alpha \frac{1}{|\xi| + |\xi|^{p_i - 1}} \rightarrow +\infty \quad \text{as } |\xi| \rightarrow \infty$$

for almost all $x, t \in Q$ and $|\eta|$ bounded;

$$\sum_{|\alpha|=m_i} (A_\alpha^i(x, t, \eta, \xi) - A_\alpha^i(x, t, \eta, \xi^*)) (\xi_\alpha - \xi_\alpha^*) > 0 \quad \text{as } \xi \neq \xi^*$$

for almost all $x, t \in Q$ and $\forall \eta$.

The following condition implies the coercivity:

$$\sum_{|\alpha|=m_i} A_\alpha^i(x, t, \eta, \xi) \xi_\alpha \geq c |\xi|^{p_i} \quad \text{for rather large } |\xi|.$$

A sufficient condition to get (47) (see [21, p. 332]) is:

$$|A_\alpha^i(x, t, \eta, \xi)| \leq c [|\eta|^{p_i - 1} + |\xi|^{p_i - 1} + k(x, t)], \quad k \in L_{q_i}(Q). \quad (52)$$

By analogy with the proof of [21, Theorem 3.2.1] and [21, Proposition 2.2.6], we get the following

Proposition 8. *Let the operators $A_i : \mathcal{V}_i \rightarrow \mathcal{V}_i^*$ ($i = 1, 2$) defined in (48) satisfy (46), (47), (49), (50) and (51). Then A_i is pseudomonotone on W_i (even on $W_{i,\sigma}$ in the classical sense). Moreover, it is bounded if (52) holds.*

From the last statement, Corollary 3, Corollary 2, Remark 13 and Remark 15, it follows that under the above listed conditions for each $f \in X^*$ there exists a generalized solution of the problem (7.3)–(42) (respectively of the problem (7.3)–(43)) $y \in W$, obtained by FG method and the representation (9) holds for all these solutions.

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(Received 10.06.2008)

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