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HYPERBOLIC DIFFERENTIAL FUNCTIONAL EQUATIONS WITH UNBOUNDED DELAY

Abstract. The present paper deals with the existence theory of initial and initial boundary value problems for the first order partial functional differential equations with unbounded delay. Strongly coupled quasilinear functional differential systems in the Schauder canonic form and nonlinear equations are considered. We give sufficient conditions for the existence, uniqueness and continuous dependence on data of generalized or classical solutions. In the case of quasilinear systems we apply the method of bicharacteristics. Existence theorems for nonlinear initial problems which are global with respect to spatial variables and for nonlinear mixed problems are proved by using the method of successive approximations. Results for nonlinear initial problems on the Haar pyramid are based on the fixed point method.

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Preface

Up to now numerous papers have been published on first order partial differential functional equations. It is not our aim to show a full review of papers concerning initial or initial boundary value problems for differential functional equations with first order partial derivatives. We will mention only those which contain such reviews. Differential inequalities were considered in [**4**], [**5**], [**20**], [**38**]. Uniqueness of solutions and continuous dependence of solutions on data were investigated in [**1**], [**6**], [**27**], [**37**]. Existence of classical or generalized solutions was studied in [**3**], [**34**], [**39**], [**40**]. In all these problems the initial or boundary functions are given on sets which are subsets of $(a, b) \times R^n$ where a, b are finite. The monograph [**22**] contains an exposition of recent developments of hyperbolic functional differential equations.

There are various concepts of a solution of a functional differential equation. Generalized solutions in the Carathéodory sense were considered in [**13**], [**39**], classical solutions were studied in [**3**], [**14**]. Cinquini Cibrario solutions ([**9**], [**10**], [**31**]) form a class of solutions placed between classical solutions and Carathéodory solutions and both inclusions are strict. The assumptions that the right-hand side of an equation is continuous is sufficient to prove that a C-C solution of such equation is classical. Continuous functions satisfying integral systems obtained by integrating differential equations along bicharacteristics were investigated in [**32**]. Viscosity solutions of Hamilton–Jacobi equations were considered in [**37**], [**38**].

We mention a few methods of proving the existence of generalized or classical solutions. The method of bicharacteristics for quasilinear differential systems was introduced in [**7**], [**8**]. It was adopted in [**14**] for functional differential problems. The method of quasilinearization of nonlinear differential problems was treated in [**9**], [**10**]. This method is used in [**6**], [**31**] for functional differential problems. The idea of successive approximations was first introduced for differential systems in [**41**]. By means of this method, the first results on classical solutions to functional differential problems were obtained ([**3**], [**21**], [**36**]). The fixed point method is based on the Banach fixed point theorem. This method was used in [**19**] for classical solutions to nonlinear problems.

Partial differential equations with unbounded delay were first investigated in [**23**]. In that paper, a system of axioms for the phase space is formulated and existence results for initial problems to quasilinear systems are obtained. Methods used in [**23**] are extended in [**15**] to initial boundary value problems. The paper [**16**] initiated investigations of nonlinear hyperbolic functional differential equations with unbounded delay.

The present paper deals with first order partial functional differential equations with unbounded delay. Our aim is to give a systematic presentation of the existence theory of initial and initial boundary value problems. Strongly coupled quasilinear functional differential systems in the Schauder canonic form and nonlinear equations are considered. The first type theorems in the paper deal with initial problems which are global or local with respect to spatial variables, while the theorems of the second type are concerned with initial boundary value problems.

In this paper we use general ideas concerning axiomatic approach to equations with unbounded delay which were introduced for ordinary differential equations in [**18**], [**30**] (see also [**11**]). In the case of quasilinear systems we apply the method of bicharacteristics. It was widely studied in non-functional setting in [**7**], [**8**]. Initial and mixed problems for weakly coupled quasilinear systems with unbounded delay were investigated in [**26**]. We extend these results to strongly coupled quasilinear Schauder systems (compare [**17**]). For nonlinear initial problems which are global with respect to spatial variables and for nonlinear mixed problems we exploit the ideas introduced in [**28**], [**29**] and used in [**12**], [**24**]. Results for nonlinear initial problems on the Haar pyramid are based on the fixed point method. The set of axioms for phase spaces which we use to initial, global with respect to spatial variables, problems and mixed problems was introduced in [**26**]. The systems of axioms for initial problems on the Haar pyramid are new.

The present work is organized in the following way. In Chapter 1 we give a system of axioms and examples of phase spaces. We consider initial problems which are global with respect to spatial variables for quasilinear systems in the Schauder canonic form. We prove theorems on existence and uniqueness of weak solutions and continuous dependence upon initial data. The same set of axioms of phase spaces is used in Chapters 2, 3 and 4. In Chapter 2 initial problems for nonlinear equations which are global with respect to spatial variables and classical unbounded solutions are studied. Chapter 3 deals with initial boundary value problems for Schauder systems. Results on the existence of Carathédory solutions are proved. In Chapter 4 we consider mixed problems for nonlinear equations. We prove a theorem on solutions in the Cinquini Cibrario sense. Chapter 5 is devoted to initial problems on the Haar pyramid. In the case of quasilinear Schauder systems we give sufficient conditions for existence of generalized solutions as well as results on continuous dependence on initial functions. Finally, we prove theorems on classical solutions to nonlinear weakly coupled systems. Examples of differential equations with a deviated argument and differential integral equations can be derived from a general model by specializing the given functions.

CHAPTER 1

Initial Problems for Quasilinear Systems

1.1. Introduction

Let $B = (-\infty, 0] \times [-r, r]$, $r \in R_+^n$, $R_+ = [0, +\infty)$. Given a function $z : (-\infty, a] \times R^n \to R^k$, $a > 0$, and a point $(t, x) \in (-\infty, a] \times R^n$, we consider the function $z_{(t,x)} : B \to R^k$ defined by

$$z_{(t,x)}(s, y) = z(t + s, x + y), \quad (s, y) \in B.$$

The function $z_{(t,x)}$ is the restriction of z to the set $(-\infty, t] \times [x - r, x + r]$ shifted to the set B. We denote by $M_{k\times m}$ the space of all $k \times m$ matrices with real elements. For $x = (x_1, \ldots, x_n) \in R^n$, $p = (p_1, \ldots, p_k) \in R^k$ and $C = [c_{ij}]_{i=1,\ldots,k,\, j=1,\ldots,m} \in M_{k\times m}$, we put

$$\|x\| = \sum_{i=1}^{n} |x_i|, \quad \|p\|_\infty = \max\Big\{|p_i| : \ 1 \le i \le k\Big\},$$

$$\|C\|_\infty = \max\Big\{\sum_{j=1}^{m} |c_{ij}| : \ 1 \le i \le k\Big\}.$$

If $C \in M_{k\times m}$, then C^T denotes the transposed matrix. For $C, D \in M_{k\times k}$, $C = [c_{ij}]_{i,j=1,\ldots,k}$, $D = [d_{ij}]_{i,j=1,\ldots,k}$, we define

$$C * D = [d_1, \ldots, d_k]^T, \quad d_i = \sum_{j=1}^{k} c_{ij} d_{ji}, \ \ 1 \le i \le k.$$

Vectorial inequalities are understood to hold componentwise. Let X be a linear space with the norm $\|\cdot\|_X$ consisting of functions mapping the set B into R^k. Suppose that

$$A : [0, a] \times R^n \times X \to M_{k\times k}, \quad A = [A_{ij}]_{i,j=1,\ldots,k},$$

$$\varrho : [0, a] \times R^n \times X \to M_{k\times n}, \quad \varrho = [\varrho_{ij}]_{i=1,\ldots,k,\, j=1,\ldots,n},$$

$$f : [0, a] \times R^n \times X \to R^k, \quad f = (f_1, \ldots, f_k), \quad \varphi : (-\infty, 0] \times R^n \to R^k$$

are given functions. Assume that $\psi : [0, a] \times R^n \to R^{n+1}$, $\psi = (\psi_0, \psi')$, $\psi' = (\psi_1, \ldots, \psi_n)$, and we require that $\psi_0(t, x) \le t$ for $(t, x) \in [0, a] \times R^n$. Let us denote by $z = (z_1, \ldots, z_k)$ an unknown function of the variables (t, x), $x = (x_1, \ldots, x_n)$. We consider the system of differential functional equations in the Schauder canonic form

$$\sum_{j=1}^{k} A_{ij}(t,x,z_{\psi(t,x)})\Big(\partial_t z_j(t,x) + \sum_{\nu=1}^{n} \varrho_{i\nu}(t,x,z_{\psi(t,x)})\partial_{x_\nu} z_j(t,x)\Big) = \\ = f_i(t,x,z_{\psi(t,x)}), \quad 1 \le i \le k, \tag{1.1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \ \text{ for } \ (t,x) \in (-\infty,0] \times R^n. \tag{1.2}$$

Here X denotes an abstract linear space satisfying suitable axioms. The elements of X are functions from B into R^k and X is called phase space. Further assumptions on X are given in next parts of the paper. The set B and the function ψ are such that the functional dependence in the above problems is of Volterra type.

We consider weak solutions of the problem (1.1), (1.2). A function $\overline{z} : (-\infty,c] \times R^n \to R^k$, $c \in (0,a]$, is a solution of (1.1), (1.2) provided

(i) $\overline{z}_{\psi(t,x)} \in X$ for $(t,x) \in [0,c] \times R^n$,
(ii) the derivatives $\partial_t \overline{z}_i$, $\partial_x \overline{z}_i = (\partial_{x_1}\overline{z}_i, \dots, \partial_{x_n}\overline{z}_i)$, $1 \le i \le k$, exist almost everywhere on $[0,c] \times R^n$,
(iii) $\overline{z}$ satisfies the differential system for almost all $(t,x) \in [0,c] \times R^n$ and the initial condition holds.

Note that (1.1) is a strongly coupled system in the following sense: each equation in (1.1) contains the partial derivatives of all unknown functions $(z_1,\dots,z_k)$. If $A = E$, where $E \in M_{k\times k}$ is the identity matrix, then (1.1) reduces to the quasilinear system

$$\partial_t z_i(t,x) + \sum_{\nu=1}^{n} \varrho_{i\nu}(t,x,z_{\psi(t,x)})\partial_{x_\nu} z_i(t,x) = f_i(t,x,z_{\psi(t,x)}), \tag{1.3}$$

where $1 \le i \le k$. The above system is weakly coupled because each equation in (1.3) contains the unknown function $z = (z_1,\dots,z_k)$ and the partial derivatives of only one scalar function z_i.

The classical theory of quasilinear systems in the Schauder canonic form without functional dependence is presented in [**2**], [**7**].

1.2. Phase Spaces

Assume that $c > 0$, $w : (-\infty,c] \times [-r,r] \to R^k$ and $t \in (-\infty,c]$. We define the function $w_{(t,\mathbf{0})} : B \to R^k$ by $w_{(t,\mathbf{0})}(s,y) = w(t+s,y)$, $(s,y) \in B$. If the above w is continuous on $[0,c] \times [-r,r]$, then we write

$$\|w\|_{[0,t]} = \max\Big\{\|w(s,y)\|_\infty : \ (s,y) \in [0,t] \times [-r,r]\Big\}, \ \ t \in [0,c].$$

The main assumption on the space X is the following.

Assumption H$[X]$. The space $(X, \|\cdot\|_X)$ is a Banach space of functions from B into R^k and

1) there is $\chi \in R_+$ such that for each function $w \in X$ we have

$$\|w(0,x)\|_\infty \le \chi \|w\|_X, \quad x \in [-r,r];$$

2) if $w : (-\infty, c] \times [-r,r] \to R^k,\ c > 0$, is such that $w_{(0,\mathbf{0})} \in X$ and w is continuous on $[0,c] \times [-r,r]$, then $w_{(t,\mathbf{0})} \in X$ for $t \in [0,c]$ and
   (i) the function $t \mapsto w_{(t,\mathbf{0})}$ is continuous on $[0,c]$,
   (ii) there are $K_1, K_0 \in R_+$ independent of w such that

$$\|w_{(t,\mathbf{0})}\|_X \le K_1 \|w\|_{[0,t]} + K_0 \|w_{(0,\mathbf{0})}\|_X,\ t \in [0,c]. \tag{1.4}$$

We give examples of $(X, \|\cdot\|_X)$.

Example 1.1. Let X be the class of all functions $w : B \to R^k$ which are bounded and uniformly continuous on B. For $w \in X$ we put

$$\|w\|_X = \sup\big\{\|w(s,y)\|_\infty :\ (s,y) \in B\big\}. \tag{1.5}$$

Then Assumption H$[X]$ is satisfied with $\chi = K_1 = K_0 = 1$.

Example 1.2. Let X be the class of all continuous functions $w : B \to R^k$ such that there exists $\lim\limits_{t \to -\infty} w(t,x) = w_0(x)$ uniformly with respect to $x \in [-r,r]$. Then Assumption H$[X]$ is satisfied with the norm defined by (1.5) and $\chi = K_1 = K_0 = 1$.

Example 1.3. Let $\gamma : (-\infty, 0] \to (0, +\infty)$ be a continuous and nonincreasing function. Let X be the class of all continuous functions $w : B \to R^k$ such that

$$\lim_{t \to -\infty} \frac{w(t,x)}{\gamma(t)} = \mathbf{0}, \quad x \in [-r,r],$$

with the norm of w defined by

$$\|w\|_X = \sup\Big\{\frac{\|w(t,x)\|_\infty}{\gamma(t)} :\ (t,x) \in B\Big\}.$$

Then Assumption H$[X]$ is satisfied with $\chi = \gamma(0)$, $K_1 = \frac{1}{\gamma(0)}$, $K_0 = 1$.

Example 1.4. Let $p \ge 1$ be fixed. Denote by Y the class of all functions $w : B \to R^k$ such that

(i) w is continuous on $\{0\} \times [-r,r]$,
(ii) for $x \in [-r,r]$ we have

$$\int_{-\infty}^{0} \|w(s,x)\|_\infty^p \, ds < +\infty,$$

(iii) for each $t \in (-\infty, 0]$ the function $w(t,\cdot) : [-r,r] \to R^k$ is continuous.

For $w \in Y$ we define the norm of w by

$$\|w\|_Y = \max\Big\{\|w(0,x)\|_\infty :\ x \in [-r,r]\Big\} +$$

$$+\sup\Big\{\Big(\int_{-\infty}^{0}\|w(s,x)\|_\infty^p\,ds\Big)^{\frac{1}{p}}:\ x\in[-r,r]\Big\}.$$

Let us denote by X the closure of Y with the above norm. Then Assumption H$[X]$ is satisfied with $\chi=1$, $K_1=1$, $K_0=1+c^{\frac{1}{p}}$.

Example 1.5. Denote by Y the class of all functions $w: B\to R^k$ satisfying the conditions:

(i) w is bounded and it is continuous on $\{0\}\times[-r,r]$,

(ii) for $x\in[-r,r]$ we have

$$I(x)=\sup\Big\{\int_{-(m+1)}^{-m}\|w(s,x)\|_\infty\,ds:\ m\in N\Big\}<+\infty,$$

(iii) for each $t\in(-\infty,0]$ the function $w(t,\cdot):[-r,r]\to R^k$ is continuous.

For $w\in Y$ we define the norm of w by

$$\|w\|_Y=\max\big\{\|w(0,x)\|_\infty:\ x\in[-r,r]\big\}+\sup\big\{I(x):\ x\in[-r,r]\big\}.$$

Let us denote by X the closure of Y with the above norm. Then Assumption H$[X]$ is satisfied with $\chi=1$, $K_1=1+c$, $K_0=2$.

For a function $z:(-\infty,c]\times R^n\to R^k$, $c>0$, which is continuous on $[0,c]\times R^n$, we define

$$\|z\|_0^{[t,x]}=\max\Big\{\|z(s,y)\|_\infty:\ (s,y)\in[0,t]\times[x-r,x+r]\Big\},\quad (t,x)\in[0,c]\times R^n.$$

If a function $z:(-\infty,c]\times R^n\to R^k$, $c>0$, satisfies the Lipschitz condition with respect to x on the set $[0,c]\times R^n$, then we write

$$\|z\|_{L.0}^{[t]}=\sup\Big\{\frac{\|z(s,y)-z(s,\overline{y})\|_\infty}{\|y-\overline{y}\|}:\ (s,y),(s,\overline{y})\in[0,t]\times R^n, y\neq\overline{y}\Big\},$$

where $t\in[0,c]$.

Lemma 1.1. *Suppose that Assumption* H$[X]$ *is satisfied and* $z:(-\infty,c]\times R^n\to R^k$, $c>0$. *If* $z_{(0,x)}\in X$ *for* $x\in R^n$ *and* z *is continuous on* $[0,c]\times R^n$, *then* $z_{(t,x)}\in X$, $(t,x)\in[0,c]\times R^n$, *and*

$$\|z_{(t,x)}\|_X\le K_1\|z\|_0^{[t,x]}+K_0\|z_{(0,x)}\|_X,\quad (t,x)\in[0,c]\times R^n. \tag{1.6}$$

If we assume additionally that z *satisfies the Lipschitz condition with respect to* x *on* $[0,c]\times R^n$, *then*

$$\|z_{(t,x)}-z_{(t,\overline{x})}\|_X\le K_1\|z\|_{L.0}^{[t]}\cdot\|x-\overline{x}\|+K_0\|z_{(0,x)}-z_{(0,\overline{x})}\|_X, \tag{1.7}$$

where (t,x), $(t,\overline{x})\in[0,c]\times R^n$.

Proof. Let $w : (-\infty, c] \times [-r, r] \to R^k$ be given by $w(s, y) = z(s, x + y)$, where $x \in R^n$ is fixed. Then $w_{(t,\mathbf{0})} = z_{(t,x)}$, $t \in [0, c]$. It follows from Assumption H$[X]$ that $z_{(t,x)} \in X$ and (1.6) holds. To prove (1.7) suppose that $(t, x), (t, \overline{x}) \in [0, c] \times R^n$ and $\widetilde{z} : (-\infty, c] \times R^n \to R^k$ is defined by $\widetilde{z}(s, y) = z(s, y + \overline{x} - x)$, $(s, y) \in (-\infty, c] \times R^n$. Then $\widetilde{z}_{(t,x)} = z_{(t,\overline{x})}$. It follows from (1.6) that

$$\|z_{(t,x)} - z_{(t,\overline{x})}\|_X = \|(z - \widetilde{z})_{(t,x)}\|_X \le K_1 \|z - \widetilde{z}\|_0^{[t,x]} + K_0 \|(z - \widetilde{z})_{(0,x)}\|_X \le$$
$$\le K_1 \|z\|_{L.0}^{[t]} \cdot \|x - \overline{x}\| + K_0 \|z_{(0,x)} - z_{(0,\overline{x})}\|_X.$$

This completes the proof. □

If a function $z : (-\infty, c] \times R^n \to R^k$, $c > 0$, satisfies the Lipschitz condition with respect to (t, x) on the set $[-c, c] \times R^n$, then we write

$$\|z\|_{L.1}^{[t]} =$$
$$= \sup \Big\{ \frac{\|z(s, y) - z(\overline{s}, \overline{y})\|_\infty}{|s - \overline{s}| + \|y - \overline{y}\|} : \ (s, y),\ (\overline{s}, \overline{y}) \in [-t, t] \times R^n,\ (s, y) \ne (\overline{s}, \overline{y}) \Big\},$$

where $t \in [0, c]$.

Lemma 1.2. *Suppose that Assumption* H$[X]$ *is satisfied and* $z : (-\infty, c] \times R^n \to R^k$, $c > 0$. *If* $z_{(t,x)} \in X$ *for* $(t, x) \in [-c, 0] \times R^n$ *and* z *satisfies the Lipschitz condition with respect to* (t, x) *on* $[-c, c] \times R^n$, *then*

$$\|z_{(t,x)} - z_{(\overline{t},\overline{x})}\|_X \le$$
$$\le K_1 \|z\|_{L.1}^{[t]} \cdot \big(|t - \overline{t}| + \|x - \overline{x}\|\big) + K_0 \|z_{(0,x)} - z_{(\overline{t}-t,\overline{x})}\|_X, \tag{1.8}$$

where (t, x), $(\overline{t}, \overline{x}) \in [0, c] \times R^n$, $t > \overline{t}$.

Proof. Let (t, x), $(\overline{t}, \overline{x}) \in [0, c] \times R^n$, $t > \overline{t}$ and let $\widetilde{z} : (-\infty, c] \times R^n \to R^k$ be defined by $\widetilde{z}(s, y) = z(s + \overline{t} - t, y + \overline{x} - x)$, $(s, y) \in (-\infty, c] \times R^n$. Then $\widetilde{z}_{(t,x)} = z_{(\overline{t},\overline{x})}$ and $\widetilde{z}_{(0,x)} = z_{(\overline{t}-t,x)} \in X$. It follows from Lemma 1.1 that $(z - \widetilde{z})_{(t,x)} \in X$ and by (1.6) we have

$$\|(z - \widetilde{z})_{(t,x)}\|_X \le K_1 \|z - \widetilde{z}\|_0^{[t,x]} + K_0 \|(z - \widetilde{z})_{(0,x)}\|_X.$$

Thus

$$\|z_{(t,x)} - z_{(\overline{t},\overline{x})}\|_X = \|(z - \widetilde{z})_{(t,x)}\|_X \le$$
$$\le K_1 \|z\|_{L.1}^{[t]} \cdot \big(|t - \overline{t}| + \|x - \overline{x}\|\big) + K_0 \|z_{(0,x)} - z_{(\overline{t}-t,\overline{x})}\|_X,$$

which proves (1.8). □

1.3. Bicharacteristics for Quasilinear Systems

We begin with the following definitions. For any metric spaces Y and Z we denote by $C(Y, Z)$ the class of all continuous functions from Y into Z. We will denote by $L([0, c], R_+)$, $c > 0$, the class of all functions $\gamma : [0, c] \to R_+$ which are integrable on $[0, c]$. Let Δ denote the set of all functions $\alpha : [0, a] \times R_+ \to R_+$ such that $\alpha(\cdot, t) \in L([0, a], R_+)$ for $t \in R_+$ and the function $\alpha(t,\cdot):R_+\to R_+$ is continuous, nondecreasing and $\alpha(t,0)=0$ for almost all $t\in[0,a]$. Let Σ denote the set of all functions $\alpha\in C(R_+,R_+)$ which are nondecreasing and $\alpha(0)=0$.

For a linear normed space $(Y,\|\cdot\|_Y)$ we write

$$Y[\mu]=\big\{w\in Y:\ \|w\|_Y\le\mu\big\},\ \ \mu\in R_+. \tag{1.9}$$

Let $\mathcal{J}_L[X]$ denote the class of all initial functions $\varphi:(-\infty,0]\times R^n\to R^k$ satisfying the conditions:

1) $\varphi_{(t,x)}\in X$ for $(t,x)\in(-\infty,0]\times R^n$,
2) there are $b_0,b_1\in R_+$ such that

$$\|\varphi_{(t,x)}\|_X\le b_0,\quad \big\|\varphi_{(t,x)}-\varphi_{(\bar t,\bar x)}\big\|_X\le b_1\big(|t-\bar t|+\|x-\bar x\|\big),$$

where (t,x), $(\bar t,\bar x)\in(-\infty,0]\times R^n$.

Fix $\varphi\in\mathcal{J}_L[X]$ and $c\in(0,a]$, $d=(d_0,d_1)\in R_+^2$ and denote by $C^L_{\varphi.c}[d]$ the class of all functions $z:(-\infty,c]\times R^n\to R^k$ such that

(i) $z(t,x)=\varphi(t,x)$ for $(t,x)\in(-\infty,0]\times R^n$,
(ii) the estimates

$$\|z(t,x)\|_\infty\le d_0,\quad \big\|z(t,x)-z(\bar t,\bar x)\big\|_\infty\le d_1\big(|t-\bar t|+\|x-\bar x\|\big)$$

hold on $[0,c]\times R^n$.

The assumptions on ψ and ϱ are following.

Assumption H$_L[\psi]$. The function $\psi:[0,a]\times R^n\to R^{n+1}$, $\psi=(\psi_0,\psi')$, $\psi'=(\psi_1,\dots,\psi_n)$, satisfies the conditions:

1) $\psi_0(t,x)\le t$ for $t\in[0,a]$, $x\in R^n$,
2) there is $s_1\in R_+$ such that

$$\big|\psi_0(t,x)-\psi_0(\bar t,\bar x)\big|+\big\|\psi'(t,x)-\psi'(\bar t,\bar x)\big\|\le s_1\big(|t-\bar t|+\|x-\bar x\|\big)$$

on $[0,a]\times R^n$.

Assumption H$_L[\varrho]$. The function $\varrho(\cdot,x,w):[0,a]\to M_{k\times n}$ is measurable for every $(x,w)\in R^n\times X$ and there are $\alpha_1\in\Sigma$, $\beta_1\in\Delta$ such that

$$\|\varrho(t,x,w)\|_\infty\le\alpha_1(\mu),$$
$$\big\|\varrho(t,x,w)-\varrho(t,\bar x,\bar w)\big\|_\infty\le\beta_1(t,\mu)\big(\|x-\bar x\|+\|w-\bar w\|_X\big)$$

for (x,w), $(\bar x,\bar w)\in R^n\times X[\mu]$ and for almost all $t\in[0,a]$.

Suppose that Assumptions H$[X]$, H$_L[\psi]$, H$_L[\varrho]$ are satisfied and $\varphi\in\mathcal{J}_L[X]$, $z\in C^L_{\varphi.c}[d]$. Consider the Cauchy problem

$$\eta'(\tau)=\varrho_i\big(\tau,\eta(\tau),z_{\psi(\tau,\eta(\tau))}\big),\ \ \eta(t)=x, \tag{1.10}$$

where $(t,x)\in[0,c]\times R^n$ and $1\le i\le k$ are fixed, while $\varrho_i=(\varrho_{i1},\dots,\varrho_{in})$. Let us denote by $g_i[z](\cdot,t,x)$ the solution of (1.10). The function $g_i[z]$ is the i-th bicharacteristic of the system (1.1) corresponding to z.

For functions $\varphi \in \mathcal{J}_L[X]$ and $z \in C^L_{\varphi.c}[d]$ we write

$$\|\varphi\|^*_X = \sup\Big\{\|\varphi_{(t,x)}\|_X :\ (t,x) \in (-\infty,0] \times R^n\Big\},$$
$$\|z\|_t = \sup\Big\{\|z(s,y)\|_\infty :\ (s,y) \in [0,t] \times R^n\Big\},\ \ t \in [0,c].$$

We prove a lemma on existence, uniqueness and regularity of bicharacteristics.

Lemma 1.3. *Suppose that Assumptions* H$[X]$, H$_L[\psi]$, H$_L[\varrho]$ *are satisfied and assume that* $\varphi, \overline{\varphi} \in \mathcal{J}_L[X]$, $z \in C^L_{\varphi.c}[d]$, $\overline{z} \in C^L_{\overline{\varphi}.c}[d]$, $c \in (0,a]$. *Then for each* $1 \le i \le k$, $(t,x) \in [0,c] \times R^n$ *the solutions* $g_i[z](\cdot,t,x)$ *and* $g_i[\overline{z}](\cdot,t,x)$ *exist on* $[0,c]$ *and they are unique. Moreover,*

$$\big\|g_i[z](\tau,t,x) - g_i[z](\tau,\overline{t},\overline{x})\big\| \le Q_c\alpha_1^+(\mu_0)\big(|t-\overline{t}| + \|x-\overline{x}\|\big) \tag{1.11}$$

for $(\tau,t,x) \in [0,c]^2 \times R^n$, *and*

$$\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\| \le$$
$$\le Q_c\Big|\int_t^\tau \beta_1(\xi,\mu_0)\,d\xi\Big|\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|^*_X\Big) \tag{1.12}$$

for (τ,t,x), $(\tau,\overline{t},\overline{x}) \in [0,c]^2 \times R^n$, *where* $\alpha_1^+(\mu_0) = 1 + \alpha_1(\mu_0)$ *and*

$$\mu_0 = K_1d_0 + K_0b_0, \quad Q_c = \exp\Big(\Lambda\int_0^c \beta_1(\xi,\mu_0)\,d\xi\Big),$$
$$\Lambda = 1 + s_1(K_1d_1 + K_0b_1). \tag{1.13}$$

Proof. It follows from Assumptions H$[X]$, H$_L[\psi]$ and from Lemma 1.1 that

$$\|z_{\psi(\tau,y)}\|_X \le \mu_0, \quad (\tau,y) \in [0,c] \times R^n.$$

We prove that Lemma 1.2 implies the estimate

$$\big\|z_{\psi(\tau,y)} - z_{\psi(\tau,\overline{y})}\big\|_X \le s_1(K_1d_1 + K_0b_1)\|y-\overline{y}\|,$$

where (τ,y), $(\tau,\overline{y}) \in [0,c] \times R^n$. It is obvious in the cases (i) $\psi_0(\tau,y) \le 0$, $\psi_0(\tau,\overline{y}) \le 0$ and (ii) $\psi_0(\tau,y) \ge 0$, $\psi_0(\tau,\overline{y}) \ge 0$. Consider the case (iii) $\psi_0(\tau,y) \le 0$, $\psi_0(\tau,\overline{y}) > 0$. There are $q \in (0,1)$, $\eta \in R^n$ such that

$$q\psi(\tau,y) + (1-q)\psi(\tau,\overline{y}) = (0,\eta),$$

which yields

$$|\psi_0(\tau,y)| + \|\psi'(\tau,y) - \eta\| + |\psi_0(\tau,\overline{y})| + \|\eta - \psi'(\tau,\overline{y})\| =$$
$$= \big|\psi_0(\tau,y) - \psi_0(\tau,\overline{y})\big| + \big\|\psi'(\tau,y) - \psi'(\tau,\overline{y})\big\|.$$

Thus

$$\big\|z_{\psi(\tau,y)} - z_{\psi(\tau,\overline{y})}\big\|_X = \big\|\varphi_{\psi(\tau,y)} - z_{\psi(\tau,\overline{y})}\big\|_X \le$$
$$\le \big\|\varphi_{\psi(\tau,y)} - \varphi_{(0,\eta)}\big\|_X + \big\|z_{(0,\eta)} - z_{\psi(\tau,\overline{y})}\big\|_X \le$$

$$\leq (K_0 b_1 + K_1 d_1)\Big(\big|\psi_0(\tau,y)-\psi_0(\tau,\overline{y})\big| + \big\|\psi'(\tau,y)-\psi'(\tau,\overline{y})\big\|\Big) \leq$$
$$\leq s_1(K_0 b_1 + K_1 d_1)\|y-\overline{y}\|.$$

Now it is easy to see that the following Lipschitz condition is satisfied

$$\big\|\varrho_i(\tau,y,z_{\psi(\tau,y)}) - \varrho_i(\tau,\overline{y},z_{\psi(\tau,\overline{y})})\big\| \leq$$
$$\leq \beta_1(\tau,\mu_0)\Lambda\|y-\overline{y}\|, \ \ \tau\in[0,c],\ y,\ \overline{y}\in R^n,$$

and there exists exactly one Carathéodory solution of (1.10) defined on $[0,c]$.

We prove the estimates (1.11) and (1.12). The function $g_i[z](\cdot,t,x)$ satisfies the integral equation

$$g_i[z](\tau,t,x) = x + \int_t^\tau \varrho_i\big(P_i[z](\xi,t,x)\big)\,d\xi,$$

where

$$P_i[z](\xi,t,x) = \big(\xi, g_i[z](\xi,t,x), z_{\psi(\xi,g_i[z](\xi,t,x))}\big). \tag{1.14}$$

If (τ,t,x), $(\tau,\overline{t},\overline{x})\in[0,c]^2\times R^n$, then we have

$$\big\|g_i[z](\tau,t,x)-g_i[z](\tau,\overline{t},\overline{x})\big\| \leq \|x-\overline{x}\| + \bigg|\int_{\overline{t}}^{t}\big\|\varrho_i\big(P_i[z](\xi,\overline{t},\overline{x})\big)\big\|\,d\xi\bigg| +$$
$$+\bigg|\int_t^\tau \Big\|\varrho_i\big(P_i[z](\xi,\overline{t},\overline{x})\big)-\varrho_i\big(P_i[z](\xi,t,x)\big)\Big\|\,d\xi\bigg| \leq$$
$$\leq \alpha_1^+(\mu_0)\big(|t-\overline{t}|+\|x-\overline{x}\|\big)+\Lambda\bigg|\int_\tau^t \beta_1(\xi,\mu_0)\big\|g_i[z](\xi,t,x)-g_i[z](\xi,\overline{t},\overline{x})\big\|\,d\xi\bigg|.$$

We obtain (1.11) from the Gronwall inequality. If $z\in C_{\varphi.c}[d,\lambda]$, $\overline{z}\in C_{\overline{\varphi}.c}[d,\lambda]$, $(\tau,t,x)\in[0,c]^2\times R^n$, then we have

$$\big\|z_{\psi(\xi,g_i[z](\xi,t,x))}-\overline{z}_{\psi(\xi,g_i[\overline{z}](\xi,t,x))}\big\|_X \leq$$
$$\leq s_1(K_1 d_1+K_0 b_1)\big\|g_i[z](\xi,t,x)-g_i[\overline{z}](\xi,t,x)\big\|+K_1\|z-\overline{z}\|_\xi+K_0\|\varphi-\overline{\varphi}\|_X^*,$$

where $\xi\in[0,c]$, and thus

$$\big\|g_i[z](\tau,t,x)-g_i[\overline{z}](\tau,t,x)\big\| \leq$$
$$\leq \bigg|\int_t^\tau \Big\|\varrho_i\big(P_i[z](\xi,t,x)\big)-\varrho_i\big(P_i[\overline{z}](\xi,t,x)\big)\Big\|\,d\xi\bigg| \leq$$
$$\leq \bigg|\int_t^\tau \beta_1(\xi,\mu_0)\,d\xi\bigg|\Big(K_1\|z-\overline{z}\|_c+K_0\|\varphi-\overline{\varphi}\|_X^*\Big)+$$
$$+\Lambda\bigg|\int_t^\tau \beta_1(\xi,\mu_0)\big\|g_i[z](\xi,t,x)-g_i[\overline{z}](\xi,t,x)\big\|\,d\xi\bigg|.$$

Using the Gronwall inequality we obtain (1.12). □

1.4. Existence and Uniqueness of Lipschitz Continuous Solutions

We define an integral operator corresponding to the problem (1.1), (1.2). First we formulate the following assumptions on f and A.

Assumption $\mathbf{H}_L[f]$. The function $f(\cdot,x,w):[0,a]\to R^k$ is measurable for every $(x,w)\in R^n\times X$ and there are $\alpha_2\in\Sigma$, $\beta_2\in\Delta$ such that

$$\|f(t,x,w)\|_\infty\le\alpha_2(\mu),$$

$$\big\|f(t,x,w)-f(t,\overline{x},\overline{w})\big\|_\infty\le\beta_2(t,\mu)\big(\|x-\overline{x}\|+\|w-\overline{w}\|_X\big)$$

for (x,w), $(\overline{x},\overline{w})\in R^n\times X[\mu]$ and for almost all $t\in[0,a]$.

Assumption $\mathbf{H}_L[A]$. The function $A:[0,a]\times R^n\times X\to M_{k\times k}$ satisfies the conditions:

1) there are α, $\beta\in\Sigma$ such that on $[0,a]\times R^n\times X[\mu]$

$$\|A(t,x,w)\|_\infty\le\alpha(\mu),$$

$$\big\|A(t,x,w)-A(\overline{t},\overline{x},\overline{w})\big\|_\infty\le\beta(\mu)\big(|t-\overline{t}|+\|x-\overline{x}\|+\|w-\overline{w}\|_X\big),$$

2) for each $(t,x,w)\in[0,a]\times R^n\times X[\mu]$ there exists an inverse matrix $A^{-1}(t,x,w)$ and there are α_0, $\beta_0\in\Sigma$ such that

$$\|A^{-1}(t,x,w)\|_\infty\le\alpha_0(\mu),$$

$$\big\|A^{-1}(t,x,w)-A^{-1}(\overline{t},\overline{x},\overline{w})\big\|_\infty\le\beta_0(\mu)\big(|t-\overline{t}|+\|x-\overline{x}\|+\|w-\overline{w}\|_X\big)$$

for (t,x,w), $(\overline{t},\overline{x},\overline{w})\in[0,a]\times R^n\times X[\mu]$.

Remark 1.1. If $A:[0,a]\times R^n\times X\to M_{k\times k}$ satisfies the condition 1) of Assumption $\mathrm{H}_L[A]$ and there exists $\sigma:R_+\to(0,+\infty)$ such that

$$detA(t,x,w)\ge\sigma(\mu)\ \text{ for }\ (t,x,w)\in[0,a]\times R^n\times X[\mu],$$

then the condition 2) of Assumption $\mathrm{H}_L[A]$ is satisfied.

Let us fix $\varphi\in\mathcal{J}_L[X]$, $c\in(0,a]$, $z\in C^L_{\varphi.c}[d]$. Suppose that $(t,x)\in[0,c]\times R^n$ and $g_i[z](\cdot,t,x)$, $1\le i\le k$, is the solution of (1.10). It follows from (1.1) that for $(t,x)\in[0,c]\times R^n$

$$\sum_{j=1}^k A_{ij}\big(P_i[z](\tau,t,x)\big)\,\frac{d}{d\tau}\,z_j\big(\tau,g_i[z](\tau,t,x)\big)=f_i\big(P_i[z](\tau,t,x)\big),$$

where $1\le i\le k$ and $P_i[z](\cdot,t,x)$ is given by (1.14). Integrating from 0 to t, we obtain

$$\sum_{j=1}^k A_{ij}(t,x,z_{\psi(t,x)})z_j(t,x)=\sum_{j=1}^k A_{ij}\big(P_i[z](0,t,x)\big)\varphi_j(0,g_i[z](0,t,x))+$$

$$+\int_0^t\sum_{j=1}^k\frac{d}{d\tau}\,A_{ij}\big(P_i[z](\tau,t,x)\big)z_j(\tau,g_i[z](\tau,t,x))\,d\tau+\int_0^t f_i\big(P_i[z](\tau,t,x)\big)\,d\tau.$$

The above relation allows us to construct the following integral operator. Write

$$A[z](\tau,t,x)=\Big[A_{ij}(P_i[z](\tau,t,x))\Big]_{i,j=1,\ldots,k},$$
$$\Phi[z](\tau,t,x)=\Big[\varphi_i(0,g_j[z](\tau,t,x))\Big]_{i,j=1,\ldots,k},$$
$$Z[z](\tau,t,x)=\Big[z_i(\tau,g_j[z](\tau,t,x))\Big]_{i,j=1,\ldots,k},$$
$$f[z](\tau,t,x)=\Big[f_i(P_i[z](\tau,t,x))\Big]^T_{i=1,\ldots,k}.$$

For $z\in C^L_{\varphi.c}[d]$ define $T_\varphi(z):(-\infty,c]\times R^n\to R^k$ in the following way

$$T_\varphi(z)(t,x)=A^{-1}(t,x,z_{\psi(t,x)})\Big\{A[z](0,t,x)*\Phi[z](0,t,x)+$$
$$+\int_0^t\Big(\frac{d}{d\tau}A[z](\tau,t,x)*Z[z](\tau,t,x)+f[z](\tau,t,x)\Big)\,d\tau\Big\},\quad (t,x)\in[0,c]\times R^n,$$
$$T_\varphi(z)(t,x)=\varphi(t,x),\quad (t,x)\in(-\infty,0]\times R^n. \tag{1.15}$$

We can write the above relation as follows

$$T_\varphi(z)(t,x)=\varphi(0,x)+A^{-1}(t,x,z_{\psi(t,x)})\sum_{i=1}^{3}\Delta_i[z](t,x), \tag{1.16}$$

where $(t,x)\in[0,c]\times R^n$ and

$$\Delta_1[z](t,x)=\int_0^t f[z](\tau,t,x)\,d\tau,$$
$$\Delta_2[z](t,x)=A[z](0,t,x)*\big(\Phi[z](0,t,x)-\Phi[z](t,t,x)\big),$$
$$\Delta_3[z](t,x)=\int_0^t\frac{d}{d\tau}A[z](\tau,t,x)*\big(Z[z](\tau,t,x)-\Phi[z](t,t,x)\big)\,d\tau.$$

We formulate the following lemmas on the operator T_φ.

Lemma 1.4. *If Assumptions* H$[X]$, H$_L[\psi]$, H$_L[\varrho]$, H$_L[f]$, H$_L[A]$ *are satisfied, then there are* $c\in(0,a]$, $d=(d_0,d_1)\in R^2_+$ *such that for each* $\varphi\in\mathcal{J}_L[X]$ *the operator* T_φ *maps the set* $C^L_{\varphi.c}[d]$ *into itself.*

Proof. Let $\varphi\in\mathcal{J}_L[X]$ and for $z\in C^L_{\varphi.c}[d]$ let $T_\varphi(z)$ be defined by (1.15), (1.16). We will show that $T_\varphi(z)\in C^L_{\varphi.c}[d]$. It follows from the assumptions of the lemma that

$$\Big|z_j\big(\tau,g_i[z](\tau,t,x)\big)-\varphi_j(0,x)\Big|\le$$
$$\le\Big|z_j\big(\tau,g_i[z](\tau,t,x)\big)-z_j\big(0,g_i[z](\tau,t,x)\big)\Big|+$$

$$+\Big|\varphi_j(0,g_i[z](\tau,t,x))-\varphi_j(0,x)\Big|\le cd_1+c\alpha_1(\mu_0)\chi b_1,$$

$$\Big\|\frac{d}{d\tau}\,z_{\psi(\tau,g_i[z](\tau,t,x))}\Big\|_X\le (K_1d_1+K_0b_1)s_1\alpha_1^+(\mu_0),$$

$$\Big\|\frac{d}{d\tau}\,A[z](\tau,t,x))\Big\|_\infty\le p^* \;\text{ with }\; p^*=\beta(\mu_0)\Lambda\alpha_1^+(\mu_0).$$

Thus

$$\sum_{i=1}^{3}\|\Delta_i[z](t,x)\|_\infty\le c\delta^*,$$

where $\delta^*=\alpha_2(\mu_0)+\alpha(\mu_0)\chi b_1\alpha_1(\mu_0)+cp^*d_1\alpha_1^+(\mu_0)$. Therefore

$$\|T_\varphi(z)(t,x)\|_\infty\le \chi b_0+c\alpha_0(\mu_0)\delta^* \;\text{ on }\; [0,c]\times R^n.$$

We assume that

$$d_0\ge \chi b_0+c\alpha_0(\mu_0)\delta^*. \tag{1.17}$$

Then

$$\|T_\varphi(z)(t,x)\|_\infty\le d_0 \;\text{ for }\; (t,x)\in[0,c]\times R^n.$$

To estimate $\|T_\varphi(z)(t,x)-T_\varphi(z)(\bar t,\overline{x})\|_\infty$, we first note that

$$\big\|g_i[z](\tau,t,x)-g_i[z](\tau,\bar t,\overline{x})\big\|+\big\|z_{\psi(\tau,g_i[z](\tau,t,x))}-z_{\psi(\tau,g_i[z](\tau,\bar t,\overline{x}))}\big\|_X\le$$
$$\le \Lambda Q_c\alpha_1^+(\mu_0)\big(|t-\bar t|+\|x-\overline{x}\|\big).$$

We have

$$\big\|\Delta_1[z](t,x)-\Delta_1[z](\bar t,\overline{x})\big\|_\infty\le$$
$$\le\Big\|\int_0^t\big(f[z](\tau,t,x)-f[z](\tau,\bar t,\overline{x})\big)\,d\tau\Big\|_\infty+\Big\|\int_t^{\bar t}f[z](\tau,\bar t,\overline{x})\,d\tau\Big\|_\infty\le$$
$$\le d_{1.c}\big(|t-\bar t|+\|x-\overline{x}\|\big),$$

where

$$d_{1.c}=\alpha_2(\mu_0)+\Lambda Q_c\alpha_1^+(\mu_0)\int_0^c\beta_2(\xi,\mu_0)\,d\xi.$$

Moreover,

$$\big\|\Delta_2[z](t,x)-\Delta_2[z](\bar t,\overline{x})\big\|_\infty\le$$
$$\le\Big\|\Big(A[z](0,t,x)-A[z](0,\bar t,\overline{x})\Big)*\Big(\Phi[z](0,t,x)-\Phi[z](t,t,x)\Big)\Big\|_\infty+$$
$$+\Big\|A[z](0,\bar t,\overline{x})*\Big(\Phi[z](0,t,x)-\Phi[z](0,\bar t,\overline{x})-\Phi[z](t,t,x)+\Phi[z](\bar t,\bar t,\overline{x})\Big)\Big\|_\infty\le$$
$$\le d_{2.c}\big(|t-\bar t|+\|x-\overline{x}\|\big),$$

where

$$d_{2.c}=\alpha(\mu_0)\chi b_1\big(Q_c\alpha_1^+(\mu_0)+1\big)+c\beta(\mu_0)\Lambda Q_c\alpha_1^+(\mu_0)\chi b_1\alpha_1(\mu_0).$$

Finally,

$$\big\|\Delta_3[z](t,x)-\Delta_3[z](\bar t,\overline{x})\big\|_\infty\le$$

$$\leq \Big\| \int_t^{\bar t} \frac{d}{d\tau} A[z](\tau,\bar t,\bar x) * \Big(Z[z](\tau,\bar t,\bar x) - \Phi[z](\bar t,\bar t,\bar x)\Big)\, d\tau \Big\|_\infty +$$

$$+\Big\| \Big(A[z](\tau,t,x) - A[z](\tau,\bar t,\bar x)\Big) * \Big(Z[z](\tau,t,x) - \Phi[z](t,t,x)\Big)\Big|_{\tau=0}^{\tau=t} \Big\|_\infty +$$

$$+\Big\| \int_0^{t} \Big(A[z](\tau,t,x) - A[z](\tau,\bar t,\bar x)\Big) * \frac{d}{d\tau} Z[z](\tau,t,x)\, d\tau \Big\|_\infty +$$

$$+\Big\| \int_0^{t} \frac{d}{d\tau} A[z](\tau,\bar t,\bar x) *$$

$$* \Big(Z[z](\tau,t,x) - Z[z](\tau,\bar t,\bar x) - \Phi[z](t,t,x) + \Phi[z](\bar t,\bar t,\bar x)\Big)\, d\tau \Big\|_\infty \leq$$

$$\leq d_{3.c}\big(|t-\bar t| + \|x-\bar x\|\big),$$

where

$$d_{3.c} = cp^*\Big(d_1\big(Q_c\alpha_1^+(\mu_0)+1\big) + \chi b_1\alpha_1^+(\mu_0)\Big) +$$
$$+ c\beta(\mu_0)\alpha_1^+(\mu_0)\Big(d_1\Lambda\big(Q_c\alpha_1^+(\mu_0)+1\big) + (1+s_1b_1)\chi b_1\alpha_1(\mu_0)Q_c\Big).$$

Suppose that d_1 satisfies the condition

$$d_1 \geq \chi b_1 + c\beta(\mu_0)\Lambda\delta^* + \alpha_0(\mu_0)\sum_{i=1}^{3} d_{i.c}. \tag{1.18}$$

Since

$$T_\varphi(z)(t,x) - T_\varphi(z)(\bar t,\bar x) =$$

$$= \varphi(0,x) - \varphi(0,\bar x) + \Big(A^{-1}(t,x,z_{\psi(t,x)}) - A^{-1}(\bar t,\bar x,z_{\psi(\bar t,\bar x)})\Big)\sum_{i=1}^{3}\Delta_i[z](t,x) +$$

$$+ A^{-1}(\bar t,\bar x,z_{\psi(\bar t,\bar x)})\sum_{i=1}^{3}\Big(\Delta_i[z](t,x) - \Delta_i[z](\bar t,\bar x)\Big),$$

we obtain

$$\big\| T_\varphi(z)(t,x) - T_\varphi(z)(\bar t,\bar x)\big\|_\infty \leq d_1\big(|t-\bar t| + \|x-\bar x\|\big) \text{ on } [0,c]\times R^n.$$

In this way we have proved that $T_\varphi : C^L_{\varphi.c}[d] \to C^L_{\varphi.c}[d]$ for $c \in (0,a]$ and $d = (d_0,d_1) \in R^2_+$ satisfying the inequalities (1.17) and (1.18). □

Lemma 1.5. *Suppose that the assumptions of Lemma* 1.4 *are satisfied. If* φ, $\overline{\varphi} \in \mathcal{J}_L[X]$ *and* $z \in C^L_{\varphi.c}[d]$, $\overline{z} \in C^L_{\overline{\varphi}.c}[d]$, *then there are* $G_{1.c}$, $G_2 \in R_+$ *such that*

$$\big\| T_\varphi(z) - T_{\overline{\varphi}}(\overline{z})\big\|_c \leq G_{1.c}\|z-\overline{z}\|_c + G_2\|\varphi - \overline{\varphi}\|^*_X. \tag{1.19}$$

Proof. Let $\varphi, \overline{\varphi} \in \mathcal{J}_L[X]$, $z \in C^L_{\varphi.c}[d]$, $\overline{z} \in C^L_{\overline{\varphi}.c}[d]$. We have the following estimates

$$\|z_{\psi(t,x)} - \overline{z}_{\psi(t,x)}\|_X \leq K_1\|z - \overline{z}\|_c + K_0\|\varphi - \overline{\varphi}\|^*_X,$$

and

$$\big\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\big\| + \big\|z_{\psi(\tau,g_i[z](\tau,t,x))} - \overline{z}_{\psi(\tau,g_i[\overline{z}](\tau,t,x))}\big\|_X \leq$$
$$\leq q^*\Big(K_1\|z - \overline{z}\|_c + K_0\|\varphi - \overline{\varphi}\|^*_X\Big) \text{ with } q^* = 1 + \Lambda Q_c \int_0^c \beta_1(\xi,\mu_0)\,d\xi.$$

We conclude from Assumptions $\mathrm{H}_L[\psi]$, $\mathrm{H}_L[f]$, $\mathrm{H}_L[A]$ that

$$\big\|\Delta_1[z](t,x) - \Delta_1[\overline{z}](t,x)\big\|_\infty \leq$$
$$\leq \int_0^t \big\|f[z](\tau,t,x) - f[\overline{z}](\tau,t,x)\big\|_\infty\,d\tau \leq \sigma_{1.c}\Big(K_1\|z - \overline{z}\|_c + K_0\|\varphi - \overline{\varphi}\|^*_X\Big),$$

$$\big\|\Delta_2[z](t,x) - \Delta_2[\overline{z}](t,x)\big\|_\infty \leq$$
$$\leq \Big\|\big(A[z](0,t,x) - A[\overline{z}](0,t,x)\big) * \big(\Phi[z](0,t,x) - \Phi[z](t,t,x)\big)\Big\|_\infty +$$
$$+\Big\|A[\overline{z}](0,t,x) * \Big(\Phi[z](0,t,x) - \Phi[\overline{z}](0,t,x) - \Phi[z](t,t,x) +$$
$$+\overline{\Phi}[\overline{z}](t,t,x) + \Phi[\overline{z}](0,t,x) - \overline{\Phi}[\overline{z}](0,t,x)\Big)\Big\|_\infty \leq$$
$$\leq \sigma_{2.c}\Big(K_1\|z - \overline{z}\|_c + K_0\|\varphi - \overline{\varphi}\|^*_X\Big) + 2\alpha(\mu_0)\chi\|\varphi - \overline{\varphi}\|^*_X,$$

$$\big\|\Delta_3[z](t,x) - \Delta_3[\overline{z}](t,x)\big\|_\infty \leq$$
$$\leq \Big\| \int_0^t \frac{d}{d\tau}\,A[z](\tau,t,x) *$$
$$*\Big(Z[z](\tau,t,x) - Z[\overline{z}](\tau,t,x) - \Phi[z](t,t,x) + \overline{\Phi}[\overline{z}](t,t,x)\Big)\,d\tau\Big\|_\infty +$$
$$+\Big\|\Big(A[z](\tau,t,x) - A[\overline{z}](\tau,t,x)\Big) * \Big(Z[\overline{z}](\tau,t,x) - \overline{\Phi}[\overline{z}](t,t,x)\Big)\Big|_{\tau=0}^{\tau=t}\Big\|_\infty +$$
$$+\Big\|\int_0^t \Big(A[z](\tau,t,x) - A[\overline{z}](\tau,t,x)\Big) * \frac{d}{d\tau}\,Z[\overline{z}](\tau,t,x)\,d\tau\Big\|_\infty \leq$$
$$\leq \sigma_{3.c}\Big(K_1\|z - \overline{z}\|_c + K_0\|\varphi - \overline{\varphi}\|^*_X\Big) + cp^*\chi\|\varphi - \overline{\varphi}\|^*_X + cp^*\|z - \overline{z}\|_c,$$

where

$$\sigma_{1.c} = q^*\int_0^c \beta_2(\xi,\mu_0)\,d\xi,$$

$$\sigma_{2.c} = q^*\beta(\mu_0)\chi b_1\alpha_1(\mu_0)c + \alpha(\mu_0)\chi b_1 Q_c \int_0^c \beta_1(\xi,\mu_0)\,d\xi,$$

$$\sigma_{3.c} = cp^* d_1 Q_c \int_0^c \beta_1(\xi,\mu_0)\,d\xi + \beta(\mu_0)\Big(d_1 c + q^*\chi b_1\alpha_1(\mu_0)c + q^* d_1\alpha_1^+(\mu_0)c\Big).$$

Since

$$T_\varphi(z)(t,x) - T_{\overline{\varphi}}(\overline{z})(t,x) =$$

$$= \varphi(0,x) - \overline{\varphi}(0,x) + \Big(A^{-1}(t,x,z_{\psi(t,x)}) - A^{-1}(t,x,\overline{z}_{\psi(t,x)})\Big)\sum_{i=1}^{3}\Delta_i[z](t,x) +$$

$$+ A^{-1}(t,x,\overline{z}(t,x))\sum_{i=1}^{3}\big(\Delta_i[z](t,x) - \Delta_i[\overline{z}](t,x)\big),$$

we obtain

$$\big\|T_\varphi(z) - T_{\overline{\varphi}}(\overline{z})\big\|_c \le G_{1.c}\|z-\overline{z}\|_c + G_2\|\varphi-\overline{\varphi}\|_X^*,$$

where

$$G_{1.c} = K_1\Big(c\beta_0(\mu_0)\delta^* + \alpha_0(\mu_0)\sum_{i=1}^{3}\sigma_{i.c}\Big) + c\alpha_0(\mu_0)p^*, \tag{1.20}$$

$$G_2 = K_0\Big(c\beta_0(\mu_0)\delta^* + \alpha_0(\mu_0)\sum_{i=1}^{3}\sigma_{i.c}\Big) + \alpha_0(\mu_0)\chi(2\alpha(\mu_0) + cp^*). \tag{1.21}$$

This completes the proof of Lemma 1.5. □

Now we can give a theorem on solution of the problem (1.1), (1.2).

Theorem 1.1. *Suppose that Assumptions* H$[X]$, H$_L[\psi]$, H$_L[\varrho]$, H$_L[f]$ *and* H$_L[A]$ *are satisfied. Assume that the inequalities* (1.17), (1.18) *and*

$$G_{1.c} < 1 \tag{1.22}$$

hold, where $G_{1.c}$ *is given by* (1.20). *Then for each* $\varphi \in \mathcal{J}_L[X]$ *there exists* $z = z[\varphi] \in C^L_{\varphi.c}[d]$ *which is a unique solution of* (1.1), (1.2) *in the class* $C^L_{\varphi.c}[d]$. *Moreover, if* $\varphi, \overline{\varphi} \in \mathcal{J}_L[X]$, $z = z[\varphi]$, $\overline{z} = z[\overline{\varphi}]$, *then*

$$\|z-\overline{z}\|_c \le \frac{G_2}{1-G_{1.c}}\,\|\varphi-\overline{\varphi}\|_X^* \tag{1.23}$$

with G_2 *given by* (1.21).

Proof. It follows from the assumptions of the theorem that for each $\varphi \in \mathcal{J}_L[X]$ the operator $T_\varphi : C^L_{\varphi.c}[d] \to C^L_{\varphi.c}[d]$ and it is a contraction. Thus T_φ has a unique fixed point $z = z[\varphi] \in C^L_{\varphi.c}[d]$. We prove that $z = z[\varphi]$ is a solution of (1.1). We have shown that

$$\sum_{j=1}^{k} A_{ij}(t,x,z_{\psi(t,x)})z_j(t,x) = \sum_{j=1}^{k} A_{ij}\big(P_i[z](0,t,x)\big)\varphi_j(0,g_i[z](0,t,x)) +$$

$$+\int_0^t \sum_{j=1}^k \frac{d}{d\tau} A_{ij}\big(P_i[z](\tau,t,x)\big) z_j(\tau, g_i[z](\tau,t,x))\, d\tau + \int_0^t f_i\big(P_i[z](\tau,t,x)\big)\, d\tau$$

on $[0,c]\times R^n$. The relations

$$\eta_i = g_i[z](0,t,x) \ \text{ and } \ x = g_i[z](t,0,\eta_i)$$

are equivalent for $x,\eta_i \in R^n$. We have $g_i[z](\tau,t,g_i[z]t,0,\eta_i) = g_i[z](\tau,0,\eta_i)$. Thus

$$\sum_{j=1}^k A_{ij}\big(P_i[z](t,0,\eta_i)\big) z_j(t, g_i[z](t,0,\eta_i)) = \sum_{j=1}^k A_{ij}(0,\eta_i, z_{\psi(0,\eta_i)})\varphi_j(0,\eta_i) +$$

$$+\int_0^t \sum_{j=1}^k \frac{d}{d\tau} A_{ij}\big(P_i[z](\tau,0,\eta_i)\big) z_j\big(\tau, g_i[z](\tau,0,\eta_i)\big)\, d\tau +$$

$$+\int_0^t f_i\big(P_i[z](\tau,0,\eta_i)\big)\, d\tau.$$

By differentiating with respect to t and by using the transformations $\eta_i = g_i[z](0,t,x)$ which preserve the sets of measure zero, we obtain that z satisfies (1.1) almost everywhere on $[0,c]\times R^n$. The inequality (1.23) follows from Lemma 1.5. □

1.5. Solutions Satisfying Generalized Lipschitz Condition

In this part of the paper we consider a special case of the problem (1.1), (1.2). Suppose that $\psi_0 : [0,a] \to R$ and $\psi' : [0,a]\times R^n \to R^n$, $\psi' = (\psi_1,\dots,\psi_n)$. We require that $\psi_0(t) \le t$ for $t\in[0,a]$. Write $\psi(t,x) = (\psi_0(t),\psi_1(t,x),\dots,\psi_n(t,x))$, $t\in[0,a]$, $x\in R^n$. Assume that

$$\varrho : [0,a]\times R^n \times X \to M_{k\times n}, \quad \varrho = [\varrho_{ij}]_{i=1,\dots,k,\, j=1,\dots,n},$$

$$f : [0,a]\times R^n\times X \to R^k, \quad f = (f_1,\dots,f_k), \quad \varphi : (-\infty,0]\times R^n \to R^k$$

are given functions. Given the function

$$A : [0,a]\times R^n \times R^k \to M_{k\times k}, \ \ A = [A_{ij}]_{i,j=1,\dots,k},$$

we consider the following initial problem

$$\sum_{j=1}^k A_{ij}(t,x,z(t,x))\Big(\partial_t z_j(t,x) + \sum_{\nu=1}^n \varrho_{i\nu}(t,x,z_{\psi(t,x)})\partial_{x_\nu} z_j(t,x)\Big) =$$

$$= f_i(t,x,z_{\psi(t,x)}), \ \ 1\le i\le k, \tag{1.24}$$

$$z(t,x) = \varphi(t,x) \ \text{ for } \ (t,x)\in(-\infty,0]\times R^n. \tag{1.25}$$

There are the following differences between the problems (1.1), (1.2) and (1.24), (1.25). The matrix A in (1.24) does not depend on the functional variable $z_{\psi(t,x)}$ and the function ψ_0 depends on t only. Solutions of (1.1), (1.2) are functions satisfying the classical Lipschitz condition on

$[0,c]\times R^n$. We look for solutions of (1.24), (1.25) in the class of functions satisfying the following generalized Lipschitz condition:

$$\big\|z(t,x)-z(\bar t,\overline{x})\big\|_\infty \le \Big|\int_t^{\bar t}\lambda(\tau)\,d\tau\Big| + d_1\|x-\overline{x}\| \ \text{ on } \ [0,c]\times R^n.$$

Let us denote by $\mathcal{J}_C[X]$ the class of all initial functions $\varphi : (-\infty,0]\times R^n \to R^k$ satisfying the conditions:

1) $\varphi_{(t,x)} \in X$ for $(t,x)\in(-\infty,0]\times R^n$,
2) there are $b_0, b_1 \in R_+$ such that

$$\|\varphi_{(t,x)}\|_X \le b_0, \ \ \|\varphi_{(t,x)}-\varphi_{(t,\overline{x})}\|_X \le b_1\|x-\overline{x}\|,$$

where $(t,x),(t,\overline{x})\in(-\infty,0]\times R^n$.

Let $\varphi\in\mathcal{J}_C[X]$ and $c\in(0,a]$, $d=(d_0,d_1)\in R_+^2$, $\lambda\in L([0,c],R_+)$. Denote by $C_{\varphi.c}[d,\lambda]$ the class of all functions $z:(-\infty,c]\times R^n\to R^k$ such that

(i) $z(t,x)=\varphi(t,x)$ for $(t,x)\in(-\infty,0]\times R^n$,
(ii) the estimates

$$\|z(t,x)\|_\infty \le d_0, \quad \big\|z(t,x)-z(\bar t,\overline{x})\big\|_\infty \le \Big|\int_t^{\bar t}\lambda(\tau)\,d\tau\Big| + d_1\|x-\overline{x}\|$$

hold on $[0,c]\times R^n$.

We introduce the following assumptions on the functions ψ and ϱ.

Assumption $\mathbf{H}_C[\psi]$. The functions $\psi_0:[0,a]\to R$ and $\psi':[0,a]\times R^n\to R^n$ are continuous and satisfy the conditions:

1) $\psi_0(t)\le t$ for $t\in[0,a]$,
2) there is $s_1\in R_+$ such that

$$\|\psi'(t,x)-\psi'(t,\overline{x})\| \le s_1\|x-\overline{x}\| \ \text{ on } \ [0,a]\times R^n.$$

Assumption $\mathbf{H}_C[\varrho]$. The function $\varrho(\cdot,x,w):[0,a]\to M_{k\times n}$ is measurable for every $(x,w)\in R^n\times X$ and there are $\alpha_1,\beta_1\in\Delta$ such that

$$\|\varrho(t,x,w)\|_\infty \le \alpha_1(t,\mu),$$
$$\big\|\varrho(t,x,w)-\varrho(t,\overline{x},\overline{w})\big\|_\infty \le \beta_1(t,\mu)\big(\|x-\overline{x}\|+\|w-\overline{w}\|_X\big)$$

for (x,w), $(\overline{x},\overline{w})\in R^n\times X[\mu]$ and for almost $t\in[0,a]$.

Suppose that Assumptions H$[X]$, H$_C[\psi]$, H$_C[\varrho]$ are satisfied and $\varphi\in\mathcal{J}_C[X]$, $z\in C_{\varphi.c}[d,\lambda]$. Consider the Cauchy problem

$$\eta'(\tau)=\varrho_i\big(\tau,\eta(\tau),z_{\psi(\tau,\eta(\tau))}\big), \ \ \eta(t)=x, \tag{1.26}$$

where $(t,x)\in[0,c]\times R^n$ and $1\le i\le k$ are fixed. Let us denote by $g_i[z](\cdot,t,x)$ the solution of (1.26). The following are important properties of solutions of (1.26).

Lemma 1.6. *Suppose that Assumptions* H$[X]$, H$_C[\psi]$, H$_C[\varrho]$ *are satisfied and* $\varphi, \overline{\varphi} \in \mathcal{J}_C[X]$, $z \in C_{\varphi.c}[d,\lambda]$, $\overline{z} \in C_{\overline{\varphi}.c}[d,\lambda]$, $c \in (0,a]$. *Then, for each* $1 \le i \le k$, $(t,x) \in [0,c] \times R^n$, *the solutions* $g_i[z](\cdot,t,x)$ *and* $g_i[\overline{z}](\cdot,t,x)$ *of* (1.26) *exist on* $[0,c]$ *and they are unique. Moreover,*

$$\big\|g_i[z](\tau,t,x) - g_i[z](\tau,\overline{t},\overline{x})\big\| \le Q_c\Big(\Big|\int_t^{\overline{t}} \alpha_1(\xi,\mu_0)\,d\xi\Big| + \|x-\overline{x}\|\Big) \tag{1.27}$$

on $[0,c]^2 \times R^n$, *and*

$$\big\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\big\| \le$$
$$\le Q_c\Big|\int_t^{\tau} \beta_1(\xi,\mu_0)\,d\xi\Big|\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^*\Big) \tag{1.28}$$

on $[0,c]^2 \times R^n$, *where*

$$Q_c = \exp\Big(\Lambda\int_0^c \beta_1(\xi,\mu_0)\,d\xi\Big),$$
$$\mu_0 = K_1 d_0 + K_0 b_0, \quad \Lambda = 1 + s_1(K_1 d_1 + K_0 b_1). \tag{1.29}$$

Proof. The existence and uniqueness of Carathéodory solutions of (1.26) follows from classical theorems. It follows from the assumptions of the lemma and from the integral equation

$$g_i[z](\tau,t,x) = x + \int_t^{\tau} \varrho_i\big(P_i[z](\xi,t,x)\big)\,d\xi,$$

where

$$P_i[z](\xi,t,x) = \big(\xi, g_i[z](\xi,t,x), z_{\psi(\xi,g_i[z](\xi,t,x))}\big), \tag{1.30}$$

that the inequalities

$$\big\|g_i[z](\tau,t,x) - g_i[z](\tau,\overline{t},\overline{x})\big\| \le \|x-\overline{x}\| +$$
$$+\Big|\int_t^{\overline{t}} \alpha_1(\xi,\mu_0)\,d\xi\Big| + \Lambda\Big|\int_\tau^{t} \beta_1(\xi,\mu_0)\big\|g_i[z](\xi,t,x) - g_i[z](\xi,\overline{t},\overline{x})\big\|\,d\xi\Big|$$

and

$$\big\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\big\| \le$$
$$\le \Big|\int_t^{\tau} \beta_1(\xi,\mu_0)\,d\xi\Big|\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^*\Big) +$$
$$+ \Lambda\Big|\int_t^{\tau} \beta_1(\xi,\mu_0)\big\|g_i[z](\xi,t,x) - g_i[\overline{z}](\xi,t,x)\big\|\,d\xi\Big|$$

hold for (τ, t, x), $(\tau, \overline{t}, \overline{x}) \in [0, c]^2 \times R^n$. Using the Gronwall inequality, we obtain (1.27) and (1.28). $\square$

Now we construct an integral operator corresponding to the problem (1.24), (1.25). We will need the following assumptions on f and A.

Assumption $\mathbf{H}_C[f]$. The function $f(\cdot, x, w) : [0, a] \to R^k$ is measurable for every $(x, w) \in R^n \times X$ and there are $\alpha_2, \beta_2 \in \Delta$ such that

$$\|f(t, x, w)\|_\infty \le \alpha_2(t, \mu),$$
$$\big\|f(t, x, w) - f(t, \overline{x}, \overline{w})\big\|_\infty \le \beta_2(t, \mu)\big(\|x - \overline{x}\| + \|w - \overline{w}\|_X\big)$$

for $(x, w), (\overline{x}, \overline{w}) \in R^n \times X[\mu]$ and for almost $t \in [0, a]$.

Assumption $\mathbf{H}_C[A]$. The function $A : [0, a] \times R^n \times R^k \to M_{k\times k}$ satisfies the conditions:

1) there are $\alpha, \beta \in \Sigma$ and $\gamma \in \Delta$ such that

$$\|A(t, x, p)\|_\infty \le \alpha(\mu),$$
$$\big\|A(t, x, p) - A(\overline{t}, \overline{x}, \overline{p})\big\|_\infty \le \beta(\mu)\big(\|x - \overline{x}\| + \|p - \overline{p}\|\big) + \Big|\int_t^{\overline{t}} \gamma(\xi, \mu)\, d\xi\Big|$$

for (t, x, p), $(\overline{t}, \overline{x}, \overline{p}) \in [0, a] \times R^n \times R^k[\mu]$,

2) for each $(t, x, p) \in [0, a] \times R^n \times R^k[\mu]$ there exists the inverse matrix $A^{-1}(t, x, p)$ and there are $\alpha_0, \beta_0 \in \Sigma$ and $\gamma_0 \in \Delta$ such that

$$\|A^{-1}(t, x, p)\|_\infty \le \alpha_0(\mu),$$
$$\big\|A^{-1}(t, x, p) - A^{-1}(\overline{t}, \overline{x}, \overline{p})\big\|_\infty \le \beta_0(\mu)\big(\|x - \overline{x}\| + \|p - \overline{p}\|\big) + \Big|\int_t^{\overline{t}} \gamma_0(\xi, \mu)\, d\xi\Big|$$

for $(t, x, p), (\overline{t}, \overline{x}, \overline{p}) \in [0, a] \times R^n \times R^k[\mu]$.

Let us fix $\varphi \in \mathcal{J}_C[X]$, $c \in (0, a]$, $z \in C_{\varphi.c}[d, \lambda]$. Suppose that $(t, x) \in [0, c] \times R^n$ and $g_i[z](\cdot, t, x)$, $1 \le i \le k$, are bicharacteristics. We can write (1.24) in the form

$$\sum_{j=1}^{k} A_{ij}\big(Q_i[z](\tau, t, x)\big)\, \frac{d}{d\tau}\, z_j\big(\tau, g_i[z](\tau, t, x)\big) = f_i\big(P_i[z](\tau, t, x)\big),$$

where $P_i[z](\cdot, t, x)$ is given by (1.30) and

$$Q_i[z](\tau, t, x) = \big(\tau, g_i[z](\tau, t, x), z(\tau, g_i[z](\tau, t, x))\big). \tag{1.31}$$

Integrating from 0 to t, we obtain

$$\sum_{j=1}^{k} A_{ij}(t, x, z(t, x)) z_j(t, x) = \sum_{j=1}^{k} A_{ij}\big(Q_i[z](0, t, x)\big)\varphi_j(0, g_i[z](0, t, x)) +$$

$$+\int_0^t \sum_{j=1}^k \frac{d}{d\tau} A_{ij}\big(Q_i[z](\tau,t,x)\big) z_j(\tau, g_i[z](\tau,t,x))\, d\tau + \int_0^t f_i\big(P_i[z](\tau,t,x)\big)\, d\tau$$

which allows us to define an integral operator. Put

$$\begin{aligned} A^*[z](\tau,t,x) &= \big[A_{ij}\big(Q_i[z](\tau,t,x)\big)\big]_{i,j=1,\ldots,k}, \\ \Phi[z](\tau,t,x) &= \big[\varphi_i\big(0, g_j[z](\tau,t,x)\big)\big]_{i,j=1,\ldots,k}, \\ Z[z](\tau,t,x) &= \big[z_i\big(\tau, g_j[z](\tau,t,x)\big)\big]_{i,j=1,\ldots,k}, \\ f[z](\tau,t,x) &= \big[f_i\big(P_i[z](\tau,t,x)\big)\big]^T_{i=1,\ldots,k}. \end{aligned}$$

We define the operator $C_{\varphi.c}[d,\lambda] \ni z \longmapsto T^*_\varphi(z)$ in the following way

$$T^*_\varphi(z)(t,x) = \varphi(0,x) + A^{-1}(t,x,z(t,x)) \sum_{i=1}^3 \Delta^*_i[z](t,x) \tag{1.32}$$

for $(t,x) \in [0,c] \times R^n$ and

$$T^*_\varphi(z)(t,x) = \varphi(t,x), \quad (t,x) \in (-\infty, 0] \times R^n, \tag{1.33}$$

where

$$\begin{aligned} \Delta^*_1[z](t,x) &= \int_0^t f[z](\tau,t,x)\, d\tau, \\ \Delta^*_2[z](t,x) &= A^*[z](0,t,x) * \Big(\Phi[z](0,t,x) - \Phi[z](t,t,x)\Big), \\ \Delta^*_3[z](t,x) &= \int_0^t \frac{d}{d\tau} A^*[z](\tau,t,x) * \Big(Z[z](\tau,t,x) - \Phi[z](t,t,x)\Big)\, d\tau. \end{aligned}$$

We look for the fixed point of the operator T^*_φ.

1.6. The Existence and Uniqueness Theorem

We prove the following properties of the operator T^*_φ.

Lemma 1.7. *If Assumptions* H$[X]$, H$_C[\psi]$, H$_C[\varrho]$, H$_C[f]$, H$_C[A]$ *are satisfied, then there are* $c \in (0,a]$, $d = (d_0, d_1) \in R^2_+$, $\lambda \in L([0,c], R_+)$ *such that for each* $\varphi \in \mathcal{J}_C[X]$ *the operator* T^*_φ *maps the set* $C_{\varphi.c}[d,\lambda]$ *into itself.*

Proof. Assume that $\varphi \in \mathcal{J}_C[X]$ and $z \in C_{\varphi.c}[d,\lambda]$. Let $T^*_\varphi(z)$ be defined by (1.32), (1.33). It follows from the assumptions of the lemma that

$$\big\|g_i[z](\tau,t,x) - x\big\| \le \Big|\int_t^\tau \alpha_1(\xi,\mu_0)\, d\xi\Big|,$$

$$\Big|z_j(\tau, g_i[z](\tau,t,x)) - \varphi_j(0,x)\Big| \le \int_0^\tau \lambda(\xi)\, d\xi + \chi b_1 \Big|\int_t^\tau \alpha_1(\xi,\mu_0)\, d\xi\Big|,$$

$$\Big\|\frac{d}{d\tau}\,z\big(\tau,g_i[z](\tau,t,x)\big)\Big\|_\infty\le\lambda(\tau)+d_1\alpha_1(\tau,\mu_0),$$
$$\Big\|\frac{d}{d\tau}\,A^*[z](\tau,t,x)\Big\|_\infty\le p(\tau)$$

with $p(\tau)=\gamma(\tau,\mu_0)+\beta(\mu_0)(\lambda(\tau)+d_1\alpha_1(\tau,\mu_0))$. Thus

$$\|\Delta_i^*[z](t,x)\|_\infty\le\delta_{i.c},\quad i=1,2,3,$$

where

$$\delta_{1.c}=\int_0^c\alpha_2(\xi,\mu_0)\,d\xi,\quad \delta_{2.c}=\alpha(b_0)\chi b_1\int_0^c\alpha_1(\xi,\mu_0)\,d\xi,$$
$$\delta_{3.c}=\bigg(\int_0^c\lambda(\xi)\,d\xi+\chi b_1\int_0^c\alpha_1(\xi,\mu_0)\,d\xi\bigg)\int_0^c p(\tau)\,d\tau.$$

According to the above estimates, we have

$$\big\|T_\varphi^*(z)(t,x)\big\|_\infty\le\chi b_0+\alpha_0(b_0)\sum_{i=1}^3\delta_{i.c}\ \text{ on }\ [0,c]\times R^n.$$

We assume that the constant $c\in(0,a]$ is sufficiently small for

$$d_0\ge\chi b_0+\alpha_0(b_0)\sum_{i=1}^3\delta_{i.c}. \tag{1.34}$$

Then

$$\|T_\varphi^*(z)(t,x)\|_\infty\le d_0,\quad (t,x)\in[0,c]\times R^n.$$

To estimate $\|T_\varphi^*(z)(t,x)-T_\varphi^*(z)(\overline{t},\overline{x})\|_\infty$, we observe that

$$\big\|g_i[z](\tau,t,x)-g_i[z](\tau,\overline{t},\overline{x})\big\|+\big\|z_{\psi(\tau,g_i[z](\tau,t,x))}-z_{\psi(\tau,g_i[z](\tau,\overline{t},\overline{x}))}\big\|_X\le$$
$$\le\Lambda Q_c\bigg(\Big|\int_t^{\overline{t}}\alpha_1(\xi,\mu_0)\,d\xi\Big|+\|x-\overline{x}\|\bigg).$$

Write

$$d_{1.c}=\Lambda Q_c\int_0^c\beta_2(\tau,\mu_0)\,d\tau,\quad \lambda_{1.c}(\xi)=d_{1.c}\alpha_1(\xi,\mu_0)+\alpha_2(\xi,\mu_0),$$
$$d_{2.c}=\chi b_1\bigg(\alpha(b_0)(Q_c+1)+\beta(b_0)(1+d_1)Q_c\int_0^c\alpha_1(\tau,\mu_0)\,d\tau\bigg),$$
$$\lambda_{2.c}(\xi)=\chi b_1Q_c\bigg(\alpha(b_0)+\beta(b_0)(1+d_1)\int_0^c\alpha_1(\tau,\mu_0)\,d\tau\bigg)\alpha_1(\xi,\mu_0),$$

$$d_{3.c} = \Gamma_c + \chi b_1 \int_0^c p(\tau)\,d\tau,$$

$$\lambda_{3.c}(\xi) = \Big(\int_0^c \lambda(\tau)\,d\tau + \chi b_1 \int_0^c \alpha_1(\tau,\mu_0)\,d\tau\Big)p(\xi) + \Gamma_c\alpha_1(\xi,\mu_0),$$

where

$$\Gamma_c = \beta(d_0)(1+d_1)\Big((1+Q_c)\int_0^c \lambda(\tau)\,d\tau + 2d_1Q_c\int_0^c \alpha_1(\tau,\mu_0)\,d\tau\Big) + \\ + d_1Q_c\int_0^c p(\tau)\,d\tau,$$

and assume that

$$d_1 \geq \chi b_1 + \beta_0(d_0)(1+d_1)\sum_{i=1}^{3}\delta_{i.c} + \alpha_0(d_0)\sum_{i=1}^{3} d_{i.c}, \tag{1.35}$$

$$\lambda(\xi) \geq \big(\beta_0(d_0)\lambda(\xi) + \gamma_0(\xi,d_0)\big)\sum_{i=1}^{3}\delta_{i.c} + \alpha_0(d_0)\sum_{i=1}^{3}\lambda_{i.c}(\xi). \tag{1.36}$$

It follows easily that

$$\big\|T_\varphi^*(z)(t,x) - T_\varphi^*(z)(\overline{t},\overline{x})\big\|_\infty \leq \Big|\int_t^{\overline{t}} \lambda(\xi)\,d\xi\Big| + d_1\|x-\overline{x}\|.$$

In this way we have proved that $T_\varphi^* : C_{\varphi.c}[d,\lambda] \to C_{\varphi.c}[d,\lambda]$ for $c \in (0,a]$, $d = (d_0,d_1) \in R_+^2$, $\lambda \in L([0,c],R_+)$ satisfying the inequalities (1.34), (1.35) and (1.36). □

Lemma 1.8. *Suppose that the assumptions of Lemma* 1.7 *are satisfied. If $\varphi,\overline{\varphi} \in \mathcal{J}_C[X]$ and $z \in C_{\varphi.c}[d,\lambda]$, $\overline{z} \in C_{\overline{\varphi}.c}[d,\lambda]$, then there are $G_{1.c}, G_2 \in R_+$ such that*

$$\big\|T_\varphi^*(z) - T_{\overline{\varphi}}^*(\overline{z})\big\|_c \leq G_{1.c}\|z-\overline{z}\|_c + G_2\|\varphi-\overline{\varphi}\|_X^*. \tag{1.37}$$

Proof. Let $\varphi,\overline{\varphi} \in \mathcal{J}[X]$, $z \in C_{\varphi.c}[d,\lambda]$, $\overline{z} \in C_{\overline{\varphi}.c}[d,\lambda]$. We use the following estimates

$$\big\|z_{\psi(t,x)} - \overline{z}_{\psi(t,x)}\big\|_X \leq K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^*$$

and

$$\big\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\big\| + \big\|z_{\psi(\tau,g_i[z](\tau,t,x))} - \overline{z}_{\psi(\tau,g_i[\overline{z}](\tau,t,x))}\big\|_X \leq \\ \leq Q(\tau,t)\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^*\Big)$$

with

$$Q(\tau,t) = 1 + \Lambda Q_c \Big| \int_\tau^t \beta_1(\xi,\mu_0) d\xi \Big|.$$

We have

$$\big\| \Delta_i^*[z](t,x) - \Delta_i^*[\overline{z}](t,x) \big\|_\infty \le \sigma_{i.c} \|z - \overline{z}\|_c + \vartheta_{i.c} \|\varphi - \overline{\varphi}\|_X^*, \quad i = 1,2,3,$$

where

$$\sigma_{1.c} = K_1 \int_0^c \beta_2(\tau,\mu_0) Q(\tau,c)\, d\tau, \quad \vartheta_{1.c} = K_0 \int_0^c \beta_2(\tau,\mu_0) Q(\tau,c)\, d\tau,$$

$$\sigma_{2.c} = K_1 \Big(\beta(\mu_0) \chi b_1 Q(0,c) \int_0^c \alpha_1(\tau,\mu_0)\, d\tau + \alpha(\mu_0) \chi b_1 Q_c \int_0^c \beta_1(\tau,\mu_0)\, d\tau \Big),$$

$$\vartheta_{2.c} = K_0 \beta(\mu_0) \chi b_1 Q(0,c) \int_0^c \alpha_1(\tau,\mu_0)\, d\tau +$$

$$+ \alpha(\mu_0) \chi \Big(2 + K_0 b_1 Q_c \int_0^c \beta_1(\tau,\mu_0) d\tau \Big),$$

$$\sigma_{3.c} = K_1 \Gamma_c^*, \quad \vartheta_{3.c} = \chi \int_0^c \beta^*(\tau)\, d\tau + K_0 \Gamma_c^*,$$

where

$$\Gamma_c^* = d_1 Q_c \int_0^c \beta^*(\tau) \int_\tau^c \beta_1(\xi,\mu_0)\, d\xi\, d\tau +$$

$$+ \beta(\mu_0) \Big(\int_0^c \lambda(\tau)\, d\tau + \chi b_1 Q(0,c) \int_0^c \alpha_1(\tau,\mu_0)\, d\tau +$$

$$+ \int_0^c Q(\tau,c) \big(\lambda(\tau) + d_1 \alpha_1(\tau,\mu_0) \big)\, d\tau \Big).$$

Thus we obtain

$$\big\| T_\varphi^*(z) - T_{\overline{\varphi}}^*(\overline{z}) \big\|_c \le G_{1.c} \|z - \overline{z}\|_c + G_2 \|\varphi - \overline{\varphi}\|_X^*,$$

where

$$\begin{aligned} G_{1.c} &= K_1 \beta_0(\mu_0) \sum_{i=1}^3 \delta_{i.c} + \alpha_0(\mu_0) \sum_{i=1}^3 \sigma_{i.c}, \\ G_2 &= K_0 \beta_0(\mu_0) \sum_{i=1}^3 \delta_{i.c} + \alpha_0(\mu_0) \sum_{i=1}^3 \vartheta_{i.c}. \end{aligned} \tag{1.38}$$

This completes the proof of Lemma 1.8. □

We are now in a position to show a theorem on existence, uniqueness and continuous dependence on initial functions for the problem (1.24), (1.25).

Theorem 1.2. *Suppose that Assumptions* $\mathrm{H}[X]$, $\mathrm{H}_C[\psi]$, $\mathrm{H}_C[\varrho]$, $\mathrm{H}_C[f]$ *and* $\mathrm{H}_C[A]$ *are satisfied. Assume that* $c \in (0, a]$, $d = (d_0, d_1) \in R_+^2$, $\lambda \in L([0, c], R_+)$ *satisfy the inequalities* (1.34)–(1.36) *and*

$$G_{1.c} < 1, \tag{1.39}$$

where $G_{1.c}$ *is defined by* (1.38). *Then for each* $\varphi \in \mathcal{J}_C[X]$ *there exists* $z = z[\varphi] \in C_{\varphi.c}[d, \lambda]$ *which is a unique solution of* (1.24), (1.25) *in the class* $C_{\varphi.c}[d, \lambda]$. *Furthemore, if* $\varphi, \overline{\varphi} \in \mathcal{J}_C[X]$, $z = z[\varphi]$, $\overline{z} = z[\overline{\varphi}]$, *then*

$$\|z - \overline{z}\|_c \leq \frac{G_2}{1 - G_{1.c}} \|\varphi - \overline{\varphi}\|_X^* \tag{1.40}$$

with G_2 *given by* (1.38).

Proof. It follows from Lemmas 1.7 and 1.8 and from the inequalities (1.34)-(1.36), (1.39) that for each $\varphi \in \mathcal{J}_C[X]$ the operator $T_\varphi^* : C_{\varphi.c}[d, \lambda] \to C_{\varphi.c}[d, \lambda]$ is a contraction and thus it has a fixed point $z[\varphi] \in C_{\varphi.c}[d, \lambda]$. The assertion (1.40) immediately follows from Lemma 1.8. □

CHAPTER 2

Initial Problems for Nonlinear Equations

2.1. Introduction

We consider initial value problems for first order nonlinear partial differential functional equations. Suppose that B is the set defined in Chapter 1. Let X be a linear normed space of functions from B into R. Suppose that the functions

$$f : [0, a] \times R^n \times X \times R^n \to R, \quad \varphi : (-\infty, 0] \times R^n \to R,$$
$$\psi : [0, a] \times R^n \to R^{n+1}, \ \ \psi = (\psi_0, \psi'), \ \ \psi' = (\psi_1, \ldots, \psi_n),$$

are given. We assume that $\psi_0(t, x) \le t$ for $(t, x) \in [0, a] \times R^n$. Consider the nonlinear equation

$$\partial_t z(t, x) = f\big(t, x, z_{\psi(t,x)}, \partial_x z(t, x)\big) \tag{2.1}$$

with the initial condition

$$z(t, x) = \varphi(t, x), \ \ (t, x) \in (-\infty, 0] \times R^n. \tag{2.2}$$

We look for classical solutions of (2.1), (2.2). We use the notation introduced in Chapter 1. Additionally we use the symbol $\circ$ to denote the scalar product in R^n. We formulate the following assumption on the space X.

Assumption H$^*[X]$. The space $(X, \|\cdot\|_X)$ satisfies Assumption H$[X]$ given in Section 1.2 with $k = 1$.

Let us denote by $\mathcal{J}_N[X]$ the class of all initial functions $\varphi : (-\infty, 0] \times R^n \to R$ such that

1) $\varphi_{(t,x)} \in X$ for $(t, x) \in (-\infty, 0] \times R^n$, there exist the derivatives $\partial_t \varphi$, $\partial_x \varphi = (\partial_{x_1} \varphi, \ldots, \partial_{x_n} \varphi)$ on $(-\infty, 0] \times R^n$ and $(\partial_t \varphi)_{(t,x)}, (\partial_{x_i} \varphi)_{(t,x)} \in X$ for $(t, x) \in (-\infty, 0] \times R^n$, $1 \le i \le n$,
2) there are $b_1, b_2 \in R_+$ with the properties

$$\|\varphi_{(t,x)} - \varphi_{(\bar{t},\bar{x})}\|_X \le b_1\big(|t - \bar{t}| + \|x - \bar{x}\|\big),$$
$$\big\|(\partial_t \varphi)_{(t,x)} - (\partial_t \varphi)_{(\bar{t},\bar{x})}\big\|_X + \sum_{i=1}^{n} \big\|(\partial_{x_i} \varphi)_{(t,x)} - (\partial_{x_i} \varphi)_{(\bar{t},\bar{x})}\big\|_X \le$$
$$\le b_2\big(|t - \bar{t}| + \|x - \bar{x}\|\big),$$

where (t, x), $(\bar{t}, \bar{x}) \in (-\infty, 0] \times R^n$.

Fix $\varphi \in \mathcal{J}_N[X]$ and $c \in (0,a]$, d, p_0, $p_1 \in R_+$. Denote by $C^L_{\varphi.c}[d]$ the class of all functions $z : (-\infty, c] \times R^n \to R$ such that $z(t,x) = \varphi(t,x)$ for $(t,x) \in (-\infty, 0] \times R^n$ and the estimate

$$\big|z(t,x) - z(\overline{t},\overline{x})\big| \le d\big(|t-\overline{t}| + \|x-\overline{x}\|\big)$$

holds on $[0,c] \times R^n$. Write $C^L_{\partial_t\varphi.c}[p_0,p_1]$ to denote the class of all functions $u_0 : (-\infty,c]\times R^n \to R$ such that $u_0(t,x) = \partial_t\varphi(t,x)$ for $(t,x) \in (-\infty,0]\times R^n$ and

$$|u_0(t,x)| \le p_0, \quad \big|u_0(t,x) - u_0(\overline{t},\overline{x})\big| \le p_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big)$$

on $[0,c] \times R^n$. Let the symbol $C^L_{\partial_x\varphi.c}[p_0,p_1]$ denote the class of all functions $u : (-\infty,c]\times R^n \to R^n$ such that $u(t,x) = \partial_x\varphi(t,x)$ for $(t,x) \in (-\infty,0]\times R^n$ and

$$\|u(t,x)\| \le p_0, \quad \big\|u(t,x) - u(\overline{t},\overline{x})\big\| \le p_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big)$$

on $[0,c] \times R^n$. We will prove that for sufficiently small $c \in (0,a]$ there exists a solution $\overline{z}$ of the problem (2.1), (2.2) such that $\overline{z} \in C^L_{\varphi.c}[d]$, $\partial_t\overline{z} \in C^L_{\partial_t\varphi.c}[p_0,p_1]$ and $\partial_x\overline{z} \in C^L_{\partial_x\varphi.c}[p_0,p_1]$.

2.2. Bicharacteristics for Nonlinear Equations

We begin with the following assumptions.

Assumption $\mathbf{H}_N[\partial_q f]$. The function $f : [0,a] \times R^n \times X \times R^n \to R$ of the variables (t,x,w,q) is such that

1) the derivative $\partial_q f(t,x,w,q)$ exists for $(t,x,w,q) \in [0,a] \times R^n \times X \times R^n$,
2) the function $\partial_q f(\cdot,x,w,q) : [0,a] \to R^n$ is continuous and there are C, $L \in R_+$ such that

$$\|\partial_q f(t,x,w,q)\| \le C,$$

$$\big\|\partial_q f(t,x,w,q) - \partial_q f(t,\overline{x},\overline{w},\overline{q})\big\| \le L\Big(\|x-\overline{x}\| + \|w-\overline{w}\|_X + \|q-\overline{q}\|\Big)$$

for (t,x,w,q), $(t,\overline{x},\overline{w},\overline{q}) \in [0,a] \times R^n \times X \times R^n$.

Assumption $\mathbf{H}_N[\psi]$. The function $\psi : [0,a] \times R^n \to R^{n+1}$, $\psi = (\psi_0, \psi')$, $\psi' = (\psi_1,\dots,\psi_n)$, is such that $\psi_0(t,x) \le t$ for $(t,x) \in [0,a] \times R^n$ and

1) the partial derivatives $(\partial_{x_1}\psi_i,\dots,\partial_{x_n}\psi_i) = \partial_x\psi_i$, $0 \le i \le n$, exist on $[0,a] \times R^n$ and they are continuous,
2) there are s_1, $s_2 \in R_+$ with the properties

$$|\partial_{x_j}\psi_0(t,x)| + \|\partial_{x_j}\psi'(t,x)\| \le s_1,$$

$$\big|\partial_{x_j}\psi_0(t,x) - \partial_{x_j}\psi_0(t,\overline{x})\big| + \big\|\partial_{x_j}\psi'(t,x) - \partial_{x_j}\psi'(t,\overline{x})\big\| \le s_2\|x-\overline{x}\|$$

for (t,x), $(t,\overline{x}) \in [0,a] \times R^n$, $1 \le j \le n$.

Suppose that Assumptions $\mathrm{H}^*[X]$, $\mathrm{H}_N[\partial_q f]$, $\mathrm{H}_N[\psi]$ are satisfied and let $\varphi \in \mathcal{J}_N[X]$, $c \in (0,a]$, $z \in C^L_{\varphi.c}[d]$, $u \in C^L_{\partial_x\varphi.c}[p_0,p_1]$, $(t,x) \in [0,c] \times R^n$. Consider the Cauchy problem

$$\eta'(\tau) = -\partial_q f\big(\tau, \eta(\tau), z_{\psi(\tau,\eta(\tau))}, u(\tau,\eta(\tau))\big), \quad \eta(t) = x, \tag{2.3}$$

and denote by $g[z,u](\cdot,t,x)$ its solution in the classical sense. The function $g[z,u](\cdot,t,x)$ is the bicharacteristic of (2.1) corresponding to (z,u).

We prove a lemma on bicharacteristics. For $w : (-\infty,c] \times R^n \to R^n$ and $w_0 : (-\infty,c] \times R^n \to R$ we write

$$\|w\|_{t.n} = \sup\Big\{\|w(s,y)\| : \ (s,y) \in [0,t] \times R^n\Big\}, \quad 0 \le t \le c,$$

$$\|w_0\|_{t.1} = \sup\Big\{|w_0(s,y)| : \ (s,y) \in [0,t] \times R^n\Big\}, \quad 0 \le t \le c.$$

Put

$$\begin{gathered} Q_1 = (1+C)\exp(c\Lambda^* L), \quad Q_2 = L\exp(c\Lambda^* L), \\ \Lambda^* = 1 + s_1(K_1 d + K_0 b_1) + p_1. \end{gathered} \tag{2.4}$$

Lemma 2.1. *Suppose that Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_N[\partial_q f]$, $\mathrm{H}_N[\psi]$ *are satisfied. Let* $\varphi, \overline{\varphi} \in \mathcal{J}_N[X]$ *be such that* $\|\varphi - \overline{\varphi}\|^*_X < +\infty$ *and let* $z \in C^L_{\varphi.c}[d]$, $\overline{z} \in C^L_{\overline{\varphi}.c}[d]$, $u \in C^L_{\partial_x\varphi.c}[p_0,p_1]$, $\overline{u} \in C^L_{\partial_x\overline{\varphi}.c}[p_0,p_1]$, $c \in (0,a]$. *Then, for each* $(t,x) \in [0,c] \times R^n$ *the solutions* $g[z,u](\cdot,t,x)$ *and* $g[\overline{z},\overline{u}](\cdot,t,x)$ *exist on* $[0,c]$, *they are unique and they satisfy the conditions*

$$\big\|g[z,u](\tau,t,x) - g[z,u](\tau,\overline{t},\overline{x})\big\| \le Q_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big), \tag{2.5}$$

where (τ,t,x), $(\tau,\overline{t},\overline{x}) \in [0,c]^2 \times R^n$, *and*

$$\begin{gathered} \big\|g[z,u](\tau,t,x) - g[\overline{z},\overline{u}](\tau,t,x)\big\| \le \\ \le Q_2\Big|\int_\tau^t \Big(K_1\|z-\overline{z}\|_{\xi.1} + K_0\|\varphi-\overline{\varphi}\|^*_X + \|u-\overline{u}\|_{\xi.n}\Big)\,d\xi\Big|, \end{gathered} \tag{2.6}$$

where $(\tau,t,x) \in [0,c]^2 \times R^n$.

Proof. The existence and uniqueness of a classical solution of (2.3) follows from Assumption $\mathrm{H}_N[\partial_q f]$ and from the following Lipschitz condition

$$\Big|\partial_{q_i} f\big(\tau, y, z_{\psi(\tau,y)}, u(\tau,y)\big) - \partial_{q_i} f\big(\tau, \overline{y}, z_{\psi(\tau,\overline{y})}, u(\tau,\overline{y})\big)\Big| \le L\Lambda^*\|y-\overline{y}\|,$$

where $\tau \in [0,c]$, $y, \overline{y} \in R^n$. The bicharacteristics satisfy the integral equation

$$g[z,u](\tau,t,x) = x - \int_t^\tau \partial_q f\big(P[z,u](\xi,t,x)\big)\,d\xi,$$

where

$$\begin{gathered} P[z,u](\xi,t,x) = \\ = \Big(\xi, g[z,u](\xi,t,x), z_{\psi(\xi,g[z,u](\xi,t,x))}, u(\xi, g[z,u](\xi,t,x))\Big). \end{gathered} \tag{2.7}$$

Then we have the integral inequality

$$\big\|g[z,u](\tau,t,x)-g[z,u](\tau,\bar t,\overline{x})\big\| \le$$
$$\le (1{+}C)\big(|t{-}\bar t|+\|x{-}\overline{x}\|\big)+\Big|\int_t^\tau L\Lambda^*\Big\|g[z,u](\xi,t,x)-g[z,u](\xi,\bar t,\overline{x})\Big\|\,d\xi\Big|$$

for (τ,t,x), $(\tau,\bar t,\overline{x})\in[0,c]^2\times R^n$. Using the Gronwall inequality, we obtain (2.5).

We have

$$\big|z(t,x)-\overline{z}(t,x)\big| \le |z(t,x)-z(0,x)|+$$
$$+\big|\varphi(0,x)-\overline{\varphi}(0,x)\big|+\big|\overline{z}(0,x)-\overline{z}(t,x)\big|\le 2dc+\chi\|\varphi-\overline{\varphi}\|_X^*$$

on $[0,c]\times R^n$. Thus $\|z-\overline{z}\|_{t.1}<+\infty$, $t\in[0,c]$, and the following integral inequality

$$\Big\|g[z,u](\tau,t,x)-g[\overline{z},\overline{u}](\tau,t,x)\Big\|\le$$
$$\le\Big|\int_t^\tau L\Big(\Lambda^*\big\|g[z,u](\xi,t,x)-g[\overline{z},\overline{u}](\xi,t,x)\big\|+$$
$$+K_1\|z-\overline{z}\|_{\xi.1}+K_0\|\varphi-\overline{\varphi}\|_X^*+\|u-\overline{u}\|_{\xi.n}\Big)\,d\xi\Big|$$

holds for $(\tau,t,x)\in[0,c]^2\times R^n$. The assertion (2.6) follows from the Gronwall inequality. This completes the proof of Lemma 2.1. □

2.3. The Sequence of Successive Approximations

We formulate further assumptions on f. We denote by $CL(X,R)$ the set of all linear continuous functions from X into R and by $\|\cdot\|_*$ the norm in the space $CL(X,R)$.

Assumption $\mathbf{H}_N[f]$. The function $f:[0,a]\times R^n\times X\times R^n\to R$ satisfies Assumption $\mathrm{H}_N[\partial_q f]$ and

1) there is $\widetilde{C}\in R_+$ such that $|f(t,x,w,q)|\le\widetilde{C}$ on $[0,a]\times R^n\times X\times R^n$ and
$$\big|f(t,x,w,q)-f(\bar t,x,w,q)\big|\le C|t-\bar t|,$$
where (t,x,w,q), $(\bar t,x,w,q)\in[0,a]\times R^n\times X\times R^n$,
2) the derivative $\partial_x f(t,x,w,q)$ and the Fréchet derivative $\partial_w f(t,x,w,q)\in CL(X,R)$ exist for $(t,x,w,q)\in[0,a]\times R^n\times X\times R^n$,
3) the estimates
$$\|\partial_x f(t,x,w,q)\|\le C,\quad \|\partial_w f(t,x,w,q)\|_*\le C$$
and the Lipschitz conditions
$$\big\|\partial_x f(t,x,w,q)-\partial_x f(t,\overline{x},\overline{w},\overline{q})\big\|\le L\Big(\|x-\overline{x}\|+\|w-\overline{w}\|_X+\|q-\overline{q}\|\Big),$$

$$\big\|\partial_w f(t,x,w,q) - \partial_w f(t,\overline{x},\overline{w},\overline{q})\big\|_* \le L\Big(\|x-\overline{x}\| + \|w-\overline{w}\|_X + \|q-\overline{q}\|\Big)$$

are satisfied for (t,x,w,q), $(t,\overline{x},\overline{w},\overline{q}) \in [0,a] \times R^n \times X \times R^n$.

If $\omega = (\omega_1,\ldots,\omega_n)$ with $\omega_i \in X$, $1 \le i \le n$, and $(t,x,w,q) \in [0,a] \times R^n \times X \times R^n$, then we write

$$\partial_w f(t,x,w,q)(\omega) = \Big(\partial_w f(t,x,w,q)\omega_1,\ldots,\partial_w f(t,x,w,q)\omega_n\Big).$$

For $\varphi \in \mathcal{J}_N[X]$ and $z \in C^L_{\varphi.c}[d]$, $u,v \in C^L_{\partial_x\varphi.c}[p_0,p_1]$, $v_0 \in C^L_{\partial_t\varphi.c}[p_0,p_1]$ with $c \in (0,a]$, we define

$$F[z,u] : [0,c] \times R^n \to R, \quad G[z,u,v_0,v] : [0,c] \times R^n \to R^n$$

in the following way

$$F[z,u](t,x) = \varphi\big(0, g[z,u](0,t,x)\big) +$$

$$+\int_0^t \Big[f\big(P[z,u](\tau,t,x)\big) - \partial_q f\big(P[z,u](\tau,t,x)\big) \circ u\big(\tau, g[z,u](\tau,t,x)\big)\Big]\, d\tau, \quad (2.8)$$

$$G[z,u,v_0,v](t,x) = \partial_x\varphi\big(0, g[z,u](0,t,x)\big) +$$

$$+\int_0^t \Big[\partial_x f\big(P[z,u](\tau,t,x)\big) + \partial_w f\big(P[z,u](\tau,t,x)\big)\big(W[z,u,v_0,v](\tau,t,x)\big)\Big]\, d\tau, \quad (2.9)$$

where $P[z,u](\cdot,t,x)$ is given by (2.7) and

$$W[z,u,v_0,v](\tau,t,x) = \Big(W_1[z,u,v_0,v](\tau,t,x),\ldots,W_n[z,u,v_0,v](\tau,t,x)\Big),$$

$$W_i[z,u,v_0,v](\tau,t,x) = \sum_{j=0}^{n} \partial_{x_i}\psi_j\big(\tau, g[z,u](\tau,t,x)\big)(v_j)_{\psi(\tau,g[z,u](\tau,t,x))},$$

where $1 \le i \le n$, $v = (v_1,\ldots,v_n)$. We define the sequences $\{z^{(m)}\}$, $\{u_0^{(m)}\}$ and $\{u^{(m)}\}$, where $z^{(m)}, u_0^{(m)} : (-\infty,c] \times R^n \to R$, $u^{(m)} : (-\infty,c] \times R^n \to R^n$, as follows. Let $\widetilde{\varphi} : (-\infty,c] \times R^n \to R$ be an extention of φ such that $\widetilde{\varphi} \in C^L_{\varphi.c}[d]$, $\partial_t\widetilde{\varphi} \in C^L_{\partial_t\varphi.c}[p_0,p_1]$, $\partial_x\widetilde{\varphi} \in C^L_{\partial_x\varphi.c}[p_0,p_1]$. Put

$$z^{(0)} = \widetilde{\varphi}, \quad u_0^{(0)} = \partial_t\widetilde{\varphi} \text{ and } u^{(0)} = \partial_x\widetilde{\varphi} \text{ on } (-\infty,c] \times R^n.$$

Suppose that $z^{(m)} \in C^L_{\varphi.c}[d]$, $u_0^{(m)} \in C^L_{\partial_t\varphi.c}[p_0,p_1]$ and $u^{(m)} \in C^L_{\partial_x\varphi.c}[p_0,p_1]$ are known functions. Then

1) the function $u^{(m+1)}$ is a solution of the problem

$$u = G\big[z^{(m)}, u, u_0^{(m)}, u^{(m)}\big], \quad u = \partial_x\varphi \text{ on } (-\infty,0] \times R^n, \qquad (2.10)$$

2) the functions $z^{(m+1)}$ and $u_0^{(m+1)}$ are given by

$$\begin{aligned} &z^{(m+1)} = F[z^{(m)}, u^{(m+1)}], \quad z^{(m+1)} = \varphi \text{ on } (-\infty,0] \times R^n, \\ &u_0^{(m+1)} = f\big(\cdot, (z^{(m)})_{\psi(\cdot)}, u^{(m+1)}(\cdot)\big), \quad u_0^{(m+1)} = \partial_t\varphi \text{ on } (-\infty,0] \times R^n. \end{aligned} \qquad (2.11)$$

Remark 2.1. The above defined sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ can be called the sequence of succesive approximations for the system of functional integral equations

$$z = F[z,u], \quad u = G[z,u,u_0,u] \ \text{ on } \ [0,c]\times R^n, \tag{2.12}$$

where $u_0(t,x) = f(t,x,z_{\psi(t,x)},u(t,x))$ for $(t,x)\in[0,c]\times R^n$ and $u_0(t,x) = \partial_t\varphi(t,x)$ for $(t,x)\in(-\infty,0]\times R^n$, with the initial conditions

$$z = \varphi, \quad u = \partial_x\varphi \ \text{ on } \ (-\infty,0]\times R^n.$$

This problem is obtained in the following way. We introduce an unknown function u, where $u = \partial_x z$ and consider the linearization of (2.1)

$$\partial_t z(t,x) = f\big(t,x,z_{\psi(t,x)},u(t,x)\big) + \\ +\partial_q f\big(t,x,z_{\psi(t,x)},u(t,x)\big)\circ\big(\partial_x z(t,x) - u(t,x)\big). \tag{2.13}$$

By virtue of (2.1) we get the following differential system for the unknown function u

$$\partial_t u(t,x) = \partial_x f\big(t,x,z_{\psi(t,x)},u(t,x)\big) + \partial_q f\big(t,x,z_{\psi(t,x)},u(t,x)\big)\circ\partial_x u(t,x) + \\ +\partial_w f\big(t,x,z_{\psi(t,x)},u(t,x)\big)(V(t,x)), \tag{2.14}$$

where $V = (V_1,\dots,V_n)$ and

$$V_i(t,x) = \partial_{x_i}\psi_0(t,x)(\partial_t z)_{\psi(t,x)} + \sum_{j=1}^{n}\partial_{x_i}\psi_j(t,x)(\partial_{x_j}z)_{\psi(t,x)}, \ \ 1\le i\le n.$$

Finally we put $\partial_x z = u$ and $\partial_t z = u_0$ in (2.14) and we consider (2.13), (2.14) along the bicharacteristics $g[z,u](\cdot,t,x)$. In this way we obtain

$$\frac{d}{d\tau}\, z\big(\tau,g[z,u](\tau,t,x)\big) = \\ = f\big(P[z,u](\tau,t,x)\big) - \partial_q f\big(P[z,u](\tau,t,x)\big)\circ u\big(\tau,g[z,u](\tau,t,x)\big), \\ \frac{d}{d\tau}\, u\big(\tau,g[z,u](\tau,t,x)\big) = \\ = \partial_x f\big(P[z,u](\tau,t,x)\big) + \partial_w f\big(P[z,u](\tau,t,x)\big)\big(W[z,u,u_0,u](\tau,t,x)\big).$$

By integrating from 0 to t with respect to τ we get (2.12).

We formulate the lemmas on existence of the above defined sequences $\{z^{(m)}\}$, $\{u_0^{(m)}\}$ and $\{u^{(m)}\}$. We need the following assumption on the constants c, d, p_0, p_1. Write

$$\lambda_0 = K_1\max\{\widetilde{C},p_0\} + K_0 b_1, \quad \lambda_1 = K_1\max\big\{(1+\Lambda^*)C,p_1\big\} + K_0 b_2, \\ L_f = L\Lambda^* Q_1, \quad L_\varphi = \chi b_2 Q_1, \quad L_w = Q_1(s_2\lambda_0 + s_1^2\lambda_1).$$

Assumption H$[c,d,p_0,p_1]$. The constants $c\in(0,a]$, d, p_0, $p_1\in R_+$ satisfy the conditions

$$p_0 = d \ge \max\Big\{\chi b_1 + cC(1+s_1\lambda_0),$$

$$\chi b_1 Q_1 + \widetilde{C} + Cp_0 + c\big((C + Lp_0)\Lambda^* + CQ_1 p_1\big)\Big\}, \qquad (2.15)$$

$$p_1 \geq \max\Big\{C(1+\Lambda^*), L_\varphi + c\big(L_f + L_f s_1 \lambda_0 + CL_w\big)\Big\}. \qquad (2.16)$$

If $m \geq 1$ is fixed, and the functions $z^{(m)} \in C^L_{\varphi.c}[d]$, $u_0^{(m)} \in C^L_{\partial_t\varphi.c}[p_0, p_1]$ and $u^{(m)} \in C^L_{\partial_x\varphi.c}[p_0, p_1]$ are known, then we write

$$G^{(m)}[u] = G\big[z^{(m)}, u, u_0^{(m)}, u^{(m)}\big], \;\; u \in C^L_{\partial_x\varphi.c}[p_0, p_1]. \qquad (2.17)$$

Lemma 2.2. *If Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_N[f]$, $\mathrm{H}_N[\psi]$, $\mathrm{H}[c, d, p_0, p_1]$ *are satisfied and* $\varphi \in \mathcal{J}_N[X]$, *then* $G^{(m)} : C^L_{\partial_x\varphi.c}[p_0, p_1] \to C^L_{\partial_x\varphi.c}[p_0, p_1]$. *Moreover, there exists exactly one function* $\widetilde{u} \in C^L_{\partial_x\varphi.c}[p_0, p_1]$ *satisfying the equation* $u = G^{(m)}[u]$.

Proof. Let $u \in C^L_{\partial_x\varphi.c}[p_0, p_1]$. We prove that

$$\big\|G^{(m)}[u](t,x)\big\| \leq p_0, \;\; (t,x) \in [0,c] \times R^n \qquad (2.18)$$

and

$$\big\|G^{(m)}[u](t,x) - G^{(m)}[u](\overline{t},\overline{x})\big\| \leq p_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big), \qquad (2.19)$$

where (t,x), $(\overline{t},\overline{x}) \in [0,c] \times R^n$. It follows from the assumptions that

$$\big\|G^{(m)}[u](t,x)\big\| \leq \chi b_1 + cC(1 + s_1\lambda_0) \;\text{ on }\; [0,c] \times R^n,$$

and according to (2.15) we get (2.18).

Let $w^{(m)}[u](\tau,t,x) \in X^n$ be given by

$$w^{(m)}[u](\tau,t,x) = W\big[z^{(m)}, u, u_0^{(m)}, u^{(m)}\big](\tau,t,x).$$

Suppose that (t,x), $(\overline{t},\overline{x}) \in [0,c] \times R^n$. The terms

$$\Big\|\partial_x f\big(P[z^{(m)},u](\tau,t,x)\big) - \partial_x f\big(P[z^{(m)},u](\tau,\overline{t},\overline{x})\big)\Big\|,$$
$$\Big\|\partial_w f\big(P[z^{(m)},u](\tau,t,x)\big) - \partial_w f\big(P[z^{(m)},u](\tau,\overline{t},\overline{x})\big)\Big\|_*$$

are bounded from above by $L_f(|t-\overline{t}| + \|x-\overline{x}\|)$. We have also

$$\Big\|\partial_x\varphi\big(0, g[z^{(m)},u](0,t,x)\big) - \partial_x\varphi\big(0, g[z^{(m)},u](0,\overline{t},\overline{x})\big)\Big\| \leq$$
$$\leq L_\varphi\big(|t-\overline{t}| + \|x-\overline{x}\|\big),$$
$$\big\|w^{(m)}[u](\tau,t,x) - w^{(m)}[u](\tau,\overline{t},\overline{x})\big\|_X \leq L_w\big(|t-\overline{t}| + \|x-\overline{x}\|\big).$$

Thus we obtain (2.19) under the assumption (2.16). This proves that $G^{(m)}[u] \in C^L_{\partial_x\varphi.c}[p_0, p_1]$.

There is $\widetilde{\gamma} > 0$ such that for $u, \overline{u} \in C^L_{\partial_x\varphi.c}[p_0, p_1]$ we have

$$\big\|G^{(m)}[u](t,x) - G^{(m)}[\overline{u}](t,x)\big\| \leq \widetilde{\gamma}\int_0^t \|u-\overline{u}\|_{\xi.n}\, d\xi, \;\; (t,x) \in [0,c] \times R^n.$$

For $u \in C^L_{\partial_x\varphi.c}[p_0,p_1]$ and for $\lambda > \widetilde{\gamma}$ we define

$$\|u\|_{(\lambda)} = \max\Big\{\|u(t,x)\|e^{-\lambda t} : \ (t,x) \in [0,c] \times R^n\Big\}.$$

If $u, \overline{u} \in C^L_{\partial_x\varphi.c}[p_0,p_1]$, then

$$\big\|G^{(m)}[u](t,x) - G^{(m)}[\overline{u}](t,x)\big\| \le \widetilde{\gamma}\int_0^t \|u-\overline{u}\|_{(\lambda)} e^{\lambda\xi}\, d\xi \le \frac{\widetilde{\gamma}}{\lambda}\,\|u-\overline{u}\|_{(\lambda)}\, e^{\lambda t},$$

that is,

$$\big\|G^{(m)}[u] - G^{(m)}[\overline{u}]\big\|_{(\lambda)} \le \frac{\widetilde{\gamma}}{\lambda}\,\|u-\overline{u}\|_{(\lambda)}.$$

We have $\frac{\widetilde{\gamma}}{\lambda} < 1$ and by the Banach fixed point theorem there exists exactly one $\widetilde{u} \in C^L_{\partial_x\varphi.c}[p_0,p_1]$ satisfying the equation $u = G^{(m)}[u]$. The proof of Lemma 2.2 is complete. □

The following lemma is important in our considerations.

Lemma 2.3. *If Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_N[f]$, $\mathrm{H}_N[\psi]$, $\mathrm{H}[c,d,p_0,p_1]$ *are satisfied,* $\varphi \in \mathcal{J}_N[X]$*, then for any* $m \ge 0$ *we have*

$$\partial_t z^{(m)}(t,x) = u_0^{(m)}(t,x), \quad \partial_x z^{(m)}(t,x) = u^{(m)}(t,x) \ \ \textit{on} \ \ [0,c] \times R^n \tag{2.20}$$

and

$$z^{(m)} \in C^L_{\varphi.c}[d], \quad u_0^{(m)} \in C^L_{\partial_t\varphi.c}[p_0,p_1]. \tag{2.21}$$

Proof. First we prove (2.20) by induction. It follows from the definition of $z^{(0)}$, $u_0^{(0)}$, $u^{(0)}$ that (2.20) is satisfied for $m = 0$. Suppose that (2.20) holds for a given $m \ge 0$. We will prove that for $z^{(m+1)}$ given by (2.11) the following equalities

$$\partial_t z^{(m+1)} = u_0^{(m+1)}, \quad \partial_x z^{(m+1)} = u^{(m+1)} \ \ \text{on} \ \ [0,c] \times R^n \tag{2.22}$$

are true. Write

$$\Delta(t,\overline{t},x,\overline{x}) = $$
$$= z^{(m+1)}(\overline{t},\overline{x}) - z^{(m+1)}(t,x) - u_0^{(m+1)}(t,x)(\overline{t}-t) - u^{(m+1)}(t,x) \circ (\overline{x}-x),$$

where (t,x), $(\overline{t},\overline{x}) \in [0,c] \times R^n$. We prove that there exists $C_0 \in R_+$ such that

$$\big|\Delta(t,\overline{t},x,\overline{x})\big| \le C_0\big(|\overline{t}-t| + \|\overline{x}-x\|\big)^2. \tag{2.23}$$

According to (2.10), (2.11) and (2.17) we have

$$\Delta(t,\overline{t},x,\overline{x}) = F\big[z^{(m)},u^{(m+1)}\big](\overline{t},\overline{x}) - F\big[z^{(m)},u^{(m+1)}\big](t,x) -$$
$$- u_0^{(m+1)}(t,x)(\overline{t}-t) - G^{(m)}[u^{(m+1)}](t,x) \circ (\overline{x}-x).$$

For simplicity write

$$\begin{aligned} g(\tau,t,x) = g\big[z^{(m)},u^{(m+1)}\big](\tau,t,x), \quad w(\tau,t,x) = w^{(m)}[u^{(m+1)}](\tau,t,x),\\ Q(t,x) = (0, g(0,t,x)), \quad P(\tau,t,x) = P\big[z^{(m)},u^{(m+1)}\big](\tau,t,x). \end{aligned} \tag{2.24}$$

Let $R(s,\tau,t,\bar{t},x,\overline{x})$ be the following intermediate point

$$R(s,\tau,t,\bar{t},x,\overline{x}) = P(\tau,t,x) + s\big(P(\tau,\bar{t},\overline{x}) - P(\tau,t,x)\big), \quad 0 \le s \le 1.$$

Fix (t,x), $(\bar{t},\overline{x}) \in [0,c] \times R^n$. To formulate the properties of Δ, we define

$$\mathcal{A}(t,\bar{t},x,\overline{x}) = \varphi(Q(\bar{t},\overline{x})) - \varphi(Q(t,x)) - \partial_x\varphi(Q(t,x)) \circ \big(g(0,\bar{t},\overline{x}) - g(0,t,x)\big),$$

$$\mathcal{B}(t,\bar{t},x,\overline{x}) = \partial_x\varphi(Q(t,x)) \circ \big(g(0,\bar{t},\overline{x}) - g(0,t,x) - (\overline{x} - x)\big),$$

$$\delta_{f.x}(s,\tau,t,\bar{t},x,\overline{x}) = \partial_x f\big(R(s,\tau,t,\bar{t},x,\overline{x})\big) - \partial_x f(P(\tau,t,x)),$$

$$\delta_{f.w}(s,\tau,t,\bar{t},x,\overline{x}) = \partial_w f\big(R(s,\tau,t,\bar{t},x,\overline{x})\big) - \partial_w f(P(\tau,t,x)),$$

$$\delta_{f.q}(s,\tau,t,\bar{t},x,\overline{x}) = \partial_q f\big(R(s,\tau,t,\bar{t},x,\overline{x})\big) - \partial_q f\big(P(\tau,\bar{t},\overline{x})\big).$$

We have

$$\Delta(t,\bar{t},x,\overline{x}) = \Delta_1(t,\bar{t},x,\overline{x}) + \Delta_2(t,\bar{t},x,\overline{x}),$$

where

$$\Delta_1(t,\bar{t},x,\overline{x}) = \mathcal{A}(t,\bar{t},x,\overline{x}) +$$
$$+ \int_0^t \int_0^1 \Big[\delta_{f.x}(s,\tau,t,\bar{t},x,\overline{x}) \circ \big(g(\tau,\bar{t},\overline{x}) - g(\tau,t,x)\big) +$$
$$+ \delta_{f.w}(s,\tau,t,\bar{t},x,\overline{x})\Big((z^{(m)})_{\psi(\tau,g(\tau,\bar{t},\overline{x}))} - (z^{(m)})_{\psi(\tau,g(\tau,t,x))}\Big) +$$
$$+ \delta_{f.q}(s,\tau,t,\bar{t},x,\overline{x}) \circ \Big(u^{(m+1)}(\tau,g(\tau,\bar{t},\overline{x})) - u^{(m+1)}(\tau,g(\tau,t,x))\Big)\Big]\, ds\, d\tau +$$
$$+ \int_0^t \partial_w f(P(\tau,t,x))\Big((z^{(m)})_{\psi(\tau,g(\tau,\bar{t},\overline{x}))} - (z^{(m)})_{\psi(\tau,g(\tau,t,x))} -$$
$$- w(\tau,t,x) \circ \big(g(\tau,\bar{t},\overline{x}) - g(\tau,t,x)\big)\Big)\, d\tau$$

and

$$\Delta_2(t,\bar{t},x,\overline{x}) = \mathcal{B}(t,\bar{t},x,\overline{x}) +$$
$$+ \int_0^t \Big[\Big(\partial_x f(P(\tau,t,x)) + \partial_w f(P(\tau,t,x))(w(\tau,t,x))\Big) \circ$$
$$\circ \big(g(\tau,\bar{t},\overline{x}) - g(\tau,t,x) - (\overline{x} - x)\big) -$$
$$- \big(\partial_q f(P(\tau,\bar{t},\overline{x})) - \partial_q f(P(\tau,t,x))\big) \circ u^{(m+1)}(\tau,g(\tau,t,x))\Big]\, d\tau +$$
$$+ \int_t^{\bar{t}} \Big(f(P(\tau,\bar{t},\overline{x})) - \partial_q f(P(\tau,\bar{t},\overline{x})) \circ u^{(m+1)}(\tau,g(\tau,\bar{t},\overline{x}))\Big)\, d\tau -$$
$$- u_0^{(m+1)}(t,x)(\bar{t} - t).$$

Substituting the relation

$$g(\tau,\overline{t},\overline{x})-g(\tau,t,x)-(\overline{x}-x)=$$
$$=\int_{\tau}^{t}\big(\partial_q f(P(\tau,\overline{t},\overline{x}))-\partial_q f(P(\tau,t,x))\big)\,d\tau+\int_{t}^{\overline{t}}\partial_q f(P(\tau,\overline{t},\overline{x}))\,d\tau$$

into $\Delta_2(t,\overline{t},x,\overline{x})$ and changing the order of integration, we obtain

$$\Delta_2(t,\overline{t},x,\overline{x})=\mathcal{C}(t,\overline{t},x,\overline{x})+$$
$$+\int_0^t\Big(\partial_q f(P(\tau,\overline{t},\overline{x}))-\partial_q f(P(\tau,t,x))\Big)\circ\mathcal{D}(\tau,t,x)\,d\tau,$$

where

$$\mathcal{C}(t,\overline{t},x,\overline{x})=\int_t^{\overline{t}}\Big[\partial_q f(P(\tau,\overline{t},\overline{x}))\circ$$
$$\circ\Big(u^{(m+1)}(\tau,g(\tau,t,x))-u^{(m+1)}(\tau,g(\tau,\overline{t},\overline{x}))\Big)+f\big(P(\tau,\overline{t},\overline{x})\big)-u_0^{(m+1)}(t,x)\Big]\,d\tau,$$

$$\mathcal{D}(\tau,t,x)=-u^{(m+1)}(\tau,g(\tau,t,x))+\partial_x\varphi(Q(t,x))+$$
$$+\int_0^t\Big(\partial_x f(P(\xi,t,x))+\partial_w f(P(\xi,t,x))(w(\xi,t,x))\Big)\,d\xi.$$

Since $g(s,\tau,g(\tau,t,x))=g(s,t,x)$ for $(\tau,t,x)\in[0,c]^2\times R^n$, we have

$$u^{(m+1)}(\tau,g(\tau,t,x))=\partial_x\varphi(Q(t,x))+$$
$$+\int_0^{\tau}\Big(\partial_x f(P(s,t,x))+\partial_w f(P(s,t,x))(w(s,t,x))\Big)\,ds.$$

Thus

$$\mathcal{D}(\tau,t,x)=0,\ \ (\tau,t,x)\in[0,c]^2\times R^n.$$

Consequently,

$$\Delta_2(t,\overline{t},x,\overline{x})=\mathcal{C}(t,\overline{t},x,\overline{x})\ \text{ for }\ (t,x),(\overline{t},\overline{x})\in[0,c]\times R^n,$$

and there is $C_2\in R_+$ such that

$$\big|\Delta_2(t,\overline{t},x,\overline{x})\big|\le C_2\big(|\overline{t}-t|+\|\overline{x}-x\|\big)^2,\ \ (t,x),\ (\overline{t},\overline{x})\in[0,c]\times R^n. \quad (2.25)$$

To estimate $\Delta_1(t,\overline{t},x,\overline{x})$, we note that there exists $C_{\mathcal{A}}\in R_+$ such that

$$\big|\mathcal{A}(t,\overline{t},x,\overline{x})\big|\le C_{\mathcal{A}}\big(|\overline{t}-t|+\|\overline{x}-x\|\big)^2.$$

Moreover, the terms

$$\big\|\delta_{f.x}(s,\tau,t,\overline{t},x,\overline{x})\big\|,\ \ \big\|\delta_{f.q}(s,\tau,t,\overline{t},x,\overline{x})\big\|,\ \ \big\|\delta_{f.w}(s,\tau,t,\overline{t},x,\overline{x})\big\|_*$$

are bounded from above by $L_\delta \|g(\tau,\bar t,\bar x) - g(\tau,t,x)\|$ for some $L_\delta \in R_+$. We have also

$$\Big\|(z^{(m)})_{\psi(\tau,g(\tau,\bar t,\bar x))} - (z^{(m)})_{\psi(\tau,g(\tau,t,x))}\Big\|_X \le$$
$$\le s_1(K_1 d + K_0 b_1)\big\|g(\tau,\bar t,\bar x) - g(\tau,t,x)\big\|,$$
$$\big\|u^{(m+1)}(\tau,g(\tau,\bar t,\bar x)) - u^{(m+1)}(\tau,g(\tau,t,x))\big\| \le p_1\big\|g(\tau,\bar t,\bar x) - g(\tau,t,x)\big\|.$$

It follows from the equalities $\partial_t z^{(m)} = u_0^{(m)}$, $\partial_x z^{(m)} = u^{(m)}$ on $[0,c]\times R^n$ that

$$\Big\|(z^{(m)})_{\psi(\tau,g(\tau,\bar t,\bar x))} - (z^{(m)})_{\psi(\tau,g(\tau,t,x))} -$$
$$- w(\tau,t,x)\circ\big(g(\tau,\bar t,\bar x) - g(\tau,t,x)\big)\Big\|_X \le$$
$$\le \Big(s_1^2(K_1 p_1 + K_0 b_2) + s_2(K_1 d + K_0 b_1)\Big)\big\|g(\tau,\bar t,\bar x) - g(\tau,t,x)\big\|^2.$$

All the above estimates together with properties of bicharacteristics imply that there is $C_1 \in R_+$ such that

$$\big|\Delta_1(t,\bar t,x,\bar x)\big| \le C_1\big(|\bar t - t| + \|\bar x - x\|\big)^2,\quad (t,x),\ (\bar t,\bar x)\in[0,c]\times R^n. \quad (2.26)$$

The relations (2.25) and (2.26) give (2.23) and consequently on $[0,c]\times R^n$

$$\partial_t z^{(m+1)}(t,x) = u_0^{(m+1)}(t,x),\quad \partial_x z^{(m+1)}(t,x) = u^{(m+1)}(t,x).$$

The proof of (2.22) is complete. □

Now we prove that $z^{(m+1)} \in C^L_{\varphi.c}[d]$, where $z^{(m+1)}$ is given by (2.11). It follows from (2.20) and from Assumption H$[c,d,p_0,p_1]$ that on $[0,c]\times R^n$

$$\big\|\partial_x z^{(m+1)}(t,x)\big\| \le d.$$

Let $(t,x), (\bar t,x) \in [0,c]\times R^n$. We use the notation (2.24) and we can write the following estimates

$$\big|\varphi(Q(t,x)) - \varphi(Q(\bar t,x))\big| \le \chi b_1 Q_1 |t-\bar t|,$$

$$\Big|\int_{\bar t}^{t}\Big|f(P(\tau,t,x)) - \partial_q f(P(\tau,t,x))\circ u^{(m+1)}(\tau,g(\tau,t,x))\Big|\,d\tau\Big| \le$$
$$\le (\widetilde C + Cp_0)|t-\bar t|,$$

$$\int_0^{\bar t}\Big(\big|f(P(\tau,t,x)) - f(P(\tau,\bar t,x))\big| + \Big|\partial_q f(P(\tau,t,x))\circ u^{(m+1)}(\tau,g(\tau,t,x)) -$$
$$-\partial_q f(P(\tau,\bar t,x))\circ u^{(m+1)}(\tau,g(\tau,\bar t,x))\Big|\Big)\,d\tau \le$$
$$\le c\big((C + Lp_0)\Lambda^* + CQ_1 p_1\big)|t-\bar t|.$$

It follows from Assumption H$[c, d, p_0, p_1]$ that

$$\big|z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},x)\big| \le d|t-\bar{t}|.$$

Since $|u_0^{(m+1)}(t,x)| \le \widetilde{C}$ and

$$\big|u_0^{(m+1)}(t,x) - u_0^{(m+1)}(\bar{t},\bar{x})\big| \le (1+\Lambda^*)C\big(|t-\bar{t}| + \|x-\bar{x}\|\big)$$

on $[0,c]\times R^n$, we have

$$\big|u_0^{(m+1)}(t,x)\big| \le p_0, \quad \big|u_0^{(m+1)}(t,x) - u_0^{(m+1)}(\bar{t},\bar{x})\big| \le p_1\big(|t-\bar{t}| + \|x-\bar{x}\|\big)$$

on $[0,c]\times R^n$. The above estimates prove that $u_0^{(m+1)} \in C^L_{\partial_t\varphi.c}[p_0,p_1]$. This completes the proof of Lemma 2.3.

2.4. Existence and Uniqueness of Classical Solutions

We prove the convergence of the sequences $\{z^{(m)}\}$, $\{u_0^{(m)}\}$ and $\{u^{(m)}\}$.

Lemma 2.4. *If Assumptions* H$^*[X]$, H$_N[f]$, H$_N[\psi]$, H$[c,d,p_0,p_1]$ *are satisfied and* $\varphi \in \mathcal{J}_N[X]$, *then the sequences* $\{z^{(m)}\}$, $\{u_0^{(m)}\}$ *and* $\{u^{(m)}\}$ *are uniformly convergent on* $[0,c]\times R^n$.

Proof. For $(t,x)\in[0,c]\times R^n$, $m\ge 1$, we have the following estimates

$$\big|z^{(m)}(t,x) - z^{(m-1)}(t,x)\big| \le$$
$$\le \big|z^{(m)}(t,x) - z^{(m)}(0,x)\big| + \big|z^{(m-1)}(0,x) - z^{(m-1)}(t,x)\big| \le 2dc,$$
$$\big\|u^{(m)}(t,x) - u^{(m-1)}(t,x)\big\| \le$$
$$\le \big\|u^{(m)}(t,x) - u^{(m)}(0,x)\big\| + \big\|u^{(m-1)}(0,x) - u^{(m-1)}(t,x)\big\| \le 2p_1c.$$

Thus for $t\in[0,c]$ and $m\ge1$ we can write

$$Z_m(t) = \|z^{(m)} - z^{(m-1)}\|_{t.1} \ \text{ and } \ U_m(t) = \|u^{(m)} - u^{(m-1)}\|_{t.n}.$$

The assumptions of Lemma 2.4 imply the inequality

$$U_{m+1}(t) \le \widetilde{\Gamma}_1 \int_0^t \big(K_1 Z_m(\tau) + U_{m+1}(\tau)\big)\,d\tau +$$
$$+ K_1 C s_1 \int_0^t \Big((1+C)U_m(\tau) + CK_1 d Z_{m-1}(\tau)\Big)\,d\tau,$$

where $t\in[0,c]$ for some $\widetilde{\Gamma}_1 \in R_+$ independent of m. The above estimate and the Gronwall inequality yield

$$U_{m+1}(t) \le$$
$$\le e^{c\widetilde{\Gamma}_1} \int_0^t \Big[s_1 C^2 K_1^2 Z_{m-1}(\tau) + \widetilde{\Gamma}_1 K_1 Z_m(\tau) + s_1 K_1 C(1+C)U_m(\tau)\Big]\,d\tau. \quad (2.27)$$

An easy computation shows that there is $\widetilde{\Gamma}_2 \in R_+$ such that

$$Z_{m+1}(t) \leq \widetilde{\Gamma}_2 \int_0^t \big(K_1 Z_m(\tau) + U_{m+1}(\tau)\big)\,d\tau + C\int_0^t U_{m+1}(\tau)\,d\tau. \qquad (2.28)$$

The inequalities (2.27) and (2.28) yield

$$Z_{m+1}(t)+U_{m+1}(t) \leq A_1 \int_0^t \big(Z_m(\tau) + U_m(\tau)\big)\,d\tau + A_2\int_0^t Z_{m-1}(\tau)\,d\tau \qquad (2.29)$$

for some $A_1, A_2 \in R_+$ independent of m, $t \in [0,c]$. For $Y \in C([0,c],R)$ and for $\lambda > A_1 + A_2$ we put

$$\|Y\|_\lambda = \max\big\{|Y(t)|e^{-\lambda t} :\ t \in [0,c]\big\}.$$

It follows from (2.29) that

$$\begin{aligned}Z_{m+1}(t)+U_{m+1}(t) &\leq A_1 \int_0^t \|Z_m+U_m\|_\lambda\, e^{\lambda\tau}\,d\tau + A_2\int_0^t \|Z_{m-1}\|_\lambda\, e^{\lambda\tau}\,d\tau \leq \\ &\leq \Big(\frac{A_1}{\lambda}\,\|Z_m+U_m\|_\lambda + \frac{A_2}{\lambda}\,\|Z_{m-1}\|_\lambda\Big)e^{\lambda t},\ \ t \in [0,c],\end{aligned}$$

that is,

$$\|Z_{m+1}+U_{m+1}\|_\lambda \leq \frac{A_1}{\lambda}\,\|Z_m+U_m\|_\lambda + \frac{A_2}{\lambda}\,\|Z_{m-1}\|_\lambda,\ \ m \geq 2.$$

Let us denote $y_m = \|Z_m + U_m\|_\lambda$, $m \geq 0$. Then

$$y_{m+1} \leq \frac{A_1}{\lambda} y_m + \frac{A_2}{\lambda} y_{m-1},\ \ m \geq 1.$$

Moreover, $y_1,\, y_2 \leq 2c(d+p_1)$. It follows from the stability theory for difference equations and from the inequality $\frac{A_1}{\lambda} + \frac{A_2}{\lambda} < 1$ that there is $A_0 \in R_+$ and $q \in (0,1)$ such that

$$y_m \leq A_0 q^m,\ \ m \geq 1.$$

Consequently the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ are uniformly convergent. To prove the convergence of $\{u_0^{(m)}\}$, we put

$$V_m(t) = \big\|u_0^{(m)} - u_0^{(m-1)}\big\|_{t.1},\ \ t \in [0,c],\ \ m \geq 1.$$

It follows from the inequality

$$V_{m+1}(t) \leq C(K_1+1)\big(Z_m(t) + U_{m+1}(t)\big),\ \ t \in [0,c],$$

that

$$\|V_{m+1}\|_\lambda \leq C(K_1+1)A_0(1+q)q^m.$$

We have also the estimate $V_1(t) \leq 2cp_1$, $t \in [0,c]$. Thus the sequence $\{u_0^{(m)}\}$ is a Cauchy sequence and hence it is uniformly convergent. The proof of Lemma 2.4 is complete. $\square$

We are in a position to state the main result for the problem (2.1), (2.2). Set

$$\|\partial_t\varphi\|^*_X = \sup\Big\{\|\partial_t\varphi_{(t,x)}\|_X : \ (t,x) \in (-\infty,0]\times R^n\Big\},$$

$$\|\partial_x\varphi\|^{**}_X = \sup\Big\{\sum_{j=1}^n \|\partial_{x_j}\varphi_{(t,x)}\|_X : \ (t,x) \in (-\infty,0]\times R^n\Big\},$$

where $\varphi \in \mathcal{J}_N[X]$.

Theorem 2.1. *Suppose that Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_N[f]$, $\mathrm{H}_N[\psi]$, $\mathrm{H}[c,d,p_0,p_1]$ *are satisfied. Then for each* $\varphi \in \mathcal{J}_N[X]$ *there exists a solution* $z = z[\varphi] : (-\infty,c]\times R^n \to R$ *to the problem* (2.1), (2.2) *such that*

$$z \in C^L_{\varphi.c}[d], \quad \partial_t z \in C^L_{\partial_t\varphi.c}[p_0,p_1] \ \text{ and } \ \partial_x z \in C^L_{\partial_x\varphi.c}[p_0,p_1].$$

Moreover, if $\varphi,\overline{\varphi} \in \mathcal{J}_N[X]$ *are such that* $\|\varphi-\overline{\varphi}\|^*_X$, $\|\partial_t\varphi-\partial_t\overline{\varphi}\|^*_X$, $\|\partial_x\varphi - \partial_x\overline{\varphi}\|^{**}_X$ *are finite and* $z = z[\varphi]$, $\overline{z} = z[\overline{\varphi}]$, *then there is* $\Theta \in R_+$ *such that*

$$\|z-\overline{z}\|_{c.1} + \big\|\partial_t z - \partial_t\overline{z}\big\|_{c.1} + \big\|\partial_x z - \partial_x\overline{z}\big\|_{c.n} \le$$
$$\le \Theta\Big(\|\varphi-\overline{\varphi}\|^*_X + \big\|\partial_t\varphi - \partial_t\overline{\varphi}\big\|^*_X + \big\|\partial_x\varphi-\partial_x\overline{\varphi}\big\|^{**}_X\Big). \quad (2.30)$$

Proof. It follows from Lemmas 2.3 and 2.4 that there is $z \in C^L_{\varphi.c}[d]$ such that

$$z(t,x) = \lim_{m\to\infty} z^{(m)}(t,x),$$
$$\partial_t z(t,x) = \lim_{m\to\infty} u_0^{(m)}(t,x), \quad \partial_x z(t,x) = \lim_{m\to\infty} u^{(m)}(t,x)$$

uniformly on $[0,c]\times R^n$. Thus we get

$$z = F[z,\partial_x z], \quad \partial_x z = G[z,\partial_x z,\partial_x z] \ \text{ on } \ [0,c]\times R^n.$$

Moreover, $z = \varphi$ on $(-\infty,0]\times R^n$. Hence z is a solution of the problem (2.1), (2.2) on $(-\infty,c]\times R^n$.

We prove the assertion (2.30). There are Θ_0, $\Theta_1 \in R_+$ such that the following integral inequality

$$\|z-\overline{z}\|_{t.1} + \big\|\partial_x z - \partial_x\overline{z}\big\|_{t.n} \le$$
$$\le \Theta_0\Big(\|\varphi-\overline{\varphi}\|^*_X + \big\|\partial_t\varphi-\partial_t\overline{\varphi}\big\|^*_X + \big\|\partial_x\varphi-\partial_x\overline{\varphi}\big\|^{**}_X\Big) +$$
$$+ \Theta_1\int_0^t \Big(\|z-\overline{z}\|_{\tau.1} + \big\|\partial_x z-\partial_x\overline{z}\big\|_{\tau.n}\Big)\,d\tau$$

is satisfied for $t \in [0,c]$. Using the Gronwall inequality, we get

$$\|z-\overline{z}\|_{t.1} + \big\|\partial_x z-\partial_x\overline{z}\big\|_{t.n} \le$$
$$\le \Theta_0\, e^{c\Theta_1}\Big(\|\varphi-\overline{\varphi}\|^*_X + \big\|\partial_t\varphi-\partial_t\overline{\varphi}\big\|^*_X + \big\|\partial_x\varphi-\partial_x\overline{\varphi}\big\|^{**}_X\Big). \quad (2.31)$$

Moreover, we have

$$\left\|\partial_t z - \partial_t \overline{z}\right\|_{t.1} \leq C\Big(K_1\|z-\overline{z}\|_{t.1} + K_0\|\varphi-\overline{\varphi}\|_X^* + \left\|\partial_x z - \partial_x \overline{z}\right\|_{t.n}\Big) \leq$$
$$\leq C\Big((K_1+1)\Theta_0 e^{c\Theta_1} + K_0\Big)\Big(\|\varphi-\overline{\varphi}\|_X^* + \left\|\partial_t\varphi-\partial_t\overline{\varphi}\right\|_X^* + \left\|\partial_x\varphi-\partial_x\overline{\varphi}\right\|_X^{**}\Big),$$

which together with (2.31) yields (2.30), where

$$\Theta = \Theta_0\, e^{c\Theta_1}(1 + C(K_1+1)) + CK_0.$$

This completes the proof of Theorem 2.1. □

CHAPTER 3

Mixed Problems for Quasilinear Systems

3.1. Introduction

Let B be the set defined in Chapter 1. For $a > 0$ and $b = (\widetilde{b}_1, \dots, \widetilde{b}_n)$, $\widetilde{b}_i > 0$, $i = 1, \dots, n$, we define the sets

$$E=[0,a]\times[-b,b],\quad E_0=(-\infty,0]\times[-b-r,b+r],\quad D_0=(-\infty,0]\times[-b,b],$$

$$\partial_0 E = [0,a] \times \Big([-b-r, b+r] \setminus (-b,b)\Big).$$

For $c \in (0,a]$ we put

$$E[c] = \big\{(t,x) \in E: \ t \le c\big\}, \quad \partial_0 E[c] = \big\{(t,x) \in \partial_0 E: \ t \le c\big\},$$

$$E^*[c] = \big\{(t,x) \in E_0 \cup E \cup \partial_0 E: \ t \le c\big\}.$$

Given a function $z : E^*[c] \to R^k$, $c \in (0,a]$, and a point $(t,x) \in D_0 \cup E[c]$, we consider the function $z_{(t,x)} : B \to R^k$ defined by

$$z_{(t,x)}(s,y) = z(t+s, x+y), \quad (s,y) \in B.$$

Let $(X, \|\cdot\|_X)$ be the phase space of functions from B into R^k and suppose that Assumption H$[X]$ (see Section 1.2) is satisfied. Write $\Omega = E \times X$ and suppose that

$$A : \Omega \to M_{k\times k}, \quad A = [A_{ij}]_{i,j=1,\dots,k},$$

$$\varrho : \Omega \to M_{k\times n}, \quad \varrho = [\varrho_{ij}]_{i=1,\dots,k,\, j=1,\dots,n},$$

$$f : \Omega \to R^k, \quad f = (f_1, \dots, f_k),$$

$$\varphi : E_0 \cup \partial_0 E \to R^k \ \text{ and } \ \psi : E \to D_0 \cup E, \quad \psi = (\psi_0, \psi'), \quad \psi' = (\psi_1, \dots, \psi_n),$$

are given functions. We assume that $\psi_0(t,x) \le t$ for $(t,x) \in E$. We consider the system of differential functional equations in the Schauder canonic form

$$\sum_{j=1}^{k} A_{ij}(t,x,z_{\psi(t,x)})\Big(\partial_t z_j t,x) + \sum_{\nu=1}^{n} \varrho_{i\nu}(t,x,z_{\psi(t,x)})\partial_{x_\nu} z_j(t,x)\Big) =$$
$$= f_i(t,x,z_{\psi(t,x)}), \quad 1 \le i \le k, \tag{3.1}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x) \ \text{ for } \ (t,x) \in E_0 \cup \partial_0 E. \tag{3.2}$$

A function $\overline{z} : E^*[c] \to R^k$, $c \in (0,a]$, is a solution of the above problem if

(i) $\overline{z}_{\psi(t,x)} \in X$ for $(t,x) \in E[c]$,
(ii) the derivatives $\partial_t \overline{z}$, $\partial_x \overline{z}_i = (\partial_{x_1} \overline{z}_i, \ldots, \partial_{x_n} \overline{z}_i)$, $1 \le i \le k$, exist almost everywhere on $E[c]$,
(iii) $\overline{z}$ satisfies (3.1) almost everywhere on $E[c]$ and the condition (3.2) holds.

We use the notation introduced in Chapter 1. Let us denote by $\mathcal{J}_B[X]$ the class of all initial boundary functions $\varphi : E_0 \cup \partial_0 E \to R^k$ satisfying the following conditions:

1) $\varphi_{(t,x)} \in X$ for $(t,x) \in D_0$ and there are $b_0, b_1 \in R_+$ such that

$$\|\varphi_{(t,x)}\|_X \le b_0, \;\; \|\varphi_{(t,x)} - \varphi_{(\overline{t},\overline{x})}\|_X \le b_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big),$$

where $(t,x), (\overline{t},\overline{x}) \in D_0$,

2) $\|\varphi(t,x)\|_\infty \le q_0$ on $\partial_0 E$ and there is $q_1 \in R_+$ such that on $\partial_0 E$

$$\big\|\varphi(t,x) - \varphi(\overline{t},\overline{x})\big\|_\infty \le q_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big).$$

Let $\varphi \in \mathcal{J}_B[X]$, $c \in (0,a]$ and $d = (d_0, d_1) \in R_+^2$. Denote by $C_{\varphi.c}[d]$ the class of all functions $z : E^*[c] \to R^k$ such that $z(t,x) = \varphi(t,x)$ for $(t,x) \in E_0 \cup \partial_0 E[c]$ and the estimates

$$\|z(t,x)\|_\infty \le d_0, \;\; \big\|z(t,x) - z(\overline{t},\overline{x})\big\|_\infty \le d_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big)$$

hold on $E[c] \cup \partial_0 E[c]$. We will prove the existence and uniqueness of a solution to the problem (3.1), (3.2) in the class $C_{\varphi.c}[d]$.

3.2. Bicharacteristics and their Domains

First we will introduce assumptions on the functions ϱ and ψ. Write

$$\Delta_j^+ = \big\{x \in [-b,b] : \; x_j = \widetilde{b}_j\big\}, \;\; \Delta_j^- = \big\{x \in [-b,b] : \; x_j = -\widetilde{b}_j\big\}, \;\; 1 \le j \le n.$$

Assumption $\mathbf{H}_B[\varrho]$. The function $\varrho(\cdot, x, w) : [0,a] \to M_{k\times n}$ is measurable for every $(x,w) \in [-b,b] \times X$ and

1) there exist $\alpha_1, \beta_1 \in \Sigma$ such that

$$\big\|\varrho(t,x,w)\big\|_\infty \le \alpha_1(\mu),$$
$$\big\|\varrho(t,x,w) - \varrho(t,\overline{x},\overline{w})\big\|_\infty \le \beta_1(\mu)\big(\|x-\overline{x}\| + \|w-\overline{w}\|_X\big)$$

for $(x,w), (\overline{x},\overline{w}) \in [-b,b] \times X[\mu]$ and for almost all $t \in [0,a]$,

2) there is $\sigma : R_+ \to (0,+\infty)$ such that for $1 \le i \le k$ and $1 \le j \le n$ we have

$$\varrho_{ij}(t,x,w) \le -\sigma(\mu), \;\; (x,w) \in \Delta_j^+ \times X[\mu],$$
$$\varrho_{ij}(t,x,w) \ge \sigma(\mu), \;\; (x,w) \in \Delta_j^- \times X[\mu]$$

for almost all $t \in [0,a]$.

Assumption $\mathbf{H}_B[\psi]$. The function $\psi : E \to D_0 \cup E$, $\psi = (\psi_0, \psi')$, $\psi' = (\psi_1, \ldots, \psi_n)$, is continuous and

1) there is $s_1 \in R_+$ satisfying

$$\big|\psi_0(t,x) - \psi_0(\overline{t},\overline{x})\big| + \big\|\psi'(t,x) - \psi'(\overline{t},\overline{x})\big\| \le s_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big) \text{ on } E,$$

2) $\psi_0(t,x) \le t$ for $(t,x) \in E$.

Suppose that Assumptions H$[X]$, H$_B[\varrho]$, H$_B[\psi]$ are satisfied and let $\varphi \in \mathcal{J}_B[X]$, $z \in C_{\varphi.c}[d]$. Consider the Cauchy problem

$$\eta'(\tau) = \varrho_i\big(\tau, \eta(\tau), z_{\psi(\tau,\eta(\tau))}\big), \;\; \eta(t) = x, \tag{3.3}$$

where $(t,x) \in E[c]$, $1 \le i \le k$. Denote by $g_i[z](\cdot,t,x)$ the solution of (3.3). Let $\delta_i[z](t,x)$ be the left end of the maximal interval on which the solution $g_i[z](\cdot,t,x)$ is defined. We write

$$P_i[z](\tau,t,x) = \big(\tau, g_i[z](\tau,t,x), z_{\psi(\tau,g_i[z](\tau,t,x))}\big). \tag{3.4}$$

For $\varphi \in \mathcal{J}_B[X]$ and $z \in C_{\varphi.c}[d]$ we put

$$\|\varphi\|^b_X = \sup\big\{\|\varphi_{(t,x)}\|_X : \; (t,x) \in D_0\big\},$$
$$\|\varphi\|_{\partial.t} = \max\Big\{\|\varphi(s,y)\| : \; (s,y) \in \partial_0 E[t]\Big\}, \;\; 0 \le t \le c,$$
$$\|z\|_t = \max\Big\{\|z(s,y)\| : (s,y) \in E[t] \cup \partial_0 E[t]\Big\}, \;\; 0 \le t \le c.$$

Write

$$\begin{aligned} \alpha_1^+(\mu_0) &= \alpha_1(\mu_0) + 1, \quad \mu_0 = K_1 d_0 + K_0 b_0, \\ Q_c &= \exp\Big(c\Lambda\beta_1(\mu_0)\Big), \quad \Lambda = 1 + s_1(K_1 d_1 + K_0 b_1) \end{aligned} \tag{3.5}$$

and

$$\Gamma = \big(\{0\} \times [-b,b]\big) \cup \Big((0,c] \times \bigcup_{j=1}^{n} (\Delta_j^+ \cup \Delta_j^-)\Big).$$

Lemma 3.1. *Suppose that Assumptions* H$[X]$, H$_B[\varrho]$, H$_B[\psi]$ *are satisfied and* $\varphi, \overline{\varphi} \in \mathcal{J}_B[X]$, $z \in C_{\varphi.c}[d]$, $\overline{z} \in C_{\overline{\varphi}.c}[d]$, $c \in (0,a]$. *Then for each* $1 \le i \le k$, $(t,x) \in E[c]$ *the solutions* $g_i[z](\cdot,t,x)$ *and* $g_i[\overline{z}](\cdot,t,x)$ *exist on intervals* $I^i_{(t,x)}$ *and* $\overline{I}^i_{(t,x)}$ *such that* $(\zeta_i, g_i[z](\zeta_i,t,x))$, $(\overline{\zeta}_i, g_i[\overline{z}](\overline{\zeta}_i,t,x)) \in \Gamma$, *where* $\zeta_i = \delta_i[z](t,x)$, $\overline{\zeta}_i = \delta_i[\overline{z}](t,x)$. *The solutions of* (3.3) *are unique and they satisfy the conditions*

$$\big\|g_i[z](\tau,t,x) - g_i[z](\tau,\overline{t},\overline{x})\big\| \le Q_c\alpha_1^+(\mu_0)\big(|t-\overline{t}| + \|x-\overline{x}\|\big), \tag{3.6}$$

where (t,x), $(\overline{t},\overline{x}) \in E[c]$, $\tau \in I^i_{(t,x)} \cap I^i_{(\overline{t},\overline{x})}$, *and*

$$\begin{aligned} &\big\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\big\| \le \\ &\qquad \le Q_c\beta_1(\mu_0)\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|^b_X\Big)c, \end{aligned} \tag{3.7}$$

where $(t,x) \in E[c]$, $\tau \in I^i_{(t,x)} \cap \overline{I}^i_{(t,x)}$. *Moreover, for each* $1 \le i \le k$ *the functions* $\delta_i[z]$ *and* $\delta_i[\overline{z}]$ *are continuous on* $E[c]$ *and*

$$\big|\delta_i[z](t,x) - \delta_i[z](\overline{t},\overline{x})\big| \le \frac{2Q_c\alpha_1^+(\mu_0)}{\sigma(\mu_0)}\big(|t-\overline{t}| + \|x-\overline{x}\|\big), \tag{3.8}$$

$$\left|\delta_i[z](t,x)-\delta_i[\overline{z}](t,x)\right| \le c\,\frac{2Q_c\beta_1(\mu_0)}{\sigma(\mu_0)}\Big(K_1\|z-\overline{z}\|_c+K_0\|\varphi-\overline{\varphi}\|_X^b\Big), \quad (3.9)$$

where (t,x), $(\overline{t},\overline{x}) \in E[c]$.

Proof. The existence and uniqueness of solutions of (3.3) follows from classical theorems on Carathéodory solutions of ordinary initial problems. The proof of (3.6) and (3.7) is similar to the proof of Lemma 1.3. We omit the details.

The continuity of $\delta_i[z]$ and $\delta_i[\overline{z}]$ follows from theorems on continuous dependence on initial data for Carathéodory solutions of ordinary differential systems. Let $(t,x), (\overline{t},\overline{x}) \in E[c]$, $\zeta = \delta_i[z](t,x)$, $\overline{\zeta} = \delta_i[z](\overline{t},\overline{x})$. The estimate (3.8) is obvious in the case where $\zeta = \overline{\zeta} = 0$. Suppose that $0 \le \zeta < \overline{\zeta}$. We have

$$g_i[z](\overline{\zeta},\overline{t},\overline{x}) \in \bigcup_{j=1}^{n}(\Delta_j^+ \cup \Delta_j^-).$$

Consider the case where $g_i[z](\overline{\zeta},\overline{t},\overline{x}) \in \Delta_j^+$ for some $j \in \{1,\ldots,n\}$. Then $g_{ij}[z](\overline{\zeta},\overline{t},\overline{x}) = \widetilde{b}_j$. Let $y=(y_1,\ldots,y_n)$, $\overline{y}=(y_1,\ldots,y_{j-1},\widetilde{b}_j,y_{j+1},\ldots,y_n)$. We have

$$\left|\varrho_{ij}(\tau,y,z_{\psi(\tau,y)}) - \varrho_{ij}(\tau,\overline{y},z_{\psi(\tau,\overline{y})})\right| \le \beta_1(\mu_0)\Lambda(\widetilde{b}_j - y_j)$$

for $y \in [-b,b]$ and for almost all $\tau \in [0,c]$. Thus

$$\varrho_{ij}(\tau,y,z_{\psi(\tau,y)}) \le -\frac{1}{2}\sigma(\mu_0)$$

for $y \in [-b,b]$ such that $\widetilde{b}_j - y_j \le \varepsilon_0$ with $\varepsilon_0 = \frac{\sigma(\mu_0)}{2\beta_1(\mu_0)\Lambda}$. If the points (t,x), $(\overline{t},\overline{x})$ are such that

$$|t-\overline{t}| + \|x-\overline{x}\| < \widetilde{\delta}_1 \;\text{ with }\; \widetilde{\delta}_1 = \frac{\sigma(\mu_0)}{2\beta_1(\mu_0)\Lambda\alpha_1^+(\mu_0)Q_c}, \quad (3.10)$$

then

$$\widetilde{b}_j - g_{ij}[z](\overline{\zeta},t,x) = g_{ij}[z](\overline{\zeta},\overline{t},\overline{x}) - g_{ij}[z](\overline{\zeta},t,x) \le \varepsilon_0.$$

We get also

$$\varrho_{ij}\big(P_i[z](\overline{\zeta},t,x)\big) \le -\frac{1}{2}\sigma(\mu_0) < 0,$$

and consequently $g_{ij}[z](\cdot,t,x)$ is decreasing on $(\zeta,\overline{\zeta})$. Therefore

$$\widetilde{b}_j - g_{ij}[z](\tau,t,x) \le \varepsilon_0 \;\text{ and }\; \varrho_{ij}\big(P_i[z](\tau,t,x)\big) \le -\frac{1}{2}\sigma(\mu_0)$$

for almost all $\tau \in (\zeta,\overline{\zeta})$. Then

$$-\frac{1}{2}\sigma(\mu_0)(\overline{\zeta}-\zeta) \ge \int_{\zeta}^{\overline{\zeta}} \varrho_{ij}\big(P_i[z](\tau,t,x)\big)\,d\tau =$$

$$= g_{ij}[z](\overline{\zeta},t,x) - g_{ij}[z](\zeta,t,x) \ge g_{ij}[z](\overline{\zeta},t,x) - g_{ij}[z](\overline{\zeta},\overline{t},\overline{x}) \ge$$

$$\ge -Q_c\alpha_1^+(\mu_0)\big(|t-\overline{t}| + \|x-\overline{x}\|\big),$$

that is,

$$\overline{\zeta} - \zeta \le \frac{2Q_c \alpha_1^+(\mu_0)}{\sigma(\mu_0)} \big(|t - \overline{t}| + \|x - \overline{x}\|\big). \tag{3.11}$$

In the case where $g_{ij}[z](\overline{\zeta}, \overline{t}, \overline{x}) = -\widetilde{b}_j$, we proceed in a similar way. If $(t,x), (\overline{t}, \overline{x})$ do not satisfy (3.10), then we consider the points (t_0, x_0), (t_1, x_1), ..., (t_p, x_p) such that $(t_0, x_0) = (t, x)$, $(t_p, x_p) = (\overline{t}, \overline{x})$ and

$$|t - \overline{t}| + \|x - \overline{x}\| = \sum_{j=0}^{p-1} \big(|t_j - t_{j+1}| + \|x_j - x_{j+1}\|\big)$$

and

$$|t_j - t_{j+1}| + \|x_j - x_{j+1}\| < \widetilde{\delta}_1 \ \text{ for } \ 0 \le j \le p-1.$$

We have

$$\begin{aligned}\big|\delta_i[z](t,x) - \delta_i[z](\overline{t}, \overline{x})\big| &\le \sum_{j=0}^{p-1} \big|\delta_i[z](t_j, x_j) - \delta_i[z](t_{j+1}, x_{j+1})\big| \le \\ &\le \frac{2Q_c \alpha_1^+(\mu_0)}{\sigma(\mu_0)} \sum_{j=0}^{p-1} \big(|t_j - t_{j+1}| + \|x_j - x_{j+1}\|\big) = \\ &= \frac{2Q_c \alpha_1^+(\mu_0)}{\sigma(\mu_0)} \big(|t - \overline{t}| + \|x - \overline{x}\|\big).\end{aligned}$$

To prove (3.9), suppose that $(t,x) \in E[c]$ and $0 \le \delta_i[z](t,x) < \delta_i[\overline{z}](t,x)$. Let $\overline{\xi} = \delta_i[\overline{z}](t,x)$, $\xi = \delta_i[z](t,x)$. We have

$$g_i[\overline{z}](\overline{\xi}, t, x) \in \bigcup_{j=1}^{n} (\Delta_j^+ \cup \Delta_j^-).$$

Consider the case where $g_i[\overline{z}](\overline{\xi}, t, x) \in \Delta_j^+$ for some $j \in \{1, \ldots, n\}$. We have $g_{ij}[\overline{z}](\overline{\xi}, t, x) = \widetilde{b}_j$. If $(t,x) \in E[c]$ and (φ, z), $(\overline{\varphi}, \overline{z})$ are such that

$$K_1 \|z - \overline{z}\|_c + K_0 \|\varphi - \overline{\varphi}\|_X^b < \widetilde{\delta}_2 \ \text{ with } \ \widetilde{\delta}_2 = \frac{\sigma(\mu_0)}{2c\beta_1(\mu_0)\Lambda Q_c \beta_1(\mu_0)}\,, \tag{3.12}$$

then

$$\widetilde{b}_j - g_{ij}[z](\overline{\xi}, t, x) = g_{ij}[\overline{z}](\overline{\xi}, t, x) - g_{ij}[z](\overline{\xi}, t, x) \le \varepsilon_0.$$

Thus $\widetilde{b}_j - g_{ij}[z](\tau, t, x) \le \varepsilon_0$ and

$$\varrho_{ij}\big(P_i[z](\tau, t, x)\big) \le -\frac{1}{2}\sigma(\mu_0) < 0$$

for almost all $\tau \in (\xi, \overline{\xi})$. Then

$$\begin{aligned}-\frac{1}{2}\sigma(\mu_0)(\overline{\xi} - \xi) \ge \int_{\xi}^{\overline{\xi}} \varrho_{ij}\big(P_i[z](\tau, t, x)\big)\, d\tau = \\ = g_{ij}[z](\overline{\xi}, t, x) - g_{ij}[z](\xi, t, x) \ge g_{ij}[z](\overline{\xi}, t, x) - g_{ij}[\overline{z}](\overline{\xi}, t, x) \ge\end{aligned}$$

$$\geq -Q_c\beta_1(\mu_0)\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^b\Big)c,$$

that is,

$$\overline{\xi}-\xi \leq \frac{2Q_c\beta_1(\mu_0)}{\sigma(\mu_0)}\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^b\Big)c. \tag{3.13}$$

If (φ, z), $(\overline{\varphi}, \overline{z})$ do not satisfy (3.12), then to obtain (3.13) we use the functions $z_0, z_1, \ldots, z_\nu$ with $z_0 = z$, $z_\nu = \overline{z}$, $z_j \in C_{\varphi_j.c}[d]$, where $\varphi_j \in \mathcal{J}_B[X]$, $0 \leq j \leq \nu$, $\varphi_0 = \varphi$, $\varphi_\nu = \overline{\varphi}$, satisfying the conditions

$$K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^b = \sum_{j=0}^{\nu-1}\Big(K_1\|z_j - z_{j+1}\|_c + K_0\|\varphi_j-\varphi_{j+1}\|_X^b\Big),$$

$$K_1\|z_j - z_{j+1}\|_c + K_0\|\varphi_j-\varphi_{j+1}\|_X^b < \widetilde{\delta}_2, \text{ for } 0 \leq j \leq \nu-1.$$

The proof of Lemma 3.1 is complete. □

3.3. Existence and Uniqueness of Weak Solutions

We formulate assumptions on the functions f and A.

Assumption $\mathbf{H}_B[f]$. The function $f(\cdot, x, w) : [0,a] \to R^k$ is measurable for every $(x, w) \in [-b, b] \times X$ and there are $\alpha_2 \in \Sigma$, $\beta_2 \in \Delta$ such that

$$\|f(t,x,w)\|_\infty \leq \alpha_2(\mu),$$
$$\big\|f(t,x,w) - f(t,\overline{x},\overline{w})\big\|_\infty \leq \beta_2(t,\mu)\big(\|x-\overline{x}\| + \|w-\overline{w}\|_X\big)$$

for $(x,w), (\overline{x},\overline{w}) \in [-b,b]\times X[\mu]$ and for almost $t \in [0,a]$.

Assumption $\mathbf{H}_B[A]$. The function $A : \Omega \to M_{k\times k}$ satisfies the conditions:

1) there are $\alpha, \beta \in \Sigma$ such that

$$\|A(t,x,w)\|_\infty \leq \alpha(\mu),$$
$$\big\|A(t,x,w) - A(\overline{t},\overline{x},\overline{w})\big\|_\infty \leq \beta(\mu)\big(|t-\overline{t}| + \|x-\overline{x}\| + \|w-\overline{w}\|_X\big)$$

for $(t,x,w), (\overline{t},\overline{x},\overline{w}) \in E \times X[\mu]$,

2) for each $(t,x,w) \in E \times X[\mu]$ there exists the inverse matrix $A^{-1}(t,x,w)$ and there are α_0, $\beta_0 \in \Sigma$ such that

$$\|A^{-1}(t,x,w)\|_\infty \leq \alpha_0(\mu),$$
$$\big\|A^{-1}(t,x,w) - A^{-1}(\overline{t},\overline{x},\overline{w})\big\|_\infty \leq \beta_0(\mu)\big(|t-\overline{t}| + \|x-\overline{x}\| + \|w-\overline{w}\|_X\big)$$

for (t,x,w), $(\overline{t},\overline{x},\overline{w}) \in E \times X[\mu]$.

Now we construct the integral operator corresponding to (3.1), (3.2). Suppose that $\varphi \in \mathcal{J}_B[X]$, $c \in (0,a]$, $z \in C_{\varphi.c}[d]$. Let $I^i_{(t,x)}$ be the domain of $g_i[z](\cdot,t,x)$ with the left end $\delta_i[z](t,x)$, where $1 \le i \le k$, $(t,x) \in E[c]$. It follows from (3.1) that for $(t,x) \in E[c]$ we have

$$\sum_{j=1}^{k} A_{ij}\big(P_i[z](\tau,t,x)\big)\,\frac{d}{d\tau}\,z_j\big(\tau,g_i[z](\tau,t,x)\big) = f_i\big(P_i[z](\tau,t,x)\big),$$

where $P_i[z](\tau,t,x)$ is given by (3.4). After integration from $\delta_i[z](t,x)$ to t we obtain

$$\sum_{j=1}^{k} A_{ij}\big(t,x,z_{\psi(t,x)}\big) z_j(t,x) =$$
$$= \sum_{j=1}^{k} A_{ij}\Big(P_i[z]\big(\delta_i[z](t,x),t,x\big)\Big)\varphi_j\big(Q_i[z](t,x)\big)+$$
$$+\int_{\delta_i[z](t,x)}^{t} \sum_{j=1}^{k} \frac{d}{d\tau}\,A_{ij}\big(P_i[z](\tau,t,x)\big) z_j\big(\tau,g_i[z](\tau,t,x)\big)\,d\tau+$$
$$+\int_{\delta_i[z](t,x)}^{t} f_i\big(P_i[z](\tau,t,x)\big)\,d\tau,$$

where

$$Q_i[z](t,x) = \Big(\delta_i[z](t,x), g_i[z]\big(\delta_i[z](t,x),t,x\big)\Big). \tag{3.14}$$

For $z \in C_{\varphi.c}[d]$ we define $U = T_\varphi(z)$ as follows

$$\sum_{j=1}^{k} A_{ij}\big(t,x,z_{\psi(t,x)}\big) U_j(t,x) =$$
$$= \sum_{j=1}^{k} A_{ij}\Big(P_i[z]\big(\delta_i[z](t,x),t,x\big)\Big)\varphi_j\big(Q_i[z](t,x)\big)+$$
$$+\int_{\delta_i[z](t,x)}^{t} \sum_{j=1}^{k} \frac{d}{d\tau}\,A_{ij}\big(P_i[z](\tau,t,x)\big) z_j\big(\tau,g_i[z](\tau,t,x)\big)\,d\tau+$$
$$+\int_{\delta_i[z](t,x)}^{t} f_i\big(P_i[z](\tau,t,x)\big)\,d\tau,$$

where $1 \le i \le k$, $(t,x) \in E[c]$, and

$$T_\varphi(z)(t,x) = \varphi(t,x), \quad (t,x) \in E_0 \cup \partial_0 E[c]. \tag{3.15}$$

We can write on $E[c]$ the following equality

$$T_\varphi(z)(t,x) = \varphi(0,x) + A^{-1}\big(t,x,z_{\psi(t,x)}\big) \sum_{i=1}^{3} W_i[z](t,x), \tag{3.16}$$

where

$$W_1[z](t,x) = \Big[\int_{\delta_i}^{t} f_i\big(P_i[z](\tau,t,x)\big)\, d\tau \Big]^T_{i=1,\dots,k},$$

$$W_2[z](t,x) =$$

$$= \Big[\sum_{j=1}^{k} A_{ij}\Big(P_i[z]\big(\delta_i,t,x\big)\Big)\Big(\varphi_j\big(Q_i[z](t,x)\big) - \varphi_j(0,x)\Big)\Big]^T_{i=1,\dots,k},$$

$$W_3[z](t,x) =$$

$$= \Big[\sum_{j=1}^{k} \int_{\delta_i}^{t} \frac{d}{d\tau} A_{ij}\big(P_i[z](\tau,t,x)\big)\Big(z_j\big(\tau, g_i[z](\tau,t,x)\big) - \varphi_j(0,x)\Big)\, d\tau\Big]^T_{i=1,\dots,k}$$

and $\delta_i = \delta_i[z](t,x)$, $1 \le i \le k$. Now we give lemmas on the operator T_φ.

Lemma 3.2. *If Assumptions* $\mathrm{H}[X]$, $\mathrm{H}_B[\psi]$, $\mathrm{H}_B[\varrho]$, $\mathrm{H}_B[f]$, $\mathrm{H}_B[A]$ *are satisfied, then there are* $c \in (0,a]$, $d = (d_0,d_1) \in R^2_+$ *such that for each* $\varphi \in \mathcal{J}_B[X]$ *the operator* T_φ *maps the set* $C_{\varphi.c}[d]$ *into itself.*

Proof. Assume that $\varphi \in \mathcal{J}_B[X]$ and $z \in C_{\varphi.c}[d]$ with some $c \in (0,a]$, $d = (d_0,d_1) \in R^2_+$. Let us define $T_\varphi(z)$ by the relations (3.15) and (3.16). We assume that the constants $c \in (0,a]$, $d = (d_0,d_1) \in R^2_+$ satisfy the condition

$$d_0 \ge q_0 + c\alpha_0(\mu_0)S_0, \tag{3.17}$$

where

$$S_0 = \alpha_2(\mu_0) + \alpha_1^+(\mu_0)\big(q_1\alpha(\mu_0) + cd_1\beta^*\big), \quad \beta^* = \beta(\mu_0)\Lambda\alpha_1^+(\mu_0).$$

We prove that

$$\big\|T_\varphi(z)(t,x)\big\|_\infty \le d_0,\ (t,x) \in E[c]. \tag{3.18}$$

It follows from Lemma 1.2 that

$$\big\|z_{\psi(t,x)} - z_{\psi(\overline{t},\overline{x})}\big\|_X \le (K_1d_1 + K_0b_1)s_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big),$$

$$\Big\|\frac{d}{d\tau}\, z_{\psi(\tau,g_i[z](\tau,t,x))}\Big\|_X \le \alpha_1^+(\mu_0)(K_1d_1 + K_0b_1)s_1.$$

Thus

$$\Big|\sum_{j=1}^{k} \frac{d}{d\tau} A_{ij}\big(P_i[z](\tau,t,x)\Big| \le \beta^*.$$

We have also

$$\Big|\varphi_j\big(Q_i[z](t,x)\big) - \varphi_j(0,x)\Big| \le cq_1\alpha_1^+(\mu_0),$$

$$\Big|z_j(\tau, g_i[z](\tau,t,x)) - \varphi_j(0,x)\Big| \le cd_1\alpha_1^+(\mu_0).$$

The above estimates together with (3.17) give (3.18).

To prove that

$$\left\|T_\varphi(z)(t,x) - T_\varphi(z)(\bar t,\overline{x})\right\|_\infty \le d_1\big(|t-\bar t| + \|x-\overline{x}\|\big) \text{ on } E[c], \tag{3.19}$$

we assume an additional condition for the constants $c \in (0,a]$, $d = (d_0, d_1) \in R_+^2$. Let $(t,x), (\bar t, \overline{x}) \in E[c]$. Put $\delta_i = \delta_i[z](t,x)$ and $\overline{\delta_i} = \delta_i[z](\bar t,\overline{x})$. Consider the case $\overline{\delta_i} \le \delta_i \le t \le \bar t$. We have

$$\Big|f_i\big(P_i[z](\tau,t,x)\big) - f_i\big(P_i[z](\tau,\bar t,\overline{x})\big)\Big| \le \beta_2(\tau,\mu_0)\Lambda Q_c\alpha_1^+(\mu_0)\big(|t-\bar t| + \|x-\overline{x}\|\big),$$

and thus

$$\Big|\int_{\delta_i}^{t} f_i\big(P_i[z](\tau,t,x)\big)\,d\tau - \int_{\overline{\delta_i}}^{\bar t} f_i\big(P_i[z](\tau,\bar t,\overline{x})\big)\,d\tau\Big| \le$$

$$\le \int_{\delta_i}^{t}\Big|f_i\big(P_i[z](\tau,t,x)\big) - f_i\big(P_i[z](\tau,\bar t,\overline{x})\big)\Big|\,d\tau + \int_{\overline{\delta_i}}^{\delta_i}\Big|f_i\big(P_i[z](\tau,\bar t,\overline{x})\big)\Big|\,d\tau +$$

$$+ \int_{t}^{\bar t}\big|f_i\big(P_i[z](\tau,\bar t,\overline{x})\big)\big|\,d\tau \le S_{1.c}\big(|t-\bar t| + \|x-\overline{x}\|\big),$$

where

$$S_{1.c} = \int_0^c \beta_2(\xi,\mu_0)\,d\xi \cdot \Lambda Q_c \alpha_1^+(\mu_0) + \alpha_2(\mu_0)\xi_c, \quad \xi_c = 1 + \frac{2Q_c\alpha_1^+(\mu_0)}{\sigma(\mu_0)}\,.$$

The same estimate is obtained in the case where $\overline{\delta_i} \le \delta_i \le \bar t \le t$. If $\overline{\delta_i} \le \bar t \le \delta_i \le t$, then we define $(t_0,x_0),\dots,(t_s,x_s)$ such that $(t_0,x_0) = (\bar t,\overline{x})$, $(t_s,x_s) = (t,x)$ and

$$\sum_{j=0}^{s-1}\big(|t_{j+1}-t_j| + \|x_{j+1}-x_j\|\big) = |t-\bar t| + \|x-\overline{x}\|,$$

and $\delta_{i.j} \le \delta_{i,j+1} \le t_j \le t_{j+1}$ for $j = 0,1,\dots,s-1$, where $\delta_{i.j} = \delta_i[z](t_j,x_j)$, $j = 0,1,\dots,s$. Then

$$\Big|\int_{\overline{\delta_i}}^{\bar t} f_i\big(P_i[z](\tau,\bar t,\overline{x})\big)\,d\tau - \int_{\delta_i}^{t} f_i\big(P_i[z](\tau,t,x)\big)\,d\tau\Big| \le$$

$$\le \sum_{j=0}^{s-1}\Big|\int_{\delta_{i.j}}^{t_j} f_i\big(P_i[z](\tau,t_j,x_j)\big)\,d\tau - \int_{\delta_{i.j+1}}^{t_{j+1}} f_i\big(P_i[z](\tau,t_{j+1},x_{j+1})\big)\,d\tau\Big| \le$$

$$\le S_{1.c}\sum_{j=0}^{s-1}\big(|t_{j+1}-t_j| + \|x_{j+1}-x_j\|\big) = S_{1.c}\big(|t-\bar t| + \|x-\overline{x}\|\big).$$

In each case we obtain

$$\big\|W_1[z](t,x) - W_1[z](\bar t,\bar x)\big\|_\infty \le S_{1.c}\big(|t-\bar t| + \|x-\bar x\|\big).$$

Since

$$\sum_{j=1}^k \Big|A_{ij}\big(P_i[z](\delta_i,t,x)\big) - A_{ij}\big(P_i[z](\overline{\delta_i},\bar t,\bar x)\big)\Big| \le$$
$$\le \beta(\mu_0)\Lambda Q_c\alpha_1^+(\mu_0)\xi^*\big(|t-\bar t| + \|x-\bar x\|\big),$$
$$\Big|\varphi_j\big(Q_i[z](t,x)\big) - \varphi_j\big(Q_i[z](\bar t,\bar x)\big)\Big| \le q_1 Q_c\alpha_1^+(\mu_0)\xi^*\big(|t-\bar t| + \|x-\bar x\|\big),$$
$$\xi^* = 1 + \frac{2\alpha_1^+(\mu_0)}{\sigma(\mu_0)}\,,$$

we have

$$\big\|W_2[z](t,x) - W_2[z](\bar t,\bar x)\big\|_\infty \le S_{2.c}\big(|t-\bar t| + \|x-\bar x\|\big),$$

where

$$S_{2.c} = \alpha(\mu_0)q_1\big(Q_c\alpha_1^+(\mu_0)\xi^* + 1\big) + \beta(\mu_0)\Lambda Q_c\alpha_1^+(\mu_0)\xi^* q_1\alpha_1^+(\mu_0)c.$$

Let $P_i = P_i[z](\tau,t,x)$, $\overline{P_i} = P_i[z](\tau,\bar t,\bar x)$, $g_i = g_i[z](\tau,t,x)$, $\overline{g_i} = g_i[z](\tau,\bar t,\bar x)$. In the case $\overline{\delta_i} \le \delta_i \le t \le \bar t$ we obtain

$$\Big|\int_{\delta_i}^{t}\sum_{j=1}^k \frac{d}{d\tau}\,A_{ij}(P_i)\big(z_j(\tau,g_i) - \varphi_j(0,x)\big)\,d\tau -$$
$$-\int_{\overline{\delta_i}}^{\bar t}\sum_{j=1}^k \frac{d}{d\tau}\,A_{ij}(\overline{P_i})\big(z_j(\tau,\overline{g_i}) - \varphi_j(0,\bar x)\big)\,d\tau\Big| \le$$
$$\le \int_{\overline{\delta_i}}^{\delta_i}\Big|\sum_{j=1}^k \frac{d}{d\tau}\,A_{ij}(\overline{P_i})\big(z_j(\tau,\overline{g_i}) - \varphi_j(0,\bar x)\big)\Big|\,d\tau +$$
$$+\int_{t}^{\bar t}\Big|\sum_{j=1}^k \frac{d}{d\tau}\,A_{ij}(\overline{P_i})\big(z_j(\tau,\overline{g_i}) - \varphi_j(0,\bar x)\big)\Big|\,d\tau +$$
$$+\Big|\Big[\sum_{j=1}^k \big(A_{ij}(P_i) - A_{ij}(\overline{P_i})\big)\big(z_j(\tau,g_i) - \varphi_j(0,x)\big)\Big]_{\tau=\delta_i}^{\tau=t}\Big| +$$
$$+\int_{\delta_i}^{t}\Big|\sum_{j=1}^k \big(A_{ij}(P_i) - A_{ij}(\overline{P_i})\big)\,\frac{d}{d\tau}\,z_j(\tau,g_i)\Big|\,d\tau +$$
$$+\int_{\delta_i}^{t}\Big|\sum_{j=1}^k \frac{d}{d\tau}\,A_{ij}(P_i)\big(z_j(\tau,g_i) - z_j(\tau,\overline{g_i}) - \varphi_j(0,x) + \varphi_j(0,\bar x)\big)\Big|\,d\tau.$$

Thus

$$\left\|W_3[z](t,x)-W_3[z](\overline{t},\overline{x})\right\|_\infty \le \left(S_{3.c}+S_{4.c}+S_{5.c}+S_{6.c}\right)\left(|t-\overline{t}|+\|x-\overline{x}\|\right),$$

where

$$S_{3.c} = c\beta^* d_1 \alpha_1^+(\mu_0)\xi_c, \quad S_{4.c} = cd_1\Lambda\beta(\mu_0)\alpha_1^+(\mu_0)\left(1+Q_c\alpha_1^+(\mu_0)\right),$$
$$S_{5.c} = cd_1\Lambda Q_c\beta(\mu_0)[\alpha_1^+(\mu_0)]^2, \quad S_{6.c} = c\beta^* d_1\left(1+Q_c\alpha_1^+(\mu_0)\right).$$

In the other cases we obtain for $\|W_3[z](t,x)-W_3[z](\overline{t},\overline{x})\|$ the same estimate. Finally the following inequality is true

$$\left\|T_\varphi(z)(t,x) - T_\varphi(z)(\overline{t},\overline{x})\right\|_\infty \le$$
$$\le \Big(q_1 + \beta_0(\mu_0)\Lambda c S_0 + \alpha_0(\mu_0)\sum_{j=1}^{6} S_{j.c}\Big)\left(|t-\overline{t}| + \|x-\overline{x}\|\right).$$

We assume that

$$d_1 \ge q_1 + \beta_0(\mu_0)\Lambda c S_0 + \alpha_0(\mu_0)\sum_{j=1}^{6} S_{j.c}. \tag{3.20}$$

Then the condition (3.19) is satisfied.

If we assume that the inequalities (3.17) and (3.20) hold, then $T_\varphi : C_{\varphi.c}[d] \to C_{\varphi.c}[d]$. □

Lemma 3.3. *If the assumptions of Lemma* 3.2 *are satisfied, then there are* $G_{1.c}$, G_2, $G_3 \in R_+$ *such that for each* $\varphi, \overline{\varphi} \in \mathcal{J}_B[X]$ *and* $z \in C_{\varphi.c}[d]$, $\overline{z} \in C_{\overline{\varphi}.c}[d]$ *the following inequality is true*

$$\left\|T_\varphi(z) - T_{\overline{\varphi}}(\overline{z})\right\|_c \le G_{1.c}\|z-\overline{z}\|_c + G_2\|\varphi-\overline{\varphi}\|_X^b + G_3\|\varphi-\overline{\varphi}\|_{\partial.c}. \tag{3.21}$$

Proof. Fix $\varphi, \overline{\varphi} \in \mathcal{J}_B[X]$ and $z \in C_{\varphi.c}[d]$, $\overline{z} \in C_{\overline{\varphi}.c}[d]$. It follows from the assumptions of the lemma that

$$\left\|W_1[z](t,x) - W_1[\overline{z}](t,x)\right\|_\infty \le Q_{1.c}\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^b\Big),$$
$$\left\|W_2[z](t,x) - W_2[\overline{z}](t,x)\right\|_\infty \le$$
$$\le Q_{2.c}\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^b\Big) + \alpha(\mu_0)\left(\chi\|\varphi-\overline{\varphi}\|_X^b + \|\varphi-\overline{\varphi}\|_{\partial.c}\right),$$
$$\left\|W_3[z](t,x) - W_3[\overline{z}](t,x)\right\|_\infty \le$$
$$\le Q_{3.c}\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^b\Big) + c\beta^*\|z-\overline{z}\|_c + c\beta^*\chi\|\varphi-\overline{\varphi}\|_X^b,$$

where

$$Q_{1.c} = c\,\frac{2Q_c\beta_1(\mu_0)}{\sigma(\mu_0)}\,\alpha_2(\mu_0) + \int_0^c \beta_2(\xi,\mu_0)\,d\xi\left(1+\Lambda Q_c\beta_1(\mu_0)\right),$$
$$Q_{2.c} = c\beta(\mu_0)\left(1+c\Lambda Q_c\beta_1(\mu_0)\xi^*\right)q_1\alpha_1^+(\mu_0) + c\alpha(\mu_0)q_1Q_c\beta_1(\mu_0)\xi^*,$$
$$Q_{3.c} = c^2\beta^* d_1 Q_c\beta_1(\mu_0)\xi^* + 2cd_1\beta(\mu_0)\alpha_1^+(\mu_0)\left(2+c\Lambda Q_c\beta_1(\mu_0)\right).$$

Thus

$$\big\|T_\varphi(z) - T_{\overline{\varphi}}(\overline{z})\big\|_c \le \chi\|\varphi-\overline{\varphi}\|_X^b + c\alpha_0(\mu_0)\beta^*\|z-\overline{z}\|_c +$$
$$+\Big(c\beta_0(\mu_0)S_0 + \alpha_0(\mu_0)\sum_{i=1}^{3} Q_{i.c}\Big)\Big(K_1\|z-\overline{z}\|_c + K_0\|\varphi-\overline{\varphi}\|_X^b\Big)+$$
$$+\alpha_0(\mu_0)\big(\alpha(\mu_0)+c\beta^*\big)\chi\|\varphi-\overline{\varphi}\|_X^b + \alpha_0(\mu_0)\alpha(\mu_0)\|\varphi-\overline{\varphi}\|_{\partial.c},$$

that is,

$$\big\|T_\varphi(z) - T_{\overline{\varphi}}(\overline{z})\big\|_c \le G_{1.c}\|z-\overline{z}\|_c + G_2\|\varphi-\overline{\varphi}\|_X^b + G_3\|\varphi-\overline{\varphi}\|_{\partial.c},$$

where

$$G_{1.c} = K_1\Big(c\beta_0(\mu_0)S_0 + \alpha_0(\mu_0)\sum_{i=1}^{3} Q_{i.c}\Big) + c\alpha_0(\mu_0)\beta^*, \tag{3.22}$$

$$\begin{aligned} G_2 &= K_0\Big(c\beta_0(\mu_0)S_0 + \alpha_0(\mu_0)\sum_{i=1}^{3} Q_{i.c}\Big)+ \\ &\quad + \chi + \alpha_0(\mu_0)\chi\big(\alpha(\mu_0)+c\beta^*\big), \\ G_3 &= \alpha_0(\mu_0)\alpha(\mu_0). \end{aligned} \tag{3.23}$$

The proof of Lemma 3.3 is complete. □

Now we formulate the main theorem for the mixed problem (3.1), (3.2).

Theorem 3.1. *Suppose that Assumptions* $\mathrm{H}[X]$, $\mathrm{H}_B[\psi]$, $\mathrm{H}_B[\varrho]$, $\mathrm{H}_B[f]$ *and* $\mathrm{H}_B[A]$ *are satisfied. Assume that* $c \in (0,a]$, $d=(d_0,d_1)\in R_+^2$ *satisfy the inequalities* (3.17), (3.20) *and*

$$G_{1.c} < 1,$$

where $G_{1.c}$ *is given by* (3.22). *Then for each* $\varphi \in \mathcal{J}_B[X]$ *there exists* $z = z[\varphi] \in C_{\varphi.c}[d]$ *which is a unique solution of* (3.1), (3.2). *Furthermore, if* $\varphi,\overline{\varphi}\in\mathcal{J}_B[X]$, $z=z[\varphi]$, $\overline{z}=z[\overline{\varphi}]$, *then*

$$\|z-\overline{z}\|_c \le \frac{1}{1-G_{1.c}}\big(G_2\|\varphi-\overline{\varphi}\|_X^b + G_3\|\varphi-\overline{\varphi}\|_{\partial.c}\big) \tag{3.24}$$

with G_2, G_3 *given by* (3.23).

Proof. It follows from the assumptions of the theorem that for each $\varphi \in \mathcal{J}_B[X]$ the operator T_φ has a fixed point $z[\varphi]\in C_{\varphi.c}[d]$ which is a solution of (3.1), (3.2). The assertion (3.24) follows from Lemma 3.3. □

Remark 3.1. Theorem 3.1 extends a result obtained in [**26**] to quasilinear systems in the Schauder canonic form with the functional dependence $z_{\psi(t,x)}$, where ψ_0 is a function of both variables (t,x).

CHAPTER 4

Mixed Problems for Nonlinear Equations

4.1. Introduction

Suppose that B, E, D_0, $\partial_0 E$ and $E[c]$, $\partial_0 E[c]$, $E^*[c]$ with $c \in (0,a]$ are the sets defined in Chapter 3. Let X be a linear normed space of functions from B into R. Write $\Omega_0 = E \times X \times R^n$ and suppose that the functions

$$f : \Omega_0 \to R, \quad \varphi : E_0 \cup \partial_0 E \to R,$$
$$\psi_0 : [0,a] \to R, \quad \psi' : E \to [-b,b], \quad \psi' = (\psi_1, \ldots, \psi_n),$$

are given. We write $\psi(t,x) = (\psi_0(t), \psi_1(t,x), \ldots, \psi_n(t,x))$, $t \in [0,a]$, $x \in [-b,b]$, and we assume that $\psi_0(t) \le t$ for $t \in [0,a]$. Consider the nonlinear equation

$$\partial_t z(t,x) = f\big(t, x, z_{\psi(t,x)}, \partial_x z(t,x)\big) \tag{4.1}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x), \ \ (t,x) \in E_0 \cup \partial_0 E. \tag{4.2}$$

We consider weak solutions in the Cinquini–Cibrario sense. A function $\overline{z} : E^*[c] \to R$, $c \in (0,a]$, is a C-C solution of (4.1), (4.2) provided

(i) $\overline{z}_{\psi(t,x)} \in X$ for $(t,x) \in E[c]$ and $\partial_x \overline{z}(t,x)$ exists on $E[c]$,
(ii) $\overline{z}(\cdot, x) : [0,c] \to R$ is absolutely continuous on $[0,c]$ for each $x \in [-b,b]$,
(iii) for each $x \in [-b,b]$ the equation (4.1) is satisfied for almost all $t \in [0,c]$ and the condition (4.2) holds on $E_0 \cup \partial_0 E$.

We use the notation introduced in Chapters 1 and 2. Suppose that Assumption H$^*[X]$ (see Section 2.1) is satisfied.

Let us denote by $\mathcal{J}_M[X]$ the class of all initial boundary functions $\varphi : E_0 \cup \partial_0 E \to R$ such that

1) $\varphi_{(t,x)} \in X$ for $(t,x) \in D_0$, there exists $\partial_x \varphi = (\partial_{x_1}\varphi, \ldots, \partial_{x_n}\varphi)$ on $E_0 \cup \partial_0 E$ and $(\partial_{x_i}\varphi)_{(t,x)} \in X$ for $(t,x) \in D_0$, $1 \le i \le n$,
2) there are b_1, $b_2 \in R_+$ with the properties

$$\|\varphi_{(t,x)} - \varphi_{(\overline{t},\overline{x})}\|_X \le b_1\big(|t - \overline{t}| + \|x - \overline{x}\|\big),$$
$$\sum_{i=1}^{n} \big\|(\partial_{x_i}\varphi)_{(t,x)} - (\partial_{x_i}\varphi)_{(\overline{t},\overline{x})}\big\|_X \le b_2\big(|t - \overline{t}| + \|x - \overline{x}\|\big),$$

where (t,x), $(\overline{t},\overline{x}) \in D_0$,

3) there are $q_1, q_2 \in R_+$ such that on $\partial_0 E$ the following estimates are true

$$\big|\varphi(t,x) - \varphi(\overline{t},\overline{x})\big| \le q_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big),$$
$$\big\|\partial_x\varphi(t,x) - \partial_x\varphi(\overline{t},\overline{x})\big\| \le q_2\big(|t-\overline{t}| + \|x-\overline{x}\|\big).$$

Fix $\varphi \in \mathcal{J}_M[X]$ and $c \in (0,a]$, d, p_0, $p_1 \in R_+$. Denote by $C^L_{\varphi.c}[d]$ the class of all functions $z : E^*[c] \to R$ such that $z(t,x) = \varphi(t,x)$ for $(t,x) \in E_0 \cup \partial_0 E[c]$ and the estimate

$$\big|z(t,x) - z(\overline{t},\overline{x})\big| \le d\big(|t-\overline{t}| + \|x-\overline{x}\|\big)$$

holds on $E[c] \cup \partial_0 E[c]$. Let the symbol $C^L_{\partial_x\varphi.c}[p_0,p_1]$ denote the class of all functions $u : E^*[c] \to R^n$ such that $u(t,x) = \partial_x\varphi(t,x)$ for $(t,x) \in E_0 \cup \partial_0 E[c]$ and

$$\|u(t,x)\| \le p_0, \quad \big\|u(t,x) - u(\overline{t},\overline{x})\big\| \le p_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big)$$

on $E[c] \cup \partial_0 E[c]$. We will prove that for sufficiently small $c \in (0,a]$ there exists a solution $\overline{z}$ of the problem (4.1), (4.2) such that $\overline{z} \in C^L_{\varphi.c}[d]$ and $\partial_x\overline{z} \in C^L_{\partial_x\varphi.c}[p_0,p_1]$.

4.2. Properties of Bicharacteristics

We begin with the following assumptions.

Assumption $\mathbf{H}_M[\partial_q f]$. The function $f : \Omega_0 \to R$ of the variables (t,x,w,q) is such that

1) the derivative $\partial_q f(t,x,w,q)$ exists for $(x,w,q) \in [-b,b] \times X \times R^n$ and for almost all $t \in [0,a]$,
2) the function $\partial_q f(\cdot,x,w,q) : [0,a] \to R^n$ is measurable and there are C, $L \in R_+$ such that

$$\|\partial_q f(t,x,w,q)\| \le C,$$
$$\big\|\partial_q f(t,x,w,q) - \partial_q f(t,\overline{x},\overline{w},\overline{q})\big\| \le L\big(\|x-\overline{x}\| + \|w-\overline{w}\|_X + \|q-\overline{q}\|\big)$$

for (x,w,q), $(\overline{x},\overline{w},\overline{q}) \in [-b,b]\times X\times R^n$ and for almost all $t \in [0,a]$,
3) there is $\sigma_0 > 0$ such that for $1 \le i \le n$

$$\partial_{q_i} f(t,x,w,q) \ge \sigma_0, \quad (x,w,q) \in \Delta_i^+ \times X \times R^n,$$
$$\partial_{q_i} f(t,x,w,q) \le -\sigma_0, \quad (x,w,q) \in \Delta_i^- \times X \times R^n$$

for almost all $t \in [0,a]$, where Δ_i^+, Δ_i^-, $1 \le i \le n$, are defined in Section 3.2.

Assumption $\mathbf{H}_M[\psi]$. The functions $\psi_0 : [0,a] \to R$, $\psi' : E \to [-b,b]$, $\psi' = (\psi_1,\ldots,\psi_n)$, are such that $\psi_0(t) \le t$ for $t \in [0,a]$ and

1) the partial derivatives $[\partial_{x_j}\psi_i]_{i,j=1,\ldots,n} = \partial_x\psi'$ exist on E and they are continuous,

2) there are s_1, $s_2 \in R_+$ with the properties

$$\|\partial_{x_j}\psi'(t,x)\| \le s_1, \quad \big\|\partial_{x_j}\psi'(t,x) - \partial_{x_j}\psi'(t,\overline{x})\big\| \le s_2\|x-\overline{x}\|$$

on E, $1 \le j \le n$.

Suppose that Assumptions $\mathrm{H}^*[X]$, $\mathrm{H}_M[\partial_q f]$, $\mathrm{H}_M[\psi]$ are satisfied and let $\varphi \in \mathcal{J}_M[X]$, $c \in (0,a]$, $z \in C^L_{\varphi.c}[d]$, $u \in C^L_{\partial_x\varphi.c}[p_0,p_1]$, $(t,x) \in E[c]$. Consider the Cauchy problem

$$\eta'(\tau) = -\partial_q f\big(\tau, \eta(\tau), z_{\psi(\tau,\eta(\tau))}, u(\tau,\eta(\tau))\big), \quad \eta(t) = x, \tag{4.3}$$

and denote by $g[z,u](\cdot,t,x)$ its solution in the Carathéodory sense. The function $g[z,u](\cdot,t,x)$ is the bicharacteristic of (4.1) corresponding to (z,u). Let $\delta[z,u](t,x)$ be the left end of the maximal interval on which the solution $g[z,u](\cdot,t,x)$ is defined.

We prove a lemma on bicharacteristics and their domains. For $z \in C^L_{\varphi.c}[d]$, $u \in C^L_{\partial_x\varphi.c}[p_0,p_1]$, where $\varphi \in \mathcal{J}_M[X]$, we define

$$\|z\|_{t.1} = \max\Big\{|z(s,y)| : \ (s,y) \in E[t] \cup \partial_0 E[t]\Big\}, \quad 0 \le t \le c,$$
$$\|u\|_{t.n} = \max\Big\{\|u(s,y)\| : \ (s,y) \in E[t] \cup \partial_0 E[t]\Big\}, \quad 0 \le t \le c.$$

Put

$$\begin{gathered} Q_1 = (1+C)\exp(c\Lambda^* L), \quad Q_2 = L\exp(c\Lambda^* L), \\ \Lambda^* = 1 + s_1(K_1 d + K_0 b_1) + p_1, \end{gathered} \tag{4.4}$$

$$\Delta = \bigcup_{i=1}^n (\Delta_i^+ \cup \Delta_i^-), \quad \Gamma = \big(\{0\}\times[-b,b]\big) \cup \big((0,c]\times\Delta\big).$$

Lemma 4.1. *Suppose that Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_M[\partial_q f]$, $\mathrm{H}_M[\psi]$ *are satisfied and assume that* $\varphi, \overline{\varphi} \in \mathcal{J}_M[X]$ *are such that* $\|\varphi - \overline{\varphi}\|^b_X < +\infty$ *and* $z \in C^L_{\varphi.c}[d]$, $\overline{z} \in C^L_{\overline{\varphi}.c}[d]$, $u \in C^L_{\partial_x\varphi.c}[p_0,p_1]$, $\overline{u} \in C^L_{\partial_x\overline{\varphi}.c}[p_0,p_1]$, $c \in (0,a]$. *Then for each* $(t,x) \in E[c]$ *the solutions* $g[z,u](\cdot,t,x)$ *and* $g[\overline{z},\overline{u}](\cdot,t,x)$ *exist on intervals* $I_{(t,x)}$ *and* $\overline{I}_{(t,x)}$ *such that* $(\zeta, g[z,u](\zeta,t,x))$, $(\overline{\zeta}, g[\overline{z},\overline{u}](\overline{\zeta},t,x)) \in \Gamma$, *where* $\zeta = \delta[z,u](t,x)$, $\overline{\zeta} = \delta[\overline{z},\overline{u}](t,x)$. *The solutions of the problems* (4.3) *are unique and they satisfy the conditions*

$$\big\|g[z,u](\tau,t,x) - g[z,u](\tau,\overline{t},\overline{x})\big\| \le Q_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big), \tag{4.5}$$

where (t,x), $(\overline{t},\overline{x}) \in E[c]$, $\tau \in I_{(t,x)} \cap I_{(\overline{t},\overline{x})}$, *and*

$$\begin{aligned} &\big\|g[z,u](\tau,t,x) - g[\overline{z},\overline{u}](\tau,t,x)\big\| \le \\ &\le Q_2\Big|\int_\tau^t \Big(K_1\|z-\overline{z}\|_{\xi.1} + K_0\|\varphi-\overline{\varphi}\|^b_X + \|u-\overline{u}\|_{\xi.n}\Big)\,d\xi\Big|, \end{aligned} \tag{4.6}$$

where $(t,x) \in E[c]$, $\tau \in I_{(t,x)} \cap \overline{I}_{(t,x)}$. *Moreover, the functions* $\delta[z,u]$ *and* $\delta[\overline{z},\overline{u}]$ *are continuous on* $E[c]$ *and*

$$\big|\delta[z,u](t,x) - \delta[z,u](\overline{t},\overline{x})\big| \le \frac{2Q_1}{\sigma_0}\left(|t-\overline{t}| + \|x-\overline{x}\|\right), \tag{4.7}$$

$$\big|\delta[z,u](t,x) - \delta[\overline{z},\overline{u}](t,x)\big| \le$$

$$\le \frac{2Q_2}{\sigma_0}\int_0^t \Big(K_1\|z-\overline{z}\|_{\xi.1} + K_0\|\varphi-\overline{\varphi}\|_X^b + \|u-\overline{u}\|_{\xi.n}\Big)\,d\xi \tag{4.8}$$

on $E[c]$.

Proof. The existence and uniqueness of a Carathéodory solution of (4.3) follows from Assumption $\mathrm{H}_M[\partial_q f]$ and from the following Lipschitz condition

$$\Big|\partial_{q_i} f\big(\tau, y, z_{\psi(\tau,y)}, u(\tau,y)\big) - \partial_{q_i} f\big(\tau,\overline{y}, z_{\psi(\tau,\overline{y})}, u(\tau,\overline{y})\big)\Big| \le L\Lambda^*\|y-\overline{y}\|,$$

where $\tau \in [0,c]$, y, $\overline{y} \in [-b,b]$. The bicharacteristics satisfy the integral equation

$$g[z,u](\tau,t,x) = x - \int_t^\tau \partial_q f\big(P[z,u](\xi,t,x)\big)\,d\xi,$$

where

$$P[z,u](\xi,t,x) =$$

$$= \Big(\xi, g[z,u](\xi,t,x), z_{\psi(\xi,g[z,u](\xi,t,x))}, u\big(\xi, g[z,u](\xi,t,x)\big)\Big). \tag{4.9}$$

Then we have the integral inequality

$$\big\|g[z,u](\tau,t,x) - g[z,u](\tau,\overline{t},\overline{x})\big\| \le$$

$$\le (1+C)\big(|t-\overline{t}|+\|x-\overline{x}\|\big) + \left|\int_t^\tau L\Lambda^*\big\|g[z,u](\xi,t,x) - g[z,u](\xi,\overline{t},\overline{x})\big\|\,d\xi\right|$$

for (t,x), $(\overline{t},\overline{x}) \in E[c]$, $\tau \in I_{(t,x)} \cap I_{(\overline{t},\overline{x})}$, and the inequality

$$\big\|g[z,u](\tau,t,x) - g[\overline{z},\overline{u}](\tau,t,x)\big\| \le$$

$$\le \left|\int_t^\tau L\Big(\Lambda^*\big\|g[z,u](\xi,t,x) - g[\overline{z},\overline{u}](\xi,t,x)\big\| + K_1\|z-\overline{z}\|_{\xi.1} + K_0\|\varphi-\overline{\varphi}\|_X^b + \|u-\overline{u}\|_{\xi.n}\Big)\,d\xi\right|$$

for $(t,x) \in E[c]$, $\tau \in I_{(t,x)} \cap \overline{I}_{(t,x)}$. Using the Gronwall inequality, we obtain (4.5) and (4.6).

The continuity of $\delta[z,u]$ and $\delta[\overline{z},\overline{u}]$ follows from theorems on continuous dependence on initial data for Carathéodory solutions of ordinary differential systems. Let (t,x), $(\overline{t},\overline{x}) \in E[c]$, $\zeta = \delta[z,u](t,x)$, $\overline{\zeta} = \delta[z,u](\overline{t},\overline{x})$. The estimate (4.7) is obvious in the case $\zeta = \overline{\zeta} = 0$. Suppose that $0 \le \zeta < \overline{\zeta}$. Then $g[z,u](\overline{\zeta},\overline{t},\overline{x}) \in \Delta$. Consider the case where $g[z,u](\overline{\zeta},\overline{t},\overline{x}) \in \Delta_i^+$ for some $i \in \{1,\dots,n\}$. Then $g_i[z,u](\overline{\zeta},\overline{t},\overline{x}) = \widetilde{b}_i$.

Let $y = (y_1,\dots,y_n)$, $\widetilde{y} = (y_1,\dots,y_{i-1},\widetilde{b}_i,y_{i+1},\dots,y_n)$. We have

$$\Big|\partial_{q_i} f\big(\tau,y,z_{\psi(\tau,y)},u(\tau,y)\big) - \partial_{q_i} f\big(\tau,\widetilde{y},z_{\psi(\tau,\widetilde{y})},u(\tau,\widetilde{y})\big)\Big| \le L\Lambda^*(\widetilde{b}_i - y_i)$$

for $y \in [-b,b]$ and for almost all $\tau \in [0,c]$. Thus

$$\partial_{q_i} f\big(\tau,y,z_{\psi(\tau,y)},u(\tau,y)\big) \ge \frac{1}{2}\,\sigma_0$$

for $y \in [-b,b]$ such that $\widetilde{b}_i - y_i \le \varepsilon_0$ with $\varepsilon_0 = \frac{\sigma_0}{2L\Lambda^*}$. If the points (t,x), $(\overline{t},\overline{x})$ are such that

$$|t-\overline{t}| + \|x-\overline{x}\| < \widetilde{\delta}_1 \ \text{ with } \ \widetilde{\delta}_1 = \frac{\sigma_0}{2L\Lambda^* Q_1}\,, \tag{4.10}$$

then

$$\widetilde{b}_i - g_i[z,u](\overline{\zeta},t,x) = g_i[z,u](\overline{\zeta},\overline{t},\overline{x}) - g_i[z,u](\overline{\zeta},t,x) \le \varepsilon_0.$$

We get also

$$\partial_{q_i} f\big(P[z,u](\overline{\zeta},t,x)\big) \ge \frac{1}{2}\,\sigma_0 > 0$$

and consequently $g_i[z,u](\cdot,t,x)$ is decreasing on the interval $(\zeta,\overline{\zeta})$. Therefore $\widetilde{b}_i - g_i[z,u](\tau,t,x) \le \varepsilon_0$ and

$$\partial_{q_i} f\big(P[z,u](\tau,t,x)\big) \ge \frac{1}{2}\,\sigma_0$$

for almost all $\tau \in (\zeta,\overline{\zeta})$. Then

$$-\frac{1}{2}\,\sigma_0(\overline{\zeta}-\zeta) \ge -\int_{\zeta}^{\overline{\zeta}} \partial_{q_i} f\big(P[z,u](\tau,t,x)\big)\,d\tau =$$

$$= g_i[z,u](\overline{\zeta},t,x) - g_i[z,u](\zeta,t,x) \ge g_i[z,u](\overline{\zeta},t,x) - g_i[z,u](\overline{\zeta},\overline{t},\overline{x}) \ge$$
$$\ge -Q_1\big(|t-\overline{t}| + \|x-\overline{x}\|\big),$$

that is,

$$\overline{\zeta} - \zeta \le \frac{2Q_1}{\sigma_0}\,\big(|t-\overline{t}| + \|x-\overline{x}\|\big).$$

In the case where $g_i[z,u](\overline{\zeta},\overline{t},\overline{x}) = -\widetilde{b}_i$ we proceed in the similar way. If $(t,x),(\overline{t},\overline{x}) \in E[c]$ do not satisfy (4.10), then we consider the points $(t_0,x_0),(t_1,x_1),\dots,(t_s,x_s)$ such that $(t_0,x_0) = (t,x)$, $(t_s,x_s) = (\overline{t},\overline{x})$ and

$$|t-\overline{t}| + \|x-\overline{x}\| = \sum_{j=0}^{s-1}\big(|t_j - t_{j+1}| + \|x_j - x_{j+1}\|\big),$$

$$|t_j - t_{j+1}| + \|x_j - x_{j+1}\| < \widetilde{\delta}_1 \ \text{ for } \ 0 \le j \le s-1.$$

Then we have

$$\big|\delta_i[z](t,x)-\delta_i[z](\overline{t},\overline{x})\big| \le \sum_{j=0}^{s-1}\big|\delta[z,u](t_j,x_j)-\delta[z,u](t_{j+1},x_{j+1})\big| \le$$

$$\le \frac{2Q_1}{\sigma_0}\sum_{j=0}^{s-1}\big(|t_j-t_{j+1}|+\|x_j-x_{j+1}\|\big) = \frac{2Q_1}{\sigma_0}\big(|t-\overline{t}|+\|x-\overline{x}\|\big).$$

To prove (4.8), suppose that $(t,x)\in E[c]$, $0\le \delta[z,u](t,x)<\delta[\overline{z},\overline{u}](t,x)$. Let $\overline{\xi}=\delta[\overline{z},\overline{u}](t,x)$, $\xi=\delta[z,u](t,x)$. We have $g[\overline{z},\overline{u}](\overline{\xi},t,x)\in\Delta$. Consider the case where $g[\overline{z},\overline{u}](\overline{\xi},t,x)\in\Delta_i^+$ for some $i\in\{1,\dots,n\}$. We have $g_i[\overline{z},\overline{u}](\overline{\xi},t,x)=\widetilde{b}_i$. If $(t,x)\in E[c]$ and (φ,z,u), $(\overline{\varphi},\overline{z},\overline{u})$ are such that

$$K_1\|z-\overline{z}\|_{c.1}+K_0\|\varphi-\overline{\varphi}\|_X^b+\|u-\overline{u}\|_{c.n}<\widetilde{\delta}_2 \tag{4.11}$$

with $\widetilde{\delta}_2=\frac{\sigma_0}{2cL\Lambda^*Q_2}$, then

$$\widetilde{b}_i-g_i[z,u](\overline{\xi},t,x)=g_i[\overline{z},\overline{u}](\overline{\xi},t,x)-g_{ij}[z](\overline{\xi},t,x)\le\varepsilon_0.$$

Thus $\widetilde{b}_i-g_i[z,u](\tau,t,x)\le\varepsilon_0$ and

$$\partial_{q_i}f\big(P[z,u](\tau,t,x)\big)\ge\frac{1}{2}\,\sigma_0>0$$

for almost all $\tau\in(\xi,\overline{\xi})$. Then

$$-\frac{1}{2}\,\sigma_0(\overline{\xi}-\xi)\ge-\int_{\xi}^{\overline{\xi}}\partial_{q_i}f\big(P[z,u](\tau,t,x)\big)\,d\tau=$$

$$=g_i[z,u](\overline{\xi},t,x)-g_i[z,u](\xi,t,x)\ge g_i[z,u](\overline{\xi},t,x)-g_i[\overline{z},\overline{u}](\overline{\xi},t,x)\ge$$

$$\ge -Q_2\int_0^t\Big(K_1\|z-\overline{z}\|_{\tau.1}+K_0\|\varphi-\overline{\varphi}\|_X^b+\|u-\overline{u}\|_{\tau.n}\Big)\,d\tau,$$

that is,

$$\overline{\xi}-\xi\le\frac{2Q_2}{\sigma_0}\int_0^t\Big(K_1\|z-\overline{z}\|_{\tau.1}+K_0\|\varphi-\overline{\varphi}\|_X^b+\|u-\overline{u}\|_{\tau.n}\Big)\,d\tau.$$

If (φ,z,u), $(\overline{\varphi},\overline{z},\overline{u})$ do not satisfy (4.11), then to obtain (4.8) we use the functions (φ_j,z_j,u_j), $0\le j\le\nu$, such that $\varphi_j\in\mathcal{J}_M[X]$, $z_j\in C^L_{\varphi_j.c}[d]$, $u_j\in C^L_{\partial_x\varphi_j.c}[p_0,p_1]$, $0\le j\le\nu$, $\varphi_0=\varphi$, $z_0=z$, $u_0=u$, $\varphi_\nu=\overline{\varphi}$, $z_\nu=\overline{z}$, $u_\nu=\overline{u}$ and

$$K_1\|z-\overline{z}\|_{c.1}+K_0\|\varphi-\overline{\varphi}\|_X^b+\|u-\overline{u}\|_{c.n}=$$

$$=\sum_{j=0}^{\nu-1}\Big(K_1\|z_j-z_{j+1}\|_{c.1}+K_0\|\varphi_j-\varphi_{j+1}\|_X^b+\|u_j-u_{j+1}\|_{c.n}\Big)$$

and

$$K_1\|z_j - z_{j+1}\|_{c.1} + K_0\|\varphi_j - \varphi_{j+1}\|_X^b + \|u_j - u_{j+1}\|_{c.n} < \widetilde{\delta}_2, \quad 0 \le j \le \nu - 1.$$

This completes the proof of Lemma 4.1. □

4.3. The Sequence of Successive Approximations

We formulate further assumptions on φ and f. For $\varphi \in \mathcal{J}_M[X]$ let the symbol S_φ denote the set of all functions $\omega : E^*[a] \to R$ which are continuous and $\omega(t,x) = \varphi(t,x)$ for $(t,x) \in E_0 \cup \partial_0 E$. Let us denote by $\mathcal{J}_M^+[X]$ the class of all initial boundary functions $\varphi \in \mathcal{J}_M[X]$ satisfying the condition:

1) if $\omega, \widetilde{\omega} \in S_\varphi$ then

$$f(t,x,\omega_{\psi(t,x)},q) = f(t,x,\widetilde{\omega}_{\psi(t,x)},q)$$

for $x \in \Delta$, $q \in R^n$ and for almost all $t \in [0,a]$,

2) there is $\gamma : \partial_0 E \to R^n$, $\gamma = (\gamma_1, \dots, \gamma_n)$, such that

$$\partial_t \varphi(t,x) = f(t,x,\widetilde{\varphi}_{\psi(t,x)}, \gamma(t,x)) \tag{4.12}$$

for $x \in \Delta$ and for almost all $t \in [0,a]$, where $\widetilde{\varphi} \in S_\varphi$ and $\gamma_i(t,x) = \partial_{x_i}\varphi(t,x)$ for $i \in \{j : r_j > 0\}$.

Remark 4.1. The relation (4.12) is the consistency condition and it can be considered as an assumption on φ at (t,x) such that $t \in [0,a]$, $x \in \Delta_i^+ \cup \Delta_i^-$ and $r_i > 0$. If $i \in \{j : r_j = 0\}$, then (4.12) is the equation for $\gamma_i(t,x)$, $t \in [0,a]$, $x \in \Delta_i^+ \cup \Delta_i^-$.

Assumption $\mathbf{H}_M[f]$. The function $f : \Omega_0 \to R$ satisfies Assumption $\mathrm{H}_M[\partial_q f]$ and

1) there is $\widetilde{C} \in R_+$ such that $|f(t,x,w,q)| \le \widetilde{C}$ on Ω_0 and

$$\big|f(t,x,w,q) - f(\bar{t},x,w,q)\big| \le C|t - \bar{t}|,$$

where $(t,x,w,q), (\bar{t},x,w,q) \in \Omega_0$,

2) the derivative $\partial_x f(t,x,w,q)$ and the Fréchet derivative $\partial_w f(t,x,w,q) \in CL(X,R)$ exist for $(x,w,q) \in [-b,b] \times X \times R^n$ and for almost all $t \in [0,a]$,

3) the estimates

$$\big\|\partial_x f(t,x,w,q)\big\| \le C, \quad \big\|\partial_w f(t,x,w,q)\big\|_* \le C$$

and the Lipschitz conditions

$$\big\|\partial_x f(t,x,w,q) - \partial_x f(t,\overline{x},\overline{w},\overline{q})\big\| \le L\big(\|x - \overline{x}\| + \|w - \overline{w}\|_X + \|q - \overline{q}\|\big),$$

$$\big\|\partial_w f(t,x,w,q) - \partial_w f(t,\overline{x},\overline{w},\overline{q})\big\|_* \le L\big(\|x - \overline{x}\| + \|w - \overline{w}\|_X + \|q - \overline{q}\|\big)$$

are satisfied for $(x,w,q), (\overline{x},\overline{w},\overline{q}) \in [-b,b] \times X \times R^n$ and for almost all $t \in [0,a]$.

If $\omega = (\omega_1, \ldots, \omega_n)$ with $\omega_i \in X$, $1 \le i \le n$, and $(t, x, w, q) \in \Omega_0$, then we write

$$\partial_w f(t, x, w, q)(\omega) = \Big(\partial_w f(t, x, w, q)\omega_1, \ldots, \partial_w f(t, x, w, q)\omega_n\Big).$$

For $\varphi \in \mathcal{J}_M^+[X]$ and $z \in C^L_{\varphi.c}[d]$, $u, v \in C^L_{\partial_x\varphi.c}[p_0, p_1]$ with $c \in (0, a]$ we define

$$F[z, u] : E[c] \to R,$$

$$G[z, v, u] : E[c] \to R^n, \;\; G[z, v, u] = \big(G_1[z, v, u], \ldots, G_n[z, v, u]\big)$$

in the following way

$$F[z, u](t, x) = \varphi\big(Q[z, u](t, x)\big)+$$

$$+\int_\delta^t \Big[f\big(P[z, u](\tau, t, x)\big) - \partial_q f\big(P[z, u](\tau, t, x)\big) \circ u\big(\tau, g[z, u](\tau, t, x)\big)\Big]\, d\tau, \quad (4.13)$$

$$G[z, v, u](t, x) = \partial_x\varphi\big(Q[z, u](t, x)\big) + \int_\delta^t \Big[\partial_x f\big(P[z, u](\tau, t, x)\big)+$$

$$+\partial_w f\big(P[z, u](\tau, t, x)\big)\Big(v_{\psi(\tau, g[z,u](\tau, tx))}\partial_x\psi'(\tau, g[z, u](\tau, t, x))\Big)\Big]\, d\tau, \quad (4.14)$$

where $\delta = \delta[z, u](t, x)$, $P[z, u](\cdot, t, x)$ is given by (4.9) and

$$Q[z, u](t, x) = \Big(\delta[z, u](t, x), g[z, u]\big(\delta[z, u](t, x), t, x\big)\Big). \quad (4.15)$$

We define the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$, where $z^{(m)} : E^*[c] \to R$, $u^{(m)} : E^*[c] \to R^n$, as follows. Let $\widetilde{\varphi} : E^*[c] \to R$ be an extention of φ such that $\widetilde{\varphi} \in C^L_{\varphi.c}[d]$, $\partial_x\widetilde{\varphi} \in C^L_{\partial_x\varphi.c}[p_0, p_1]$. Put

$$z^{(0)} = \widetilde{\varphi} \;\text{ and }\; u^{(0)} = \partial_x\widetilde{\varphi} \;\text{ on }\; E^*[c].$$

Suppose that $z^{(m)} \in C^L_{\varphi.c}[d]$ and $u^{(m)} \in C^L_{\partial_x\varphi.c}[p_0, p_1]$ are known functions. Then

1) the function $u^{(m+1)}$ is a solution of the problem

$$u = G\big[z^{(m)}, u^{(m)}, u\big], \;\; u = \partial_x\varphi \;\text{ on }\; E_0 \cup \partial_0 E[c], \quad (4.16)$$

2) the function $z^{(m+1)}$ is given by

$$z^{(m+1)} = F\big[z^{(m)}, u^{(m+1)}\big], \;\; z^{(m+1)} = \varphi \;\text{ on }\; E_0 \cup \partial_0 E[c]. \quad (4.17)$$

Remark 4.2. The above defined sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$ are the sequences of succesive approximations for the system of functional integral equations

$$z = F[z, u], \quad u = G[z, u, u] \;\text{ on }\; E[c] \quad (4.18)$$

with initial boundary conditions

$$z = \varphi, \quad u = \partial_x\varphi \;\text{ on }\; E_0 \cup \partial_0 E[c].$$

This problem is obtained by introducing an unknown function u with $u = \partial_x z$ and considering the linearization of (4.1)

$$\partial_t z(t,x) = f\big(t,x,z_{\psi(t,x)},u(t,x)\big)+ \\ + \partial_q f\big(t,x,z_{\psi(t,x)},u(t,x)\big) \circ \big(\partial_x z(t,x) - u(t,x)\big). \quad (4.19)$$

By virtue of (4.1) we get the following differential system for the unknown function u

$$\partial_t u(t,x) = \partial_x f\big(t,x,z_{\psi(t,x)},u(t,x)\big) + \partial_q f\big(t,x,z_{\psi(t,x)},u(t,x)\big) \circ \partial_x u(t,x) + \\ + \partial_w f\big(t,x,z_{\psi(t,x)},u(t,x)\big)\big((\partial_x z)_{\psi(t,x)} \partial_x \psi'(t,x)\big). \quad (4.20)$$

Finally we put $\partial_x z = u$ in (4.20) and we consider (4.19), (4.20) along the bicharacteristics $g[z,u](\cdot,t,x)$. Integrating from $\delta[z,u](t,x)$ to t with respect to τ, we get (4.18).

We formulate lemmas on existence of the above defined sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$. We need the following assumption on the constants c, d, p_0, p_1. Write

$$V_1 = b_1 Q_1 \Big(1 + \frac{2(1+C)}{\sigma_0}\Big), \quad V_2 = (\widetilde{C} + Cp_0)\Big(1 + \frac{2Q_1}{\sigma_0}\Big),$$
$$V_3 = c\big(\Lambda^*(C + Lp_0) + Cp_0\big)Q_1,$$
$$\widetilde{\mu}_0 = K_1 p_0 + K_0 b_1, \quad \widetilde{\mu}_1 = K_1 p_1 + K_0 b_2,$$
$$L_f = L\Lambda^* Q_1, \quad L_\varphi = q_2 Q_1 \Big(1 + \frac{2(1+C)}{\sigma_0}\Big), \quad L_w = Q_1\big(s_2 \widetilde{\mu}_0 + s_1^2 \widetilde{\mu}_1\big).$$

Assumption $\mathbf{H}_M[c,d,p_0,p_1]$. The constants $c \in (0,a]$, d, p_0, $p_1 \in R_+$ satisfy the conditions:

$$p_0 = d \geq \max\Big\{q_1 + cC(1 + s_1\widetilde{\mu}_0), \sum_{i=1}^{3} V_i\Big\}, \quad (4.21)$$

$$p_1 \geq L_\varphi + c\Big(L_f + L_f s_1 \widetilde{\mu}_0 + CL_w\Big) + C\Big(1 + \frac{2Q_1}{\sigma_0}\Big). \quad (4.22)$$

If $m \geq 1$ is fixed and the functions $z^{(m)} \in C^L_{\varphi.c}[d]$ and $u^{(m)} \in C^L_{\partial_x\varphi.c}[p_0,p_1]$ are known, then we write

$$G^{(m)}[u] = G[z^{(m)}, u^{(m)}, u], \quad u \in C^L_{\partial_x\varphi.c}[p_0,p_1]. \quad (4.23)$$

Lemma 4.2. *If Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_M[f]$, $\mathrm{H}_M[\psi]$, $\mathrm{H}_M[c,d,p_0,p_1]$ *are satisfied and* $\varphi \in \mathcal{J}^+_M[X]$, *then* $G^{(m)} : C^L_{\partial_x\varphi.c}[p_0,p_1] \to C^L_{\partial_x\varphi.c}[p_0,p_1]$. *Moreover, there exists exactly one function* $\widetilde{u} \in C^L_{\partial_x\varphi.c}[p_0,p_1]$ *satisfying the equation* $u = G^{(m)}[u]$.

Proof. Let $u \in C^L_{\partial_x\varphi.c}[p_0,p_1]$. It follows from the assumptions of the lemma that

$$\big\|G^{(m)}[u](t,x)\big\| \leq q_1 + cC(1 + s_1\widetilde{\mu}_0) \ \text{ on } \ E[c],$$

and according to (4.21) we get

$$\big\|G^{(m)}[u](t,x)\big\| \le p_0, \quad (t,x)\in E[c].$$

Let $w^{(m)}[u](\tau,t,x)\in X^n$ be given by

$$w^{(m)}[u](\tau,t,x) = (u^{(m)})_{\psi(\tau,g[z^{(m)},u](\tau,t,x))}\partial_x\psi'\big(\tau,g[z^{(m)},u](\tau,t,x)\big).$$

Suppose that (t,x), $(\bar t,\bar x)\in E[c]$. We have

$$\Big\|\partial_x f\big(P[z^{(m)},u](\tau,t,x)\big) - \partial_x f\big(P[z^{(m)},u](\tau,\bar t,\bar x)\big)\Big\| \le$$
$$\le L_f\big(|t-\bar t| + \|x-\bar x\|\big),$$
$$\Big\|\partial_w f\big(P[z^{(m)},u](\tau,t,x)\big) - \partial_w f\big(P[z^{(m)},u](\tau,\bar t,\bar x)\big)\Big\|_* \le$$
$$\le L_f\big(|t-\bar t| + \|x-\bar x\|\big),$$
$$\Big\|\partial_x \varphi\big(Q[z^{(m)},u](t,x)\big) - \partial_x \varphi\big(Q[z^{(m)},u](\bar t,\bar x)\big)\Big\| \le$$
$$\le L_\varphi\big(|t-\bar t| + \|x-\bar x\|\big),$$
$$\|w^{(m)}[u](\tau,t,x) - w^{(m)}[u](\tau,\bar t,\bar x)\|_X \le L_w\big(|t-\bar t| + \|x-\bar x\|\big).$$

Thus we obtain

$$\big\|G^{(m)}[u](t,x)-G^{(m)}[u](\bar t,\bar x)\big\| \le p_1\big(|t-\bar t|+\|x-\bar x\|\big), \quad (t,x),\ (\bar t,\bar x)\in E[c]$$

under the assumption (4.22). This proves that $G^{(m)}[u]\in C^L_{\partial_x\varphi.c}[p_0,p_1]$.

There is $\widetilde\gamma>0$ such that for $u,\overline u\in C^L_{\partial_x\varphi.c}[p_0,p_1]$

$$\big\|G^{(m)}[u](t,x) - G^{(m)}[\overline u](t,x)\big\| \le \widetilde\gamma\int_0^t \|u-\overline u\|_{\xi.n}d\xi, \quad (t,x)\in E[c].$$

For $u\in C^L_{\partial_x\varphi.c}[p_0,p_1]$ and for $\lambda>\widetilde\gamma$ we define

$$\|u\|_{(\lambda)} = \max\Big\{\|u(t,x)\|e^{-\lambda t}:\ (t,x)\in E[c]\Big\}.$$

If $u,\overline u\in C^L_{\partial_x\varphi.c}[p_0,p_1]$, then

$$\big\|G^{(m)}[u](t,x)-G^{(m)}[\overline u](t,x)\big\| \le \widetilde\gamma\int_0^t\|u-\overline u\|_{(\lambda)}\, e^{\lambda\xi}\, d\xi \le \frac{\widetilde\gamma}{\lambda}\, \|u-\overline u\|_{(\lambda)}e^{\lambda t},$$

that is,

$$\big\|G^{(m)}[u] - G^{(m)}[\overline u]\big\|_{(\lambda)} \le \frac{\widetilde\gamma}{\lambda}\, \|u-\overline u\|_{(\lambda)}.$$

We have $\frac{\widetilde\gamma}{\lambda}<1$ and hence there exists exactly one $\widetilde u\in C^L_{\partial_x\varphi.c}[p_0,p_1]$ satisfying the equation $u=G^{(m)}[u]$. The proof of Lemma 4.2 is complete. □

The next lemma is important in our considerations.

Lemma 4.3. *If Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_M[f]$, $\mathrm{H}_M[\psi]$, $\mathrm{H}_M[c,d,p_0,p_1]$ *are satisfied,* $\varphi \in \mathcal{J}^+_M[X]$, *then for any* $m \geq 0$ *we have*

$$\partial_x z^{(m)}(t,x) = u^{(m)}(t,x), \quad (t,x) \in E[c] \tag{4.24}$$

and

$$z^{(m)} \in C^L_{\varphi.c}[d]. \tag{4.25}$$

Proof. We prove (4.24) by induction. It follows from the definition of $z^{(0)}$, $u^{(0)}$ that (4.24) is satisfied for $m = 0$. Suppose that (4.24) holds for a given $m \geq 0$. We will prove that

$$\partial_x z^{(m+1)} = u^{(m+1)} \ \text{ on } \ E[c]. \tag{4.26}$$

Write

$$\Delta(t,x,\overline{x}) = z^{(m+1)}(t,\overline{x}) - z^{(m+1)}(t,x) - u^{(m+1)}(t,x) \circ (\overline{x} - x),$$

where $(t,x), (t,\overline{x}) \in E[c]$. We prove that there exists $C_0 \in R_+$ such that

$$\big|\Delta(t,x,\overline{x})\big| \leq C_0 \|\overline{x} - x\|^2. \tag{4.27}$$

According to (4.16), (4.17) and (4.23) we have

$$\Delta(t,x,\overline{x}) =$$
$$= F\big[z^{(m)}, u^{(m+1)}\big](t,\overline{x}) - F\big[z^{(m)}, u^{(m+1)}\big](t,x) - G^{(m)}[u^{(m+1)}](t,x) \circ (\overline{x} - x).$$

For simplicity of notation write

$$g(\tau,t,x) = \big[z^{(m)}, u^{(m+1)}\big](\tau,t,x), \quad \delta(t,x) = \delta\big[z^{(m)}, u^{(m+1)}\big](t,x),$$
$$w(\tau,t,x) = w^{(m)}[u^{(m+1)}](\tau,t,x), \tag{4.28}$$
$$Q(t,x) = Q\big[z^{(m)}, u^{(m+1)}\big](t,x), \quad P(\tau,t,x) = P\big[z^{(m)}, u^{(m+1)}\big](\tau,t,x).$$

Let $R(s,\tau,t,x,\overline{x})$ be the following intermediate point

$$R(s,\tau,t,x,\overline{x}) = P(\tau,t,x) + s\big(P(\tau,t,\overline{x}) - P(\tau,t,x)\big), \quad 0 \leq s \leq 1.$$

Assume that $(t,x), (t,\overline{x}) \in E[c]$. Consider the case $\delta(t,\overline{x}) \leq \delta(t,x)$. Similar arguments apply to the case $\delta(t,\overline{x}) > \delta(t,x)$. To formulate properties of Δ, we define

$$\mathcal{A}(t,x,\overline{x}) = \varphi\big(Q(t,\overline{x})\big) - \varphi(Q(t,x)) - \partial_t\varphi(Q(t,x))\big(\delta(t,\overline{x}) - \delta(t,x)\big) +$$
$$- \partial_x\varphi(Q(t,x)) \circ \big(g(\delta(t,\overline{x}),t,\overline{x}) - g(\delta(t,x),t,x)\big),$$
$$\mathcal{B}(t,x,\overline{x}) = \partial_t\varphi(Q(t,x))\big(\delta(t,\overline{x}) - \delta(t,x)\big) +$$
$$+ \partial_x\varphi(Q(t,x)) \circ \big(g(\delta(t,\overline{x}),t,\overline{x}) - g(\delta(t,x),t,x) - (\overline{x} - x)\big),$$
$$\delta_{f.x}(s,\tau,t,x,\overline{x}) = \partial_x f\big(R(s,\tau,t,x,\overline{x})\big) - \partial_x f(P(\tau,t,x)),$$
$$\delta_{f.w}(s,\tau,t,x,\overline{x}) = \partial_w f\big(R(s,\tau,t,x,\overline{x})\big) - \partial_w f(P(\tau,t,x)),$$
$$\delta_{f.q}(s,\tau,t,x,\overline{x}) = \partial_q f\big(R(s,\tau,t,x,\overline{x})\big) - \partial_q f\big(P(\tau,t,\overline{x})\big).$$

We have

$$\Delta(t,x,\overline{x}) = \Delta_1(t,x,\overline{x}) + \Delta_2(t,x,\overline{x}),$$

where

$$\Delta_1(t,x,\overline{x}) = \mathcal{A}(t,x,\overline{x})+$$

$$+\int_{\delta(t,x)}^{t}\int_{0}^{1}\Big[\delta_{f.x}(s,\tau,t,x,\overline{x})\circ\big(g(\tau,t,\overline{x})-g(\tau,t,x)\big)+$$

$$+\delta_{f.w}(s,\tau,t,x,\overline{x})\Big((z^{(m)})_{\psi(\tau,g(\tau,t,\overline{x}))}-(z^{(m)})_{\psi(\tau,g(\tau,t,x))}\Big)+$$

$$+\delta_{f.q}(s,\tau,t,x,\overline{x})\circ\Big(u^{(m+1)}(\tau,g(\tau,t,\overline{x}))-u^{(m+1)}(\tau,g(\tau,t,x))\Big)\Big]\,ds\,d\tau+$$

$$+\int_{\delta(t,x)}^{t}\partial_w f(P(\tau,t,x))\Big((z^{(m)})_{\psi(\tau,g(\tau,t,\overline{x}))}-(z^{(m)})_{\psi(\tau,g(\tau,t,x))}-$$

$$-w(\tau,t,x)\circ\big(g(\tau,t,\overline{x})-g(\tau,t,x)\big)\Big)\,d\tau$$

and

$$\Delta_2(t,x,\overline{x}) = \mathcal{B}(t,x,\overline{x})+\int_{\delta(t,x)}^{t}\Big[\Big(\partial_x f(P(\tau,t,x))+$$

$$+\partial_w f(P(\tau,t,x))(w(\tau,t,x))\Big)\circ\big(g(\tau,t,\overline{x})-g(\tau,t,x)-(\overline{x}-x)\big)-$$

$$-\Big(\partial_q f(P(\tau,t,\overline{x}))-\partial_q f(P(\tau,t,x))\Big)\circ u^{(m+1)}(\tau,g(\tau,t,x))\Big]\,d\tau+$$

$$+\int_{\delta(t,\overline{x})}^{\delta(t,x)}\Big(f(P(\tau,t,\overline{x}))-\partial_q f(P(\tau,t,\overline{x}))\circ u^{(m+1)}\big(\tau,g(\tau,t,\overline{x})\big)\Big)\,d\tau.$$

Substituting the relation

$$g(\tau,t,\overline{x})-g(\tau,t,x)-(\overline{x}-x)=\int_{\tau}^{t}\big(\partial_q f(P(\tau,t,\overline{x}))-\partial_q f(P(\tau,t,x))\big)\,d\tau$$

into $\Delta_2(t,x,\overline{x})$ and changing the order of integration, we obtain

$$\Delta_2(t,x,\overline{x}) = \mathcal{C}(t,x,\overline{x})+$$

$$+\int_{\delta(t,x)}^{t}\big(\partial_q f(P(\tau,t,\overline{x}))-\partial_q f(P(\tau,t,x))\big)\circ\mathcal{D}(\tau,t,x)\,d\tau,$$

where

$$\mathcal{C}(t,x,\overline{x}) = \int_{\delta(t,\overline{x})}^{\delta(t,x)}\big(f(P(\tau,t,\overline{x}))-\partial_t\varphi(Q(t,x))\big)\,d\tau+$$

$$+\int_{\delta(t,\overline{x})}^{\delta(t,x)}\Big(\partial_x\varphi(Q(t,x))-u^{(m+1)}\big(\tau,g(\tau,t,\overline{x})\big)\Big)\circ\partial_q f(P(\tau,t,\overline{x}))\,d\tau,$$

$$\mathcal{D}(\tau,t,x)=-u^{(m+1)}(\tau,g(\tau,t,x))+\partial_x\varphi(Q(t,x))+ \\ +\int_{\delta(t,x)}^{t}\Big(\partial_x f(P(\xi,t,x))+\partial_w f(P(\xi,t,x))(w(\xi,t,x))\Big)\,d\xi.$$

Since $g(s,\tau,g(\tau,t,x))=g(s,t,x)$ and $\delta(\tau,g(\tau,t,x))=\delta(t,x)$ for $(t,x)\in E[c]$, $\tau,s\in I_{(t,x)}$, where $I_{(t,x)}$ is the domain of $g(\cdot,t,x)$, we have

$$u^{(m+1)}(\tau,g(\tau,t,x))=\partial_x\varphi(Q(t,x))+ \\ +\int_{\delta(t,x)}^{\tau}\Big(\partial_x f(P(s,t,x))+\partial_w f(P(s,t,x))(w(s,t,x))\Big)\,ds,$$

and thus

$$\mathcal{D}(\tau,t,x)=0,\ \ (t,x)\in E[c],\ \tau\in I_{(t,x)}.$$

It follows from our assumptions that there is $C_1\in R_+$ such that

$$\big|\mathcal{C}(t,x,\overline{x})\big|\le C_1\|\overline{x}-x\|^2$$

and, consequently,

$$\big|\Delta_2(t,x,\overline{x})\big|\le C_1\|\overline{x}-x\|^2\ \text{ for }\ (t,x),(t,\overline{x})\in E[c]. \tag{4.29}$$

We estimate $\Delta_1(t,x,\overline{x})$. There exists $C_{\mathcal{A}}\in R_+$ such that

$$\big|\mathcal{A}(t,x,\overline{x})\big|\le C_{\mathcal{A}}\|\overline{x}-x\|^2.$$

The terms $\|\delta_{f.x}(s,\tau,t,x,\overline{x})\|$, $\|\delta_{f.q}(s,\tau,t,x,\overline{x})\|$, $\|\delta_{f.w}(s,\tau,t,x,\overline{x})\|_*$ are bounded from above by $L_\delta\|g(\tau,t,\overline{x})-g(\tau,t,x)\|$ for some $L_\delta\in R_+$. We have also

$$\big\|(z^{(m)})_{\psi(\tau,g(\tau,t,\overline{x}))}-(z^{(m)})_{\psi(\tau,g(\tau,t,x))}\big\|_X\le \\ \le s_1(K_1d+K_0b_1)\big\|g(\tau,t,\overline{x})-g(\tau,t,x)\big\|,$$

$$\big\|u^{(m+1)}\big(\tau,g(\tau,t,\overline{x})\big)-u^{(m+1)}(\tau,g(\tau,t,x))\big\|\le p_1\big\|g(\tau,t,\overline{x})-g(\tau,t,x)\big\|.$$

It follows from the equality $\partial_x z^{(m)}=u^{(m)}$ on $E[c]$ that

$$\Big\|(z^{(m)})_{\psi(\tau,g(\tau,t,\overline{x}))}-(z^{(m)})_{\psi(\tau,g(\tau,t,x))}-w(\tau,t,x)\circ\big(g(\tau,t,\overline{x})-g(\tau,t,x)\big)\Big\|_X\le \\ \le\Big(s_1^2(K_1p_1+K_0b_2)+s_2(K_1d+K_0b_1)\Big)\big\|g(\tau,t,\overline{x})-g(\tau,t,x)\big\|^2.$$

All the above estimates and the properties of bicharacteristics imply that there is $C_2\in R_+$ such that for $(t,x),(t,\overline{x})\in E[c]$

$$|\Delta_1(t,x,\overline{x})|\le C_2\|\overline{x}-x\|^2. \tag{4.30}$$

The inequalities (4.29) and (4.30) give (4.27) and, consequently,

$$\partial_x z^{(m+1)}(t,x) = u^{(m+1)}(t,x), \quad (t,x) \in E[c].$$

The proof of (4.26) is complete.

It follows from (4.24) that on $E[c]$

$$\|\partial_x z^{(m+1)}(t,x)\| \leq d.$$

Let $(t,x), (\bar{t},x) \in E[c]$. We use the notation (4.28) and we can write the following estimates

$$\big|\varphi(Q(t,x)) - \varphi(Q(\bar{t},x))\big| \leq V_1|t-\bar{t}|,$$

$$\Big|\int_{\delta(\bar{t},x)}^{\delta(t,x)} \Big|f(P(\tau,t,x)) - \partial_q f(P(\tau,t,x)) \circ u^{(m+1)}\big(\tau, g(\tau,t,x)\big)\Big|\, d\tau\Big| +$$

$$+\Big|\int_{\bar{t}}^{t} \Big|f(P(\tau,t,x)) - \partial_q f(P(\tau,t,x)) \circ u^{(m+1)}\big(\tau, g(\tau,t,x)\big)\Big|\, d\tau\Big| \leq$$

$$\leq V_2|t-\bar{t}|,$$

$$\Big|\int_{\delta(\bar{t},x)}^{\bar{t}} \Big(\Big|f(P(\tau,t,x)) - f(P(\tau,\bar{t},x))\Big| + \Big|\partial_q f(P(\tau,t,x)) \circ u^{(m+1)}\big(\tau, g(\tau,t,x)\big) -$$

$$-\partial_q f\big(P(\tau,\bar{t},x)\big) \circ u^{(m+1)}\big(\tau, g(\tau,\bar{t},x)\big)\Big|\Big)\, d\tau\Big| \leq V_3|t-\bar{t}|.$$

It follows from Assumption $\mathrm{H}_M[c,d,p_0,p_1]$ that

$$\big|z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},x)\big| \leq d|t-\bar{t}|.$$

Thus $z^{(m+1)} \in C^L_{\varphi.c}[d]$. This completes the proof of Lemma 4.3. □

4.4. Existence and Uniqueness of Generalized Solutions

First we prove the convergence of the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$.

Lemma 4.4. *If Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_M[f]$, $\mathrm{H}_M[\psi]$ *and* $\mathrm{H}_M[c,d,p_0,p_1]$ *are satisfied and* $\varphi \in \mathcal{J}^+_M[X]$, *then the sequences* $\{z^{(m)}\}$ *and* $\{u^{(m)}\}$ *are uniformly convergent on* $E[c]$.

Proof. For $t \in [0,c]$ and $m \geq 1$ we write

$$Z_m(t) = \big\|z^{(m)} - z^{(m-1)}\big\|_{t.1} \quad \text{and} \quad U_m(t) = \big\|u^{(m)} - u^{(m-1)}\big\|_{t.n}.$$

The assumptions of the lemma imply the inequality

$$U_{m+1}(t) \leq \widetilde{\Gamma}_1 \int_0^t \big(K_1 Z_m(\tau) + U_{m+1}(\tau)\big)\, d\tau + K_1 C s_1 \int_0^t U_m(\tau)\, d\tau,$$

where $t \in [0,c]$, for some $\widetilde{\Gamma}_1 \in R_+$ independent of m. Thus it follows from the Gronwall inequality that

$$U_{m+1}(t) \le \widetilde{\Gamma}_2 \int_0^t \big(Z_m(\tau) + U_m(\tau)\big)\, d\tau, \tag{4.31}$$

where $\widetilde{\Gamma}_2 = e^{c\widetilde{\Gamma}_1} \max\{K_1\widetilde{\Gamma}_1, K_1 C s_1\}$. We have also

$$Z_{m+1}(t) \le \widetilde{\Gamma}_3 \int_0^t \big(K_1 Z_m(\tau) + U_{m+1}(\tau)\big)\, d\tau + C \int_0^t U_{m+1}(\tau)\, d\tau,$$

where $t \in [0,c]$, for some $\widetilde{\Gamma}_3 \in R_+$. Using (4.31), we obtain

$$Z_{m+1}(t) \le \widetilde{\Gamma}_4 \int_0^t \big(Z_m(\tau) + U_m(\tau)\big)\, d\tau,$$

where $\widetilde{\Gamma}_4 = \max\{\widetilde{\Gamma}_3 K_1, c(\widetilde{\Gamma}_3 + C)\widetilde{\Gamma}_2\}$. Put $\widetilde{\Gamma}_0 = \widetilde{\Gamma}_2 + \widetilde{\Gamma}_4$ and observe that we have obtained the integral inequality

$$Z_{m+1}(t) + U_{m+1}(t) \le \widetilde{\Gamma}_0 \int_0^t \big(Z_m(\tau) + U_m(\tau)\big)\, d\tau, \quad t \in [0,c]. \tag{4.32}$$

For $Z \in C([0,c], R)$ and for $\lambda > \widetilde{\Gamma}_0$ we write

$$\|Z\|_\lambda = \max\big\{|Z(t)|e^{-\lambda t} :\ t \in [0,c]\big\}.$$

It follows from (4.32) that

$$Z_{m+1}(t) + U_{m+1}(t) \le \widetilde{\Gamma}_0 \int_0^t \big(\|Z_m\|_\lambda + \|U_m\|_\lambda\big) e^{\lambda\tau}\, d\tau \le$$

$$\le \frac{\widetilde{\Gamma}_0}{\lambda}\, e^{-\lambda t}\big(\|Z_m\|_\lambda + \|U_m\|_\lambda\big).$$

Thus

$$\|Z_{m+1}\|_\lambda + \|U_{m+1}\|_\lambda \le \frac{\widetilde{\Gamma}_0}{\lambda}\big(\|Z_m\|_\lambda + \|U_m\|_\lambda\big), \quad m \ge 1.$$

We have also

$$\|Z_1\|_\lambda + \|U_1\|_\lambda \le 2(dc + p_0).$$

Consequently, the sequences $\{Z_m\}$, $\{U_m\}$ are uniformly convergent to zero which implies the assertion of Lemma 4.4. □

We are in a position to state the main result for the problem (4.1), (4.2). We write

$$\|\varphi\|_{(t.1)} = \max\big\{|\varphi(s,y)| :\ (s,y) \in \partial_0 E[t]\big\},$$

$$\|\partial_x\varphi\|_{(t.n)} = \max\Big\{\|\partial_x\varphi(s,y)\| :\ (s,y) \in \partial_0 E[t]\Big\},$$

where $\varphi \in \mathcal{J}_M^+[X]$ and $t \in [0,a]$.

Theorem 4.1. *Suppose that Assumptions* $\mathrm{H}^*[X]$, $\mathrm{H}_M[f]$, $\mathrm{H}_M[\psi]$, $\mathrm{H}_M[c,d,p_0,p_1]$ *are satisfied. Then for each* $\varphi \in \mathcal{J}_M^+[X]$ *there exists a solution* $z = z[\varphi] : E^*[c] \to R$ *to the problem* (4.1), (4.2) *such that*

$$z \in C^L_{\varphi.c}[d] \ \ \textit{and} \ \ \partial_x z \in C^L_{\partial_x \varphi.c}[p_0,p_1].$$

Moreover, if $\varphi, \overline{\varphi} \in \mathcal{J}_M^+[X]$ *are such that* $\|\varphi - \overline{\varphi}\|_X^b < +\infty$ *and* $z = z[\varphi]$, $\overline{z} = z[\overline{\varphi}]$, *then there is* $\Theta \in R_+$ *such that*

$$\|z-\overline{z}\|_{c.1} + \|\partial_x z - \partial_x \overline{z}\|_{c.n} \le \\ \le \Theta\Big(\|\varphi - \overline{\varphi}\|_{(c.1)} + \|\partial_x \varphi - \partial_x \overline{\varphi}\|_{(c.n)} + \|\varphi - \overline{\varphi}\|_X^b \Big). \quad (4.33)$$

Proof. Lemmas 4.3 and 4.4 imply that there is $z \in C^L_{\varphi.c}[d]$ such that

$$z(t,x) = \lim_{m\to\infty} z^{(m)}(t,x), \quad \partial_x z(t,x) = \lim_{m\to\infty} u^{(m)}(t,x)$$

uniformly on $E[c]$. Thus we get

$$z = F[z, \partial_x z], \quad \partial_x z = G[z, \partial_x z, \partial_x z] \ \text{ on } \ E[c].$$

Moreover,

$$z = \varphi \ \text{ on } \ E_0 \cup \partial_0 E[c].$$

Thus z is a solution of the problem (4.1), (4.2) on $E^*[c]$.

To prove (4.33) with $\varphi, \overline{\varphi} \in \mathcal{J}_M^+[X]$ such that $\|\varphi - \overline{\varphi}\|_X^b < +\infty$, we use the Gronwall inequality to the following one

$$\|z-\overline{z}\|_{t.1} + \|\partial_x z - \partial_x \overline{z}\|_{t.n} \le L_0\Big(\|\varphi - \overline{\varphi}\|_{(t.1)} + \|\partial_x \varphi - \partial_x \overline{\varphi}\|_{(t.n)} \Big) + \\ + L_1 \int_0^t \Big(\|z-\overline{z}\|_{\tau.1} + \|\partial_x z - \partial_x \overline{z}\|_{\tau.n} + \|\varphi - \overline{\varphi}\|_X^b \Big)\, d\tau$$

for some $L_0, L_1 \in R_+$. The proof of Theorem 4.1 is complete. □

Remark 4.3. In our considerations we do not assume that

$$\partial_{q_i} f(P) \ge 0 \ \text{ for } \ 1 \le i \le \kappa, \quad \partial_{q_i} f(P) \le 0 \ \text{ for } \ \kappa+1 \le i \le n,$$

where $P = (t,x,w,q)$ for $(x,w,q) \in [-b,b] \times X \times R^n$ and for almost all $t \in [0,a]$ (see [**25**]). In virtue of that the functional variable in (4.1) is defined on the set B which is the same for initial and mixed problems.

CHAPTER 5

Initial Problems on the Haar Pyramid

5.1. Lipschitz Continuous Solutions of Quasilinear Systems

We use the notation introduced in Chapter 1. Let $\mathcal{H}$ denote the Haar pyramid

$$\mathcal{H} = \Big\{(t,x) \in R^{n+1} : \ t \in [0,a], \ -b + h(t) \le x \le b - h(t)\Big\},$$

where $a > 0$, $b \in R^n_+$ and $h \in C([0,a], R^n_+)$ is a nondecreasing function, $h(0) = 0$, $b > h(a)$. Write

$$D_0 = (-\infty, 0] \times [-b, b], \quad \mathcal{H}_t = \big\{(s,x) \in \mathcal{H} : \ s \le t\big\}, \ \ 0 \le t \le a.$$

Let X_t, $0 \le t \le a$, be a linear space consisting of functions mapping the set $D_0 \cup \mathcal{H}_t$ into R^k. Assume that

$$A : \mathcal{H} \times X_a \to M_{k \times k}, \ \ A = [A_{ij}]_{i,j=1,\ldots,k},$$
$$\varrho : \mathcal{H} \times X_a \to M_{k \times n}, \ \ \varrho = [\varrho_{ij}]_{i=1,\ldots,k, j=1,\ldots,n},$$
$$f : \mathcal{H} \times X_a \to R^k, \ \ f = (f_1, \ldots, f_k), \ \text{ and } \ \varphi : D_0 \to R^k$$

are given functions. Let $z = (z_1, \ldots, z_k)$ be an unknown function of the variables (t,x), $x = (x_1, \ldots, x_n)$. We consider the quasilinear system of differential functional equations in the Schauder canonic form

$$\sum_{j=1}^{k} A_{ij}(t,x,z)\Big(\partial_t z_j(t,x) + \sum_{\nu=1}^{n} \varrho_{i\nu}(t,x,z)\partial_{x_\nu} z_j(t,x)\Big) = f_i(t,x,z) \qquad (5.1)$$

where $1 \le i \le k$, with the initial condition

$$z(t,x) = \varphi(t,x), \ \ (t,x) \in D_0. \qquad (5.2)$$

Here the variable z represents the functional dependence. This model is suitable for initial problems considered in the Haar pyramid. We consider weak solutions of the problem (5.1), (5.2). A function $\overline{z} : D_0 \cup \mathcal{H}_c \to R^k$, $c \in (0,a]$, is a solution of (5.1), (5.2) provided

(i) $\overline{z}$ is continuous on $\mathcal{H}_c$,
(ii) the derivatives $\partial_t \overline{z}_i, \partial_x \overline{z}_i = (\partial_{x_1}\overline{z}_i, \ldots, \partial_{x_n}\overline{z}_i)$, $1 \le i \le k$, exist almost everywhere on $\mathcal{H}_c$,
(iii) $\overline{z}$ satisfies the differential system for almost all $(t,x) \in \mathcal{H}_c$ and the condition (5.2) holds.

Let $t \in (0,a]$. For $z \in C(\mathcal{H}_t, R^k)$ we write

$$\|z\|_t = \max\left\{\|z(s,x)\|_\infty : (s,x) \in \mathcal{H}_t\right\}.$$

Denote by $C^L(\mathcal{H}_t, R^k)$ the class of all $z \in C(\mathcal{H}_t, R^k)$ such that

$$\|z\|_t^L =$$

$$= \sup\left\{\frac{\|z(s,x) - z(\overline{s},\overline{x})\|_\infty}{|s-\overline{s}| + \|x-\overline{x}\|} : (s,x),\ (\overline{s},\overline{x}) \in \mathcal{H}_t,\ (s,x) \neq (\overline{s},\overline{x})\right\} < +\infty.$$

For $z \in C^L(\mathcal{H}_t, R^k)$ we define the norm of z by

$$\|z\|_{t.L} = \|z\|_t + \|z\|_t^L.$$

We formulate the following assumptions on the spaces X_t, $0 \leq t \leq a$.

Assumption H$^L[X]$. For each $t \in [0,a]$ the space $(X_t, \|\cdot\|_{X_t})$ is a Banach space of functions from $D_0 \cup \mathcal{H}_t$ into R^k and there is a linear subspace $X_{t.L} \subset X_t$ such that $(X_{t.L}, \|\cdot\|_{X_{t.L}})$ is a Banach space. For each $t \in (0,a]$ the spaces X_t and $X_{t.L}$ satisfy the following conditions:

1) if $z : D_0 \cup \mathcal{H}_t \to R^k$ and $z_{|D_0} \in X_0$, $z_{|\mathcal{H}_t} \in C(\mathcal{H}_t, R^k)$, then $z \in X_t$ and

$$\|z\|_{X_t} \leq K_1 \|z_{|\mathcal{H}_t}\|_t + K_0 \|z_{|D_0}\|_{X_0},$$

where $K_1\, K_0 \in R_+$ are constants independent of z,

2) if $z : D_0 \cup \mathcal{H}_t \to R^k$ and $z_{|D_0} \in X_{0.L}$, $z_{|\mathcal{H}_t} \in C^L(\mathcal{H}_t, R^k)$, then $z \in X_{t.L}$ and

$$\|z\|_{X_{t.L}} \leq M_1 \|z_{|\mathcal{H}_t}\|_{t.L} + M_0 \|z_{|D_0}\|_{X_{0.L}}$$

with the constants $M_1, M_0 \in R_+$ independent of z.

We give examples of spaces satisfying Assumption H$^L[X]$.

Example 5.1. Let X_0 be the class of all functions $w : D_0 \to R^k$ which are bounded and uniformly continuous on D_0. For $w \in X_0$ we put

$$\|w\|_{X_0} = \sup\left\{\|w(t,x)\|_\infty : (t,x) \in D_0\right\}. \tag{5.3}$$

Let X_t, $0 < t \leq a$, be the set of all functions $z : D_0 \cup \mathcal{H}_t \to R^k$ such that $z_{|D_0} \in X_0$ and $z_{|\mathcal{H}_t} \in C(\mathcal{H}_t, R^k)$ with the norm of z given by

$$\|z\|_{X_t} = \|z_{|D_0}\|_{X_0} + \|z_{|\mathcal{H}_t}\|_t.$$

Denote by $X_{0.L}$ the space of all $w \in X_0$ such that

$$\|w\|_{D_0}^L =$$

$$= \sup\left\{\frac{\|w(t,x) - w(\overline{t},\overline{x})\|_\infty}{|t-\overline{t}| + \|x-\overline{x}\|} : (t,x),\ (\overline{t},\overline{x}) \in D_0,\ (t,x) \neq (\overline{t},\overline{x})\right\} < +\infty$$

with the norm of w given by

$$\|w\|_{X_{0.L}} = \|w\|_{X_0} + \|w\|_{D_0}^L. \tag{5.4}$$

Let $X_{t.L}$, $0 < t \le a$, denote the space of all $z \in X_t$ such that $z_{|D_0} \in X_{0.L}$ and $z_{|\mathcal{H}_t} \in C^L(\mathcal{H}_t, R^k)$ with the norm of z given by

$$\|z\|_{X_{t.L}} = \|z_{|D_0}\|_{X_{0.L}} + \|z_{|\mathcal{H}_t}\|_{t.L}.$$

Then Assumption $\mathrm{H}^L[X]$ is satisfied with $K_1 = K_0 = M_1 = M_0 = 1$.

Example 5.2. Let $\gamma : (-\infty, 0] \to (0, +\infty)$ be continuous and nonincreasing. We define X_0 as the space of all continuous functions $w : D_0 \to R^k$ such that

$$\lim_{t\to-\infty} \frac{w(t,x)}{\gamma(t)} = 0, \quad x \in [-b, b],$$

with the norm of w given by

$$\|w\|_{X_0} = \sup\Big\{ \frac{\|w(t,x)\|_\infty}{\gamma(t)} : \ (t,x) \in D_0 \Big\}.$$

Let X_t, $0 < t \le a$, be the set of the functions $z : D_0 \cup \mathcal{H}_t \to R^k$ such that $z_{|D_0} \in X_0$ and $z_{|\mathcal{H}_t} \in C(\mathcal{H}_t, R^k)$. For $z \in X_t$ we put

$$\|z\|_{X_t} = \|z_{|D_0}\|_{X_0} + \|z_{|\mathcal{H}_t}\|_t.$$

Denote by $X_{0.L}$ the space of all $w \in X_0$ such that $\|w\|^L_{D_0} < +\infty$ with the norm of w given by (5.4). Let $X_{t.L}$, $0 < t \le a$, denote the space of all $z \in X_t$ such that $z_{|D_0} \in X_{0.L}$ and $z_{|\mathcal{H}_t} \in C^L(\mathcal{H}_t, R^k)$ with the norm of z given by

$$\|z\|_{X_{t.L}} = \|z_{|D_0}\|_{X_{0.L}} + \|z_{|\mathcal{H}_t}\|_{t.L}.$$

Then Assumption $\mathrm{H}^L[X]$ is satisfied with $K_1 = \frac{1}{\gamma(0)}$, $K_0 = M_1 = M_0 = 1$.

Suppose that Assumption $\mathrm{H}^L[X]$ is satisfied. Fix $\varphi \in X_{0.L}$, $c \in (0, a]$ and $d = (d_0, d_1) \in R^2_+$. Denote by $K^L_{\varphi.c}[d]$ the class of all functions $z : D_0 \cup \mathcal{H}_c \to R^k$ such that $z(t,x) = \varphi(t,x)$ for $(t,x) \in D_0$ and

$$\|z(t,x)\|_\infty \le d_0, \ \ \big\|z(t,x) - z(\bar t, \bar x)\big\|_\infty \le d_1\big(|t - \bar t| + \|x - \bar x\|\big) \ \text{ on } \ \mathcal{H}_c.$$

We prove that there is a solution of (5.1), (5.2) in $K^L_{\varphi.c}[d]$ for sufficiently small $c \in (0, a]$ and for some $d \in R^2_+$. Write

$$S_t = \big[-b + h(t), b - h(t) \big], \ \ t \in [0, a],$$
$$I[x] = \big\{ t \in [0, a] : \ (t,x) \in \mathcal{H} \big\}, \ \ x \in [-b, b].$$

We adopt the following assumptions on ϱ.

Assumption $\mathbf{H}^L[\varrho]$. The function $\varrho : \mathcal{H} \times X_a \to M_{k\times k}$ is such that $\varrho(\cdot, x, w)$ is measurable on $I[x]$ for every $(x,w) \in [-b,b] \times X_a$, $\varrho(t,\cdot)$ is continuous on $S_t \times X_a$ for almost all $t \in [0,a]$ and

1) there is $\delta \in C([0,a], R^n)$, $\delta = (\delta_1, \ldots, \delta_n)$ such that

$$|\varrho_{ij}(t,x,w)| \le \delta_j(t), \ \ 1 \le i \le k, \ \ 1 \le j \le n, \ \ t \in [0,a], \ \ (x,w) \in S_t \times X_t,$$

$$h(t) = \int_0^t \delta(s)\,ds, \quad t \in [0,a],$$

2) there is $\beta_1 \in \Delta$ such that

$$\big\|\varrho(t,x,w) - \varrho(t,\overline{x},\overline{w})\big\|_\infty \le \beta_1(t,\mu)\big(\|x-\overline{x}\| + \|w-\overline{w}\|_{X_t}\big)$$

for $(x,w), (\overline{x},\overline{w}) \in S_t \times X_{t.L}[\mu]$ and for almost all $t \in [0,a]$, where $X_{t.L}[\mu]$ is defined by (1.9).

Remark 5.1. If Assumption $\mathrm{H}^L[\varrho]$ is satisfied, then the function ϱ satisfies the following Volterra condition: if $(t,x) \in \mathcal{H}$, z, $\overline{z} \in X_a$ and $z(s,y) = \overline{z}(s,y)$ for $(s,y) \in D_0 \cup \mathcal{H}_t$, then $\varrho(t,x,z) = \varrho(t,x,\overline{z})$.

Let the symbol $\mathcal{J}^L[X]$ denote the class of all initial functions $\varphi : D_0 \to R^k$ such that $\varphi \in X_{0.L}$ and there are b_0, b_1, c_0, $c_1 \in R_+$ with the properties

$$\|\varphi\|_{X_0} \le b_0, \quad \|\varphi\|_{X_{0.L}} \le b_1,$$
$$\|\varphi(0,x)\|_\infty \le c_0, \quad \big\|\varphi(0,x) - \varphi(0,\overline{x})\big\|_\infty \le c_1\|x-\overline{x}\| \ \text{ on } \ [-b,b].$$

Suppose that Assumptions $\mathrm{H}^L[X]$, $\mathrm{H}^L[\varrho]$ are satisfied and $\varphi \in \mathcal{J}^L[X]$, $c \in (0,a]$, $z \in K^L_{\varphi.c}[d]$, $(t,x) \in \mathcal{H}_c$, $1 \le i \le k$. Consider the Cauchy problem

$$\eta'(\tau) = \varrho_i(\tau,\eta(\tau),z), \quad \eta(t) = x, \tag{5.5}$$

where $\varrho_i = (\varrho_{i1},\ldots,\varrho_{in})$. Let us denote by $g_i[z](\cdot,t,x)$ the solution of (5.5) and by $[0,\sigma_i[z](t,x)]$ the maximal interval on which $g_i[z](\cdot,t,x)$ exists.

Lemma 5.1. *Suppose that Assumptions* $\mathrm{H}^L[X]$, $\mathrm{H}^L[\varrho]$ *are satisfied and* φ, $\overline{\varphi} \in \mathcal{J}^L[X]$, $z \in K^L_{\varphi.c}[d]$, $\overline{z} \in K^L_{\overline{\varphi}.c}[d]$, $c \in (0,a]$. *Then for each* $(t,x) \in \mathcal{H}_c$, $1 \le i \le k$, *the unique solutions* $g_i[z](\cdot,t,x)$ *and* $g_i[\overline{z}](\cdot,t,x)$ *exist on* $[0,\sigma_i[z](t,x)]$ *and* $[0,\sigma_i[\overline{z}](t,x)]$, *respectively. Moreover,*

$$\big\|g_i[z](\tau,t,x) - g_i[z](\tau,\overline{t},\overline{x})\big\| \le \delta_0^+ Q_c\big(|t-\overline{t}| + \|x-\overline{x}\|\big) \tag{5.6}$$

on $[0,\min\{\sigma_i[z](t,x),\sigma_i[z](\overline{t},\overline{x})\}] \times \mathcal{H}_c$, *and*

$$\big\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\big\| \le Q_c\Big|\int_\tau^t \beta_1(\xi,\mu_1)\,d\xi\Big| \cdot \|z-\overline{z}\|_{X_c} \tag{5.7}$$

on $[0,\min\{\sigma_i[z](t,x),\sigma_i[\overline{z}](t,x)\}] \times \mathcal{H}_c$, *where*

$$\delta_0^+ = 1+\delta_0, \quad \delta_0 = \max\big\{\|\delta(s)\| : \ s \in [0,a]\big\},$$
$$Q_c = \exp\Big(\int_0^c \beta_1(\xi,\mu_1)\,d\xi\Big), \quad \mu_1 = M_1(d_0+d_1) + M_0 b_1. \tag{5.8}$$

Proof. Assumption $\mathrm{H}^L[\varrho]$ and the following Lipschitz condition

$$\big\|\varrho_i(\tau,y,z) - \varrho_i(\tau,\overline{y},z)\big\| \le \beta_1(\tau,\mu_1)\|y-\overline{y}\|, \quad y, \overline{y} \in S_\tau,$$

imply the existence of a unique Caratheodory solution of (5.5). It follows from the integral equation

$$g_i[z](\tau,t,x) = x + \int_t^\tau \varrho_i(\xi, g_i[z](\xi,t,x), z)\, d\xi$$

that

$$\big\|g_i[z](\tau,t,x) - g_i[z](\tau,\bar t,\bar x)\big\| \le$$

$$\le \delta_0^+\big(|t-\bar t| + \|x-\bar x\|\big) + \left|\int_t^\tau \beta_1(\xi,\mu_1)\big\|g_i[z](\xi,t,x) - g_i[z](\xi,\bar t,\bar x)\big\|\, d\xi\right|,$$

$$\big\|g_i[z](\tau,t,x) - g_i[\bar z](\tau,t,x)\big\| \le$$

$$\le \left|\int_t^\tau \beta_1(\xi,\mu_1)\Big(\|g_i[z](\xi,t,x) - g_i[\bar z](\xi,t,x)\| + \|z-\bar z\|_{X_\xi}\Big)\, d\xi\right|.$$

Thus we get (5.6) and (5.7) by using the Gronwall inequality. □

Now we formulate assumptions on f and A.

Assumption $\mathbf{H}^L[f]$. The function $f : \mathcal{H} \times X_a \to R^k$ is such that $f(\cdot,x,w)$ is measurable on $I[x]$ for every $(x,w) \in [-b,b] \times X_a$, $f(t,\cdot)$ is continuous on $S_t \times X_a$ for almost all $t \in [0,a]$ and

1) there is $\alpha_2 \in \Sigma$ such that
$$\|f(t,x,w)\|_\infty \le \alpha_2(\mu)$$
for $(x,w) \in S_t \times X_t[\mu]$ and for almost all $t \in [0,a]$,
2) there is $\beta_2 \in \Sigma$ such that
$$\big\|f(t,x,w) - f(t,\bar x,\bar w)\big\|_\infty \le \beta_2(\mu)\big(\|x-\bar x\| + \|w-\bar w\|_{X_t}\big)$$
for $(x,w),(\bar x,\bar w) \in S_t \times X_{t.L}[\mu]$ and for almost all $t \in [0,a]$.

Assumption $\mathbf{H}^L[A]$. The function $A : \mathcal{H} \times X_a \to M_{k\times k}$ satisfies the conditions:

1) there are $\alpha, \beta \in \Sigma$ such that
$$\|A(t,x,w)\|_\infty \le \alpha(\mu), \ \ t \in [0,a], \ \ (x,w) \in S_t \times X_t[\mu],$$
$$\big\|A(t,x,w) - A(\bar t,x,w)\big\|_\infty \le \beta(\mu)|t-\bar t|$$
for $t,\bar t \in [0,a]$, $(x,w) \in S_{\tilde t} \times X_{\tilde t.L}[\mu]$, $\tilde t = \max\{t,\bar t\}$,
$$\big\|A(t,x,w) - A(t,\bar x,\bar w)\big\|_\infty \le \beta(\mu)\big(\|x-\bar x\| + \|w-\bar w\|_{X_t}\big)$$
for $t \in [0,a]$, $(x,w),(\bar x,\bar w) \in S_t \times X_{t.L}[\mu]$,
2) for each $(t,x,w) \in \mathcal{H} \times X_a[\mu]$ there exists the inverse matrix $A^{-1}(t,x,w)$ and there are α_0, $\beta_0 \in \Sigma$ such that
$$\|A^{-1}(t,x,w)\|_\infty \le \alpha_0(\mu), \ \ t \in [0,a], \ \ (x,w) \in S_t \times X_t[\mu],$$

$$\left\| A^{-1}(t,x,w) - A^{-1}(\overline{t},x,w) \right\|_\infty \le \beta_0(\mu)|t-\overline{t}|$$

$$\text{for } t,\,\overline{t} \in [0,a],\ (x,w) \in S_{\widetilde{t}} \times X_{\widetilde{t}.L}[\mu],\ \widetilde{t} = \max\{t,\overline{t}\},$$

$$\left\| A^{-1}(t,x,w) - A^{-1}(t,\overline{x},\overline{w}) \right\|_\infty \le \beta_0(\mu)\big(\|x-\overline{x}\| + \|w-\overline{w}\|_{X_t}\big)$$

$$\text{for } t \in [0,a],\ (x,w),\ (\overline{x},\overline{w}) \in S_t \times X_{t.L}[\mu].$$

Assume that $\varphi \in \mathcal{J}^L[X]$, $c \in (0,a]$, $z \in K^L_{\varphi.c}[d]$, $(t,x) \in \mathcal{H}_c$, $1 \le i \le k$. Write

$$\begin{aligned} A[z](\tau,t,x) &= \big[A_{ij}(\tau, g_i[z](\tau,t,x), z)\big]_{i,j=1,\dots,k},\\ \Phi[z](\tau,t,x) &= \big[\varphi_i(0, g_j[z](\tau,t,x))\big]_{i,j=1,\dots,k},\\ Z[z](\tau,t,x) &= \big[z_i(\tau, g_j[z](\tau,t,x))\big]_{i,j=1,\dots,k},\\ f[z](\tau,t,x) &= \big[f_i(\tau, g_i[z](\tau,t,x), z)\big]^T_{i=1,\dots,k}, \end{aligned}$$

where $g_i[z](\cdot,t,x)$ is a solution of (5.5) and $\tau \in [0, \sigma_i[z](t,x)]$. Define $T_\varphi(z) : D_0 \cup \mathcal{H}_c \to R^k$ in the following way

$$T_\varphi(z)(t,x) = \varphi(t,x), \quad (t,x) \in D_0,$$

$$T_\varphi(z)(t,x) = \varphi(0,x) + A^{-1}(t,x,z) \sum_{i=1}^{3} V_i[z](t,x), \quad (t,x) \in \mathcal{H}_c,$$

where

$$V_1[z](t,x) = \int_0^t f[z](\tau,t,x)\,d\tau,$$

$$V_2[z](t,x) = A[z](0,t,x) * \big(\Phi[z](0,t,x) - \Phi[z](t,t,x)\big),$$

$$V_3[z](t,x) = \int_0^t \frac{d}{d\tau} A[z](\tau,t,x) * \big(Z[z](\tau,t,x) - \Phi[z](t,t,x)\big)\,d\tau.$$

We prove that T_φ has a fixed point $\overline{z} \in K^L_{\varphi.c}[d]$ for some c and d. This $\overline{z}$ is a solution of (5.1), (5.2).

5.2. The Theorem on Existence and Uniqueness

We formulate the following lemmas on the operator T_φ.

Lemma 5.2. *If Assumptions* $\mathrm{H}^L[X]$, $\mathrm{H}^L[\varrho]$, $\mathrm{H}^L[f]$, $\mathrm{H}^L[A]$ *are satisfied, then there are* $c \in (0,a]$, $d = (d_0,d_1) \in R^2_+$ *such that for each* $\varphi \in \mathcal{J}^L[X]$ *the operator* T_φ *maps the set* $K^L_{\varphi.c}[d]$ *into itself.*

Proof. Let $\varphi \in \mathcal{J}^L[X]$ and $z \in K^L_{\varphi.c}[d]$. It is easy to see that

$$\sum_{i=1}^{3} \|V_i[z](t,x)\|_\infty \le c\mathcal{V},$$

where

$$\mathcal{V} = \alpha_2(\mu_0) + \alpha(\mu_0)c_1\delta_0 + c\beta(\mu_1)\delta_0^+(d_1 + c_1\delta_0), \quad \mu_0 = K_1 d_0 + K_0 b_0.$$

We assume that

$$d_0 \geq c_0 + c\alpha_0(\mu_0)\mathcal{V} \tag{5.9}$$

and we obtain the estimate

$$\big\|T_\varphi(z)(t,x)\big\|_\infty \leq d_0 \ \text{ on } \ \mathcal{H}_c.$$

Let $(t,x), (\bar{t}, \overline{x}) \in \mathcal{H}_c$. The assumptions of the lemma imply that

$$\big\|V_i[z](t,x) - V_i[z](\bar{t},\overline{x})\big\|_\infty \leq v_{i.c}\big(|t-\bar{t}| + \|x-\overline{x}\|\big), \quad i = 1,2,3,$$

where

$$\begin{aligned}
v_{1.c} &= \alpha_2(\mu_0) + c\beta_2(\mu_1)\delta_0^+ Q_c, \\
v_{2.c} &= c\beta(\mu_1)\delta_0^+ Q_c c_1 \delta_0 + \alpha(\mu_0)c_1(1 + \delta_0^+ Q_c), \\
v_{3.c} &= c\beta(\mu_1)\delta_0^+(d_1 + c_1\delta_0) + c\beta(\mu_1)\delta_0^+(d_1 + Q_c c_1 \delta_0) + \\
&\quad + c\beta(\mu_1)Q_c d_1 (\delta_0^+)^2 + c\beta(\mu_1)\delta_0^+(d_1 Q_c \delta_0^+ + c_1).
\end{aligned}$$

In this way we obtain

$$\big\|T_\varphi(z)(t,x) - T_\varphi(\bar{t},\overline{x})\big\|_\infty \leq d_1\big(|t-\bar{t}| + \|x-\overline{x}\|\big),$$

where

$$d_1 \geq c_1 + c\beta_0(\mu_1)\mathcal{V} + \alpha_0(\mu_0)\sum_{i=1}^{3} v_{i.c}. \tag{5.10}$$

The above considerations imply that for $c \in (0,a]$ and $d = (d_0, d_1) \in R_+^2$ such that the inequalities (5.9), (5.10) hold the operator T_φ maps the set $K^L_{\varphi.c}[d]$ into itself. □

Put

$$\|\varphi\|_0^* = \max\big\{\|\varphi(0,x)\|_\infty : \ x \in [-b,b]\big\},$$

where $\varphi \in \mathcal{J}^L[X]$.

Lemma 5.3. *Suppose that the assumptions of Lemma* 5.2 *are satisfied. If* $\varphi, \overline{\varphi} \in \mathcal{J}^L[X]$ *and* $z \in K^L_{\varphi.c}[d]$, $\overline{z} \in K^L_{\overline{\varphi}.c}[d]$, *then there are* $G_{1.c}, G_2, G_3 \in R_+$ *such that*

$$\big\|T_\varphi(z) - T_{\overline{\varphi}}(\overline{z})\big\|_c \leq G_{1.c}\|z-\overline{z}\|_c + G_2\|\varphi-\overline{\varphi}\|_{X_0} + G_3\|\varphi-\overline{\varphi}\|_0^*. \tag{5.11}$$

Proof. Fix $\varphi, \overline{\varphi} \in \mathcal{J}^L[X]$ and $z \in K^L_{\varphi.c}[d], \overline{z} \in K^L_{\overline{\varphi}.c}[d]$. It easily follows that

$$\begin{gathered}
\big\|V_1[z](t,x) - V_1[\overline{z}](t,x)\big\|_\infty \leq \theta_{1.c}\|z-\overline{z}\|_{X_c}, \\
\big\|V_2[z](t,x) - V_2[\overline{z}](t,x)\big\|_\infty \leq \theta_{2.c}\|z-\overline{z}\|_{X_c} + 2\alpha(\mu_0)\|\varphi-\overline{\varphi}\|_0^*, \\
\big\|V_3[z](t,x) - V_3[\overline{z}](t,x)\big\|_\infty \leq \\
\leq \theta_{3.c}\|z-\overline{z}\|_{X_c} + c\beta(\mu_1)\delta_0^+\big(\|z-\overline{z}\|_c + \|\varphi-\overline{\varphi}\|_0^*\big),
\end{gathered}$$

where

$$\theta_{1.c} = c\beta_2(\mu_1)\theta^*, \quad \theta_{2.c} = c\beta(\mu_1)\theta^* c_1\delta_0 + \alpha(\mu_0)c_1 Q_c \int_0^c \beta_1(\xi,\mu_1)\, d\xi,$$

$$\theta_{3.c} = c\beta(\mu_1)\Big(\delta_0^+ d_1 Q_c \int_0^c \beta_1(\xi,\mu_1) d\xi + d_1 + \theta^*(c_1\delta_0 + d_1\delta_0^+)\Big)$$

and

$$\theta^* = 1 + Q_c \int_0^c \beta_1(\xi,\mu_1)\, d\xi.$$

Thus we obtain

$$\big\|T_\varphi(z) - T_{\overline{\varphi}}(\overline{z})\big\|_c \le \theta_c \|z-\overline{z}\|_{X_c} + c\alpha_0(\mu_0)\beta(\mu_1)\delta_0^+\|z-\overline{z}\|_c + \\ + \Big(1 + c\alpha_0(\mu_0)\beta(\mu_1)\delta_0^+ + 2\alpha(\mu_0)\alpha_0(\mu_0)\Big)\|\varphi - \overline{\varphi}\|_0^*,$$

where

$$\theta_c = \beta_0(\mu_1)c\mathcal{V} + \alpha_0(\mu_0)\sum_{i=1}^{3}\theta_{i.c}.$$

Therefore the estimate (5.11) is true for

$$G_{1.c} = K_1\theta_c + c\alpha_0(\mu_0)\beta(\mu_1)\delta_0^+, \tag{5.12}$$

$$G_2 = K_0\theta_c, \quad G_3 = 1 + c\alpha_0(\mu_0)\beta(\mu_1)\delta_0^+ + 2\alpha(\mu_0)\alpha_0(\mu_0), \tag{5.13}$$

which completes the proof of Lemma 5.3. □

Now we are ready to prove a theorem on solution of the problem (5.1), (5.2).

Theorem 5.1. *Suppose that Assumptions* $\mathrm{H}^L[X]$, $\mathrm{H}^L[\varrho]$, $\mathrm{H}^L[f]$, $\mathrm{H}^L[A]$ *are satisfied. Assume that the constants* $c \in (0,a]$, $d = (d_0, d_1) \in R_+^2$ *satisfy the inequalities* (5.9), (5.10) *and*

$$G_{1.c} < 1, \tag{5.14}$$

where $G_{1.c}$ *is given by* (5.12)*. Then for each* $\varphi \in \mathcal{J}^L[X]$ *there exists* $z = z[\varphi] \in K^L_{\varphi.c}[d]$ *which is a unique solution of* (5.1), (5.2)*. Furthermore, if* $\varphi, \overline{\varphi} \in \mathcal{J}^L[X]$, $z = z[\varphi]$, $\overline{z} = z[\overline{\varphi}]$, *then*

$$\|z - \overline{z}\|_c \le \frac{1}{1 - G_{1.c}}\Big(G_2\|\varphi - \overline{\varphi}\|_{X_0} + G_3\|\varphi - \overline{\varphi}\|_0^*\Big), \tag{5.15}$$

where G_2, G_3 *are given by* (5.13)*.*

Proof. In virtue of (5.9), (5.10) and (5.14) the operator $T_\varphi : K^L_{\varphi.c}[d] \to K^L_{\varphi.c}[d]$ is a contraction for each $\varphi \in \mathcal{J}^L[X]$. Thus it has a fixed point $z = z[\varphi] \in K^L_{\varphi.c}[d]$. The inequality (5.15) immediately follows from Lemma 5.3. □

5.3. Solutions Satisfying Generalized Lipschitz Condition

Now we consider a special case of the problem (5.1), (5.2). Assume that

$$\varrho : \mathcal{H} \times X_a \to M_{k\times k}, \quad \varrho = [\varrho_{ij}]_{i=1,\dots,k,j=1,\dots,n},$$
$$f : \mathcal{H} \times X_a \to R^k, \quad f = (f_1, \dots, f_k), \quad \varphi : D_0 \to R^k$$

and

$$A : \mathcal{H} \times R^k \to M_{k\times k}, \quad A = [A_{ij}]_{i,j=1,\dots,k}$$

are given functions. We consider the following initial problem

$$\sum_{j=1}^{k} A_{ij}(t,x,z(t,x))\Big(\partial_t z_j(t,x) + \sum_{\nu=1}^{n} \varrho_{i\nu}(t,x,z)\partial_{x_\nu} z_j(t,x)\Big) =$$
$$= f_i(t,x,z), \quad 1 \le i \le k, \tag{5.16}$$
$$z(t,x) = \varphi(t,x) \quad (t,x) \in D_0. \tag{5.17}$$

The matrix A in (5.16) does not depend on the functional variable $z(\cdot)$. We look for solutions of (5.16), (5.17) in the class of functions satisfying the Lipschitz condition with respect to x and the generalized integral Lipschitz condition with respect to t.

We formulate new assumptions on the spaces X_t, $0 \le t \le a$. Denote by $C^{L^*}(\mathcal{H}_t, R^k)$ the class of all $z \in C(\mathcal{H}_t, R^k)$ such that

$$\|z\|_t^{L^*} = \sup\Big\{ \frac{\|z(s,x) - z(s,\overline{x})\|_\infty}{\|x - \overline{x}\|} : (s,x),\, (s,\overline{x}) \in \mathcal{H}_t,\; x \ne \overline{x} \Big\} < +\infty.$$

For $z \in C^{L^*}(\mathcal{H}_t, R^k)$ we put

$$\|z\|_{t.L^*} = \|z\|_t + \|z\|_t^{L^*}.$$

Assumption $\mathbf{H}^C[X]$. For each $t \in [0,a]$ the space $(X_t, \|\cdot\|_{X_t})$ is a Banach space of functions from $D_0 \cup \mathcal{H}_t$ into R^k and there is a linear subspace $X_{t.L^*} \subset X_t$ such that $(X_{t.L^*}, \|\cdot\|_{X_{t.L^*}})$ is a Banach space. For each $t \in (0,a]$ the spaces X_t and $X_{t.L^*}$ satisfy the following conditions:

1) if $z : D_0 \cup \mathcal{H}_t \to R^k$ and $z_{|D_0} \in X_0$, $z_{|\mathcal{H}_t} \in C(\mathcal{H}_t, R^k)$, then $z \in X_t$ and
$$\|z\|_{X_t} \le K_1 \|z_{|\mathcal{H}_t}\|_t + K_0 \|z_{|D_0}\|_{X_0},$$
where K_1, $K_0 \in R_+$ are constants independent of z,
2) if $z : D_0 \cup \mathcal{H}_t \to R^k$ and $z_{|D_0} \in X_{0.L^*}$, $z_{|\mathcal{H}_t} \in C^{L^*}(\mathcal{H}_t, R^k)$, then $z \in X_{t.L^*}$ and
$$\|z\|_{X_{t.L^*}} \le M_1 \|z_{|\mathcal{H}_t}\|_{t.L^*} + M_0 \|z_{|D_0}\|_{X_{0.L^*}}$$
with the constants $M_1, M_0 \in R_+$ independent of z.

We give examples of spaces satisfying Assumption $\mathrm{H}^C[X]$.

Example 5.3. Let the spaces $(X_t, \|\cdot\|_{X_t})$, $t \in [0,a]$, be defined as in Example 5.1. Denote by $X_{0.L^*}$ the space of all $w \in X_0$ such that

$$\|w\|_{D_0}^{L^*} = \sup\left\{ \frac{\|w(t,x) - w(t,\overline{x})\|_\infty}{\|x-\overline{x}\|} :\ (t,x),\ (t,\overline{x}) \in D_0,\ x \neq \overline{x} \right\} < +\infty$$

with the norm of w given by

$$\|w\|_{X_{0.L^*}} = \|w\|_{X_0} + \|w\|_{D_0}^{L^*}.$$

Let $X_{t.L^*}$, $0 < t \leq a$, denote the space of all $z \in X_t$ such that $z_{|D_0} \in X_{0.L^*}$ and $z_{|\mathcal{H}_t} \in C^L(\mathcal{H}_t, R^k)$ with the norm of z given by

$$\|z\|_{X_{t.L^*}} = \|z_{|D_0}\|_{X_{0.L^*}} + \|z_{|\mathcal{H}_t}\|_{t.L^*}.$$

Then Assumption $\mathrm{H}^C[X]$ is satisfied with $K_1 = K_0 = M_1 = M_0 = 1$.

Example 5.4. Let the spaces $(X_t, \|\cdot\|_{X_t})$, $t \in [0,a]$, be defined as in Example 5.2. Denote by $X_{0.L^*}$ the space of all $w \in X_0$ such that

$$\|w\|_{D_0}^{\gamma.L^*} = \sup\left\{ \frac{\|w(t,x) - w(t,\overline{x})\|_\infty}{\gamma(t)\|x-\overline{x}\|} :\ (t,x),\ (t,\overline{x}) \in D_0,\ x \neq \overline{x} \right\} < +\infty$$

with the norm of w given by

$$\|w\|_{X_{0.L^*}} = \|w\|_{X_0} + \|w\|_{D_0}^{\gamma.L^*}.$$

Let $X_{t.L^*}$, $0 < t \leq a$, denote the space of all $z \in X_t$ such that $z_{|D_0} \in X_{0.L^*}$ and $z_{|\mathcal{H}_t} \in C^L(\mathcal{H}_t, R^k)$ with the norm of z given by

$$\|z\|_{X_{t.L^*}} = \|z_{|D_0}\|_{X_{0.L^*}} + \|z_{|\mathcal{H}_t}\|_{t.L^*}.$$

Then Assumption $\mathrm{H}^C[X]$ is satisfied with $K_1 = M_1 = \frac{1}{\gamma(0)}$, $K_0 = M_0 = 1$.

Let the symbol $\mathcal{J}^C[X]$ denote the class of all initial functions $\varphi : D_0 \to R^k$ such that $\varphi \in X_{0.L^*}$ and there are $b_0,\ b_1,\ c_0,\ c_1 \in R_+$ with the properties

$$\|\varphi\|_{X_0} \leq b_0, \quad \|\varphi\|_{X_{0.L^*}} \leq b_1,$$
$$\|\varphi(0,x)\|_\infty \leq c_0, \quad \|\varphi(0,x) - \varphi(0,\overline{x})\|_\infty \leq c_1\|x - \overline{x}\| \ \text{ on } \ [-b,b].$$

Suppose that Assumption $\mathrm{H}^C[X]$ is satisfied and $\varphi \in \mathcal{J}^C[X]$, $c \in (0,a]$, $d = (d_0, d_1) \in R_+^2$, $\lambda \in L([0,c], R_+)$. Let the symbol $K_{\varphi.c}[d,\lambda]$ denote the class of all functions $z : D_0 \cup \mathcal{H}_c \to R^k$ such that $z(t,x) = \varphi(t,x)$ for $(t,x) \in D_0$ and

$$\|z(t,x)\|_\infty \leq d_0, \quad \big\|z(t,x) - z(\overline{t},\overline{x})\big\|_\infty \leq \left| \int_t^{\overline{t}} \lambda(\tau)d\tau \right| + d_1\|x - \overline{x}\| \ \text{ on } \ \mathcal{H}_c.$$

We prove that there is a solution of (5.16), (5.17) in $K_{\varphi.c}[d,\lambda]$ for sufficiently small $c \in (0,a]$ and for some $d \in R_+^2$, $\lambda \in L([0,c], R_+)$. We formulate the following assumptions on ϱ.

Assumption $\mathbf{H}^C[\varrho]$. The function $\varrho : \mathcal{H} \times X_a \to M_{k\times k}$ is such that $\varrho(\cdot, x, w)$ is measurable on $I[x]$ for every $(x,w) \in [-b,b] \times X_a$, $\varrho(t,\cdot)$ is continuous on $S_t \times X_a$ for almost all $t \in [0,a]$ and

1) there is $\widetilde{\delta} \in L([0,a],R^n)$, $\widetilde{\delta} = (\widetilde{\delta}_1,\dots,\widetilde{\delta}_n)$ such that

$$|\varrho_{ij}(t,x,w)| \le \widetilde{\delta}_j(t),\ \ 1 \le i \le k,\ \ 1 \le j \le n,\ \ t \in [0,a],\ \ (x,w) \in S_t \times X_t,$$

$$h(t) = \int_0^t \widetilde{\delta}(s)\,ds,\ \ t \in [0,a],$$

2) there is $\beta_1 \in \Delta$ such that

$$\big\|\varrho(t,x,w) - \varrho(t,\overline{x},\overline{w})\big\|_\infty \le \beta_1(t,\mu)\big(\|x-\overline{x}\| + \|w-\overline{w}\|_{X_t}\big)$$

for $(x,w),(\overline{x},\overline{w}) \in S_t \times X_{t.L^*}[\mu]$ and for almost all $t \in [0,a]$.

Suppose that Assumptions $\mathrm{H}^C[X]$, $\mathrm{H}^C[\varrho]$ are satisfied. Fix $\varphi \in \mathcal{J}^C[X]$, $z \in K_{\varphi.c}[d,\lambda]$, $c \in (0,a]$, $(t,x) \in \mathcal{H}_c$, $1 \le i \le k$ and consider the Cauchy problem

$$\eta'(\tau) = \varrho_i(\tau,\eta(\tau),z),\ \ \eta(t) = x, \tag{5.18}$$

with the solution $g_i[z](\cdot,t,x)$ defined on the interval $[0,\sigma_i[z](t,x)]$.

Lemma 5.4. *Suppose that Assumptions* $\mathrm{H}^C[X]$, $\mathrm{H}^C[\varrho]$ *are satisfied and* $\varphi,\overline{\varphi} \in \mathcal{J}^C[X]$, $z \in K_{\varphi.c}[d,\lambda]$, $\overline{z} \in K_{\overline{\varphi}.c}[d,\lambda]$, $c \in (0,a]$. *Then for each* $(t,x) \in \mathcal{H}_c$, $1 \le i \le k$, *the unique solutions* $g_i[z](\cdot,t,x)$ *and* $g_i[\overline{z}](\cdot,t,x)$ *exist on* $[0,\sigma_i[z](t,x)]$ *and* $[0,\sigma_i[\overline{z}](t,x)]$, *respectively. Moreover,*

$$\big\|g_i[z](\tau,t,x) - g_i[z](\tau,\overline{t},\overline{x})\big\| \le Q_c^*\bigg(\Big|\int_t^{\overline{t}} \|\widetilde{\delta}(\xi)\|\,d\xi\Big| + \|x-\overline{x}\|\bigg)$$

on $[0,\min\{\sigma_i[z](t,x),\sigma_i[z](\overline{t},\overline{x})\}] \times \mathcal{H}_c$, *and*

$$\big\|g_i[z](\tau,t,x) - g_i[\overline{z}](\tau,t,x)\big\| \le Q_c^*\Big|\int_\tau^t \beta_1(\xi,\mu_1^*)\,d\xi\Big| \cdot \|z-\overline{z}\|_{X_c}$$

on $[0,\min\{\sigma_i[z](t,x),\sigma_i[\overline{z}](t,x)\}] \times \mathcal{H}_c$, *where*

$$Q_c^* = \exp\Big(\int_0^c \beta_1(\xi,\mu_1^*)\,d\xi\Big),\ \ \mu_1^* = M_1(d_0+d_1) + M_0 b_1.$$

We omit the simple proof of Lemma 5.4. We formulate the following assumptions on f and A.

Assumption $\mathbf{H}^C[f]$. The function $f : \mathcal{H} \times X_a \to R^k$ is such that $f(\cdot,x,w)$ is measurable on $I[x]$ for every $(x,w) \in [-b,b] \times X_a$, $f(t,\cdot)$ is continuous on $S_t \times X_a$ for almost all $t \in [0,a]$ and

1) there is $\alpha_2 \in \Delta$ such that

$$\|f(t,x,w)\|_\infty \le \alpha_2(t,\mu)$$

for $(x,w) \in S_t \times X_t[\mu]$ and for almost all $t \in [0,a]$,

2) there is $\beta_2 \in \Delta$ such that

$$\big\|f(t,x,w) - f(t,\overline{x},\overline{w})\big\|_\infty \le \beta_2(t,\mu)\big(\|x-\overline{x}\| + \|w-\overline{w}\|_{X_t}\big)$$

for (x,w), $(\overline{x},\overline{w}) \in S_t \times X_{t.L^*}[\mu]$ and for almost all $t \in [0,a]$.

Assumption $\mathbf{H}^C[A]$. The function $A : \mathcal{H} \times R^k \to M_{k\times k}$ satisfies the conditions:

1) there are α, $\beta \in \Sigma$ and $\gamma \in \Delta$ such that

$$\|A(t,x,p)\|_\infty \le \alpha(\mu),$$

$$\big\|A(t,x,p) - A(\overline{t},\overline{x},\overline{p})\big\|_\infty \le \beta(\mu)\big(\|x-\overline{x}\| + \|p-\overline{p}\|_\infty\big) + \Big|\int_t^{\overline{t}} \gamma(\xi,\mu)\,d\xi\Big|$$

for (t,x,p), $(\overline{t},\overline{x},\overline{p}) \in \mathcal{H} \times R^k[\mu]$,

2) for each $(t,x,p) \in \mathcal{H} \times R^k[\mu]$ there exists the inverse matrix $A^{-1}(t,x,p)$ and there are $\alpha_0, \beta_0 \in \Sigma$ and $\gamma_0 \in \Delta$ such that

$$\|A^{-1}(t,x,p)\|_\infty \le \alpha_0(\mu),$$

$$\big\|A^{-1}(t,x,p) - A^{-1}(\overline{t},\overline{x},\overline{p})\big\|_\infty \le \beta_0(\mu)\big(\|x-\overline{x}\| + \|p-\overline{p}\|_\infty\big) + \Big|\int_t^{\overline{t}} \gamma_0(\xi,\mu)\,d\xi\Big|$$

for (t,x,p), $(\overline{t},\overline{x},\overline{p}) \in \mathcal{H} \times R^k[\mu]$.

Fix $\varphi \in \mathcal{J}^C[X]$, $c \in (0,a]$, $z \in K_{\varphi.c}[d,\lambda]$ and $(t,x) \in \mathcal{H}_c$. Write

$$Q_i[z](\tau,t,x) = \Big(\tau, g_i[z](\tau,t,x), z(\tau, g_i[z](\tau,t,x)\Big), \quad 1 \le i \le k,$$

$$A^*[z](\tau,t,x) = \big[A_{ij}(Q_i[z](\tau,t,x))\big]_{i,j=1,\dots,k},$$

$$\Phi[z](\tau,t,x) = \big[\varphi_i(0, g_j[z](\tau,t,x))\big]_{i,j=1,\dots,k},$$

$$Z[z](\tau,t,x) = \big[z_i(\tau, g_j[z](\tau,t,x))\big]_{i,j=1,\dots,k},$$

$$f[z](\tau,t,x) = \big[f_i(\tau, g_i[z](\tau,t,x), z)\big]^T_{i=1,\dots,k},$$

where $g_i[z](\cdot,t,x)$ is a solution of (5.18) and $\tau \in [0, \sigma_i[z](t,x)]$. We define

$$T^*_\varphi(z)(t,x) = \varphi(t,x), \quad (t,x) \in D_0,$$

$$T^*_\varphi(z)(t,x) = \varphi(0,x) + A^{-1}(t,x,z(t,x)) \sum_{i=1}^{3} W_i[z](t,x), \quad (t,x) \in \mathcal{H}_c,$$

where

$$W_1[z](t,x) = \int_0^t f[z](\tau,t,x)\,d\tau,$$

$$W_2[z](t,x) = A^*[z](0,t,x) * \big(\Phi[z](0,t,x) - \Phi[z](t,t,x)\big),$$

$$W_3[z](t,x) = \int_0^t \frac{d}{d\tau}\, A^*[z](\tau,t,x) * \big(Z[z](\tau,t,x) - \Phi[z](t,t,x)\big)\, d\tau.$$

5.4. The Existence and Uniqueness Result

We begin with formulation of lemmas on the integral operator.

Lemma 5.5. *If Assumptions* $\mathrm{H}^C[X]$, $\mathrm{H}^C[\varrho]$, $\mathrm{H}^C[f]$, $\mathrm{H}^C[A]$ *are satisfied, then there are* $c \in (0,a]$, $d = (d_0,d_1) \in R_+^2$ *and* $\lambda \in L([0,c],R_+)$ *such that for each* $\varphi \in \mathcal{J}^C[X]$ *the operator* T_φ^* *maps the set* $K_{\varphi.c}[d,\lambda]$ *into itself.*

Proof. Let $\varphi \in \mathcal{J}^C[X]$ and $z \in K_{\varphi.c}[d,\lambda]$. In the sequel we use the following estimates

$$\big|z_j(\tau, g_i[z](\tau,t,x)) - \varphi_j(0,x)\big| \le \int_0^\tau \lambda(\xi)\, d\xi + c_1 \Big| \int_t^\tau \|\widetilde{\delta}(\xi)\|\, d\xi \Big|,$$

$$\Big\| \frac{d}{d\tau}\, z\big(\tau, g_i[z](\tau,t,x)\big) \Big\|_\infty \le \lambda(\tau) + d_1 \|\widetilde{\delta}(\tau)\|,$$

$$\Big\| \frac{d}{d\tau} A^*[z](\tau,t,x) \Big\|_\infty \le q(\tau),$$

where $q(\tau) = \gamma(\tau,d_0) + \beta(d_0)((1+d_1)\|\widetilde{\delta}(\tau)\| + \lambda(\tau))$. Since

$$\sum_{i=1}^3 \|W_i[z](t,x)\|_\infty \le \mathcal{W}_c,$$

where

$$\mathcal{W}_c = \int_0^c \alpha_2(\xi,\mu_0^*)\, d\xi + \alpha(c_0)c_1 \int_0^c \|\widetilde{\delta}(\xi)\|\, d\xi +$$
$$+ \int_0^c q(\tau) \Big(\int_0^\tau \lambda(\xi)\, d\xi + c_1 \int_\tau^c \|\widetilde{\delta}(\xi)\|\, d\xi \Big)\, d\tau,$$
$$\mu_0^* = K_1 d_0 + K_0 b_0,$$

we obtain

$$\|T_\varphi^*(z)(t,x)\|_\infty \le c_0 + \alpha_0(d_0)\mathcal{W}_c \ \text{ on } \ \mathcal{H}_c.$$

Fix $(t,x), (\overline{t},\overline{x}) \in \mathcal{H}_c$. It easily follows that

$$\big\|W_i[z](t,x) - W_i[z](\overline{t},\overline{x})\big\|_\infty \le \Big| \int_t^{\overline{t}} \lambda_{i.c}(\tau)\, d\tau \Big| + w_{i.c}\|x - \overline{x}\|, \ \ i = 1,2,3,$$

where

$$w_{1.c} = Q_c^* \int_0^c \beta_2(\xi,\mu_1^*)\, d\xi, \quad \lambda_{1.c}(\tau) = w_{1.c}\|\widetilde{\delta}(\tau)\| + \alpha_2(\tau,\mu_0^*),$$

$$w_{2.c} = c_1\bigg(\alpha(c_0)(1+Q_c^*) + \beta(c_0)(1+c_1)Q_c^*\int\limits_0^c \|\widetilde{\delta}(\xi)\|\,d\xi\bigg),$$

$$\lambda_{2.c}(\tau) = c_1Q_c^*\bigg(\alpha(c_0) + \beta(c_0)(1+c_1)\int\limits_0^c \|\widetilde{\delta}(\xi)\|\,d\xi\bigg)\|\widetilde{\delta}(\tau)\|$$

and

$$w_{3.c} = \eta_c + c_1\int\limits_0^c q(\xi)\,d\xi,$$

$$\lambda_{3.c}(\tau) = \eta_c\|\widetilde{\delta}(\tau)\| + q(\tau)\int\limits_0^c \big(\lambda(\xi) + c_1\|\widetilde{\delta}(\xi)\|\big)\,d\xi,$$

$$\eta_c = \beta(d_0)\int\limits_0^c \Big((1+d_1)\lambda(\xi) + (1+c_1)c_1Q_c^*\|\widetilde{\delta}(\xi)\|\Big)\,d\xi +$$

$$+\beta(d_0)(1+d_1)Q_c^*\int\limits_0^c \big(\lambda(\xi) + d_1\|\widetilde{\delta}(\xi)\|\big)\,d\xi + d_1Q_c^*\int\limits_0^c q(\xi)\,d\xi.$$

Thus

$$\big\|T_\varphi^*(z)(t,x) - T_\varphi^*(\bar{t},\overline{x})\big\|_\infty \le c_1\|x-\overline{x}\| +$$

$$+\bigg(\beta_0(d_0)(1+d_1)\|x-\overline{x}\| + \beta_0(d_0)\Big|\int\limits_t^{\bar{t}} \lambda(\tau)\,d\tau\Big| + \Big|\int\limits_t^{\bar{t}} \gamma_0(\tau,d_0)\,d\tau\Big|\bigg)\mathcal{W}_c +$$

$$+\alpha_0(d_0)\bigg(\Big|\int\limits_t^{\bar{t}} \sum_{i=1}^3 \lambda_{i.c}(\tau)\,d\tau\Big| + \sum_{i=1}^3 w_{i.c}\|x-\overline{x}\|\bigg).$$

Assume that the constants $c \in (0,a]$, $d = (d_0,d_1) \in R_+^2$ and the function $\lambda \in L([0,c],R_+)$ satisfy the conditions

$$d_0 \ge c_0 + \alpha_0(d_0)\mathcal{W}_c, \tag{5.19}$$

$$d_1 \ge c_1 + \beta_0(d_0)(1+d_1)\mathcal{W}_c + \alpha_0(d_0)\sum_{i=1}^3 w_{i.c}, \tag{5.20}$$

$$\lambda(\tau) \ge \big(\beta_0(d_0)\lambda(\tau) + \gamma_0(\tau,d_0)\big)\mathcal{W}_c + \alpha_0(d_0)\sum_{i=1}^3 \lambda_{i.c}(\tau). \tag{5.21}$$

Then $T_\varphi^*(z) \in K_{\varphi.c}[d,\lambda]$ and the proof of Lemma 5.5 is complete. □

Lemma 5.6. *Suppose that the assumptions of Lemma* 5.5 *are satisfied. If* $\varphi, \overline{\varphi} \in \mathcal{J}^C[X]$ *and* $z \in K_{\varphi.c}[d,\lambda]$, $\overline{z} \in K_{\overline{\varphi}.c}[d,\lambda]$, *then there are*

$G^*_{1.c}, G^*_2, G^*_3 \in R_+$ *such that*

$$\left\|T^*_\varphi(z) - T^*_{\overline{\varphi}}(\overline{z})\right\|_c \leq G^*_{1.c}\|z-\overline{z}\|_c + G^*_2\|\varphi-\overline{\varphi}\|_{X_0} + G^*_3\|\varphi-\overline{\varphi}\|^*_0. \quad (5.22)$$

Proof. Let $\varphi, \overline{\varphi} \in \mathcal{J}^C[X]$ and $z \in K_{\varphi.c}[d,\lambda]$, $\overline{z} \in K_{\overline{\varphi}.c}[d,\lambda]$. Put

$$\sigma_{1.c} = \left(1 + Q^*_c \int_0^c \beta_1(\xi,\mu^*_1)\, d\xi\right) \int_0^c \beta_2(\xi,\mu_1)\, d\xi,$$

$$\sigma_{2.c} = c_1 Q^*_c \int_0^c \beta_1(\xi,\mu^*_1)\, d\xi\left(\beta(c_0)(1+c_1)\int_0^c \|\widetilde{\delta}(\xi)\|\, d\xi + \alpha(c_0)\right),$$

$$\sigma_{3.c} = Q^*_c \int_0^c \beta_1(\xi,\mu^*_1)\, d\xi\left(\beta(c_0)(1+c_1)c_1\int_0^c \|\widetilde{\delta}(\xi)\|\, d\xi + \right.$$

$$\left. + d_1\int_0^c q(\xi) d\xi + \beta(d_0)(1+d_1)\int_0^c (\lambda(\xi) + d_1\|\widetilde{\delta}(\xi)\|)\, d\xi\right),$$

$$\eta_c = \int_0^c q(\xi)\, d\xi + \beta(d_0)\int_0^c \left(2\lambda(\xi) + d_1\|\widetilde{\delta}(\xi)\|\right) d\xi,$$

$$\eta = \int_0^c q(\xi)\, d\xi + \beta(c_0)c_1\int_0^c \|\widetilde{\delta}(\xi)\|\, d\xi.$$

It follows from the estimates

$$\left\|W_1[z](t,x) - W_1[\overline{z}](t,x)\right\|_\infty \leq \sigma_{1.c}\|z-\overline{z}\|_{X_c},$$
$$\left\|W_2[z](t,x) - W_2[\overline{z}](t,x)\right\|_\infty \leq \sigma_{2.c}\|z-\overline{z}\|_{X_c} + 2\alpha(c_0)\|\varphi-\overline{\varphi}\|^*_0,$$
$$\left\|W_3[z](t,x) - W_3[\overline{z}](t,x)\right\|_\infty \leq \sigma_{3.c}\|z-\overline{z}\|_{X_c} + \eta_c\|z-\overline{z}\|_c + \eta\|\varphi-\overline{\varphi}\|^*_0$$

that

$$\left\|T^*_\varphi(z) - T^*_{\overline{\varphi}}(\overline{z})\right\|_c \leq \|\varphi-\overline{\varphi}\|^*_0 + \beta_0(d_0)\mathcal{W}_c\|z-\overline{z}\|_c +$$
$$+ \alpha_0(\mu_0)\left(\|z-\overline{z}\|_{X_c}\sum_{i=1}^3 \sigma_{i.c} + \eta_c\|z-\overline{z}\|_c + (2\alpha(c_0)+\eta)\|\varphi-\overline{\varphi}\|^*_0\right).$$

Thus the assertion (5.22) holds for the following constants

$$G^*_{1.c} = \beta_0(d_0)\mathcal{W}_c + \eta_c + K^*_1\alpha_0(d_0)\sum_{i=1}^3 \sigma_{i.c} \quad (5.23)$$

and

$$G^*_2 = K_0\alpha_0(d_0)\sum_{i=1}^3 \sigma_{i.c}, \quad G^*_3 = 1 + 2\alpha_0(d_0)\alpha(c_0) + \eta. \quad (5.24)$$

Now we prove a theorem on solution of the problem (5.16), (5.17). □

Theorem 5.2. *Suppose that Assumptions* $\mathrm{H}^C[X]$, $\mathrm{H}^C[\varrho]$, $\mathrm{H}^C[f]$, $\mathrm{H}^C[A]$ *are satisfied. Assume that* $c \in (0, a]$, $d = (d_0, d_1) \in R_+^2$, $\lambda \in L([0, c], R_+)$ *satisfy the inequalities* (5.19)–(5.21) *and*

$$G_{1.c}^* < 1, \tag{5.25}$$

where $G_{1.c}^*$ *is given by* (5.23). *Then for each* $\varphi \in \mathcal{J}^C[X]$ *there exists* $z = z[\varphi] \in K_{\varphi.c}[d, \lambda]$ *which is a unique solution of* (5.16), (5.17). *Furthermore, if* $\varphi, \overline{\varphi} \in \mathcal{J}^C[X]$, $z = z[\varphi]$, $\overline{z} = z[\overline{\varphi}]$, *then*

$$\|z - \overline{z}\|_c \le \frac{1}{1 - G_{1.c}^*} \Big(G_2^* \|\varphi - \overline{\varphi}\|_{X_0} + G_3^* \|\varphi - \overline{\varphi}\|_0^* \Big) \tag{5.26}$$

with G_2, G_3 *given by* (5.24).

Proof. It follows from Lemmas 5.5 and 5.6 that for each $\varphi \in \mathcal{J}^C[X]$ and for c, d, λ satisfying (5.19)–(5.21), (5.25) the operator $T_\varphi^* : K_{\varphi.c}[d, \lambda] \to K_{\varphi.c}[d, \lambda]$ is a contraction. Thus there exists $z = z[\varphi] \in K_{\varphi.c}[d, \lambda]$ such that $z = T_\varphi^*(z)$ and this z is a solution of (5.16), (5.17). Lemma 5.6 implies the inequality (5.26) and the proof of Theorem 5.2 is complete. □

5.5. Initial Problems for Nonlinear Systems

Let $\mathcal{H}$, D_0 and S_t, $\mathcal{H}_t$, $0 \le t \le a$, be the sets defined in Section 5.1 with $h(t) = Mt$, where $M = (\widetilde{M}_1, \dots, \widetilde{M}_n) \in R_+^n$. Let X_t, $0 \le t \le a$, be a linear space of functions from $D_0 \cup \mathcal{H}_t$ into R^k. Suppose that

$$F : \mathcal{H} \times X_a \times R^n \to R^k, \quad F = (F_1, \dots, F_k), \quad \text{and} \quad \varphi : D_0 \to R^k$$

are given functions. We deal with the Cauchy problem for the nonlinear partial differential functional system

$$\partial_t z_i(t, x) = F_i\big(t, x, z, \partial_x z_i(t, x)\big), \quad 1 \le i \le k, \tag{5.27}$$

$$z(t, x) = \varphi(t, x), \quad (t, x) \in D_0, \tag{5.28}$$

where $z = (z_1, \dots, z_k)$ is an unknown function of the variables (t, x), $x = (x_1, \dots, x_n)$. We consider classical solutions of the problem (5.27), (5.28).

Let $C^1(\mathcal{H}_t, R^k)$ denote the class of all functions $z : \mathcal{H}_t \to R^k$ which are of class C^1 on $\mathcal{H}_t$. For $z \in C^1(\mathcal{H}_t, R^k)$, $z = (z_1, \dots, z_k)$, $1 \le i \le k$ and $(s, x) \in \mathcal{H}_t$ we put

$$\partial z_i(s, x) = \big(\partial_t z_i(s, x), \partial_x z_i(s, x)\big), \quad \|\partial z_i(s, x)\| = |\partial_t z_i(s, x)| + \|\partial_x z_i(s, x)\|,$$

$$\|z\|_t^I = \|z\|_t + \max\Big\{ \|\partial z_i(s, x)\| : \ 1 \le i \le k, \ (s, x) \in \mathcal{H}_t \Big\}.$$

Denote by $C^{1.L}(\mathcal{H}_t, R^k)$ the set of all functions $z \in C^1(\mathcal{H}_t, R^k)$ such that

$$Lip[\partial z]_t =$$

$$= \sup\Big\{ \frac{\|\partial z_i(s, x) - \partial z_i(\overline{s}, \overline{x})\|}{|s - \overline{s}| + \|x - \overline{x}\|} : \ 1 \le i \le k, \ (s, x), \ (\overline{s}, \overline{x}) \in \mathcal{H}_t, \ (s, x) \ne (\overline{s}, \overline{x}) \Big\}$$

is finite. For $z \in C^{1.L}(\mathcal{H}_t, R^k)$ we define the norm of z by

$$\|z\|_t^{I.L} = \|z\|_t^I + Lip[\partial z]_t.$$

We formulate the following assumptions on the spaces X_t, $0 \le t \le a$.

Assumption $\mathbf{H}^N[X]$. For each $t \in [0,a]$ the space $(X_t, \|\cdot\|_{X_t})$ is a Banach space of functions from $D_0 \cup \mathcal{H}_t$ into R^k and there are linear subspaces $X_t^{I.L} \subset X_t^I \subset X_t$ such that $(X_t^I, \|\cdot\|_{X_t^I})$, $(X_t^{I.L}, \|\cdot\|_{X_t^{I.L}})$ are Banach spaces. For each $t \in (0,a]$ the spaces X_t, X_t^I and $X_t^{I.L}$ satisfy the following conditions:

1) if $z : D_0 \cup \mathcal{H}_t \to R^k$ and $z_{|D_0} \in X_0$, $z_{|\mathcal{H}_t} \in C(\mathcal{H}_t, R^k)$, then $z \in X_t$ and
$$\|z\|_{X_t} \le K_1 \|z_{|\mathcal{H}_t}\|_t + K_0 \|z_{|D_0}\|_{X_0},$$
where $K_1, K_0 \in R_+$ are constants independent of z,
2) if $z : D_0 \cup \mathcal{H}_t \to R^k$ and $z_{|D_0} \in X_0^I$, $z_{|\mathcal{H}_t} \in C^1(\mathcal{H}_t, R^k)$, then $z \in X_t^I$ and
$$\|z\|_{X_t^I} \le M_1 \|z_{|\mathcal{H}_t}\|_t^I + M_0 \|z_{|D_0}\|_{X_0^I},$$
where $M_1, M_0 \in R_+$ are constants independent of z,
3) if $z : D_0 \cup \mathcal{H}_t \to R^k$ and $z_{|D_0} \in X_0^{I.L}$, $z_{|\mathcal{H}_t} \in C^{1.L}(\mathcal{H}_t, R^k)$, then $z \in X_t^{I.L}$ and
$$\|z\|_{X_t^{I.L}} \le N_1 \|z_{|\mathcal{H}_t}\|_t^{I.L} + N_0 \|z_{|D_0}\|_{X_0^{I.L}},$$
where $N_1, N_0 \in R_+$ are constants independent of z.

Examples of spaces satisfying Assumption $\mathrm{H}^N[X]$ are the following.

Example 5.5. Let $(X_t, \|\cdot\|_{X_t})$, $0 \le t \le a$, be defined as in Example 5.1. Denote by X_0^I the set of all $w \in X_0$, $w = (w_1, \ldots, w_k)$, such that the derivatives $\partial w_i = (\partial_t w_i, \partial_x w_i)$, $1 \le i \le k$, exist and they are bounded and uniformly continuous on D_0. For $w \in X_0^I$ we put
$$\|w\|_{X_0^I} = \|w\|_{X_0} + \sup\Big\{ \|\partial w_i(t,x)\| :\ 1 \le i \le k, (t,x) \in D_0 \Big\}.$$
Let X_t^I, $0 < t \le a$, be the set of all $z \in X_t$ such that $z_{|D_0} \in X_0^I$ and $z_{|\mathcal{H}_t} \in C^1(\mathcal{H}_t, R^k)$ with the norm of z given by
$$\|z\|_{X_t^I} = \|z_{|D_0}\|_{X_0^I} + \|z_{|\mathcal{H}_t}\|_t^I.$$
Let $X_0^{I.L}$ be the space of all $w \in X_0^I$ such that
$$Lip[\partial w] =$$
$$= \sup\Big\{ \frac{\|\partial w_i(t,x) - \partial w_i(\bar{t},\bar{x})\|}{|t-\bar{t}| + \|x-\bar{x}\|} :\ 1 \le i \le k,\ (t,x),\ (\bar{t},\bar{x}) \in D_0,\ (t,x) \ne (\bar{t},\bar{x}) \Big\}$$
is finite. We define the norm of $w \in X_0^{I.L}$ by
$$\|w\|_{X_0^{I.L}} = \|w\|_{X_0^I} + Lip[\partial w].$$
Denote by $X_t^{I.L}$, $0 < t \le a$, the space of all $z \in X_t^I$ such that $z_{|D_0} \in X_0^{I.L}$ and $z_{|\mathcal{H}_t} \in C^{1.L}(\mathcal{H}_t, R^k)$. For $z \in X_t^{I.L}$ we write
$$\|z\|_{X_t^{I.L}} = \|z_{|D_0}\|_{X_0^{I.L}} + \|z_{|\mathcal{H}_t}\|_t^{I.L}.$$

Then Assumption $H^N[X]$ is satisfied with $K_1 = K_0 = M_1 = M_0 = N_1 = N_0 = 1$.

Example 5.6. Let $(X_t, \|\cdot\|_{X_t})$, $0 \le t \le a$, be defined as in Example 5.2. Denote by X_0^I the class of all $w \in X_0$, $w = (w_1, \ldots, w_k)$, such that the derivatives $\partial w_i = (\partial_t w_i, \partial_x w_i)$, $1 \le i \le k$, exist and they are continuous on D_0 and

$$\lim_{t \to -\infty} \frac{\partial w_i(t,x)}{\gamma(t)} = \mathbf{0}, \quad x \in [-b,b], \quad 1 \le i \le k.$$

For $w \in X_0^I$ we put

$$\|w\|_{X_0^I} = \|w\|_{X_0} + \sup\Big\{ \frac{\|\partial w_i(t,x)\|}{\gamma(t)} \; : \; 1 \le i \le k, \; (t,x) \in D_0 \Big\}.$$

Let X_t^I, $0 < t \le a$, be the set of all $z \in X_t$ such that $z_{|D_0} \in X_0^I$ and $z_{|\mathcal{H}_t} \in C^1(\mathcal{H}_t, R^k)$ with the norm of z given by

$$\|z\|_{X_t^I} = \|z_{|D_0}\|_{X_0^I} + \|z_{|\mathcal{H}_t}\|_t^I.$$

Let $X_0^{I.L}$ be the space of all $w \in X_0^I$ such that $Lip[\partial w] < +\infty$. We define the norm of $w \in X_0^{I.L}$ by

$$\|w\|_{X_0^{I.L}} = \|w\|_{X_0^I} + Lip[\partial w].$$

Denote by $X_t^{I.L}$, $0 < t \le a$, the space of all $z \in X_t^I$ such that $z_{|D_0} \in X_0^{I.L}$ and $z_{|\mathcal{H}_t} \in C^{1.L}(\mathcal{H}_t, R^k)$. For $z \in X_t^{I.L}$ we write

$$\|z\|_{X_t^{I.L}} = \|z_{|D_0}\|_{X_0^{I.L}} + \|z_{|\mathcal{H}_t}\|_t^{I.L}.$$

Then Assumption $H^N[X]$ is satisfied with $K_1 = M_1 = \frac{1}{\gamma(0)}$, $K_0 = M_0 = N_1 = N_0 = 1$.

5.6. Existence and Uniqueness of Classical Solutions

The method used in the existence result for (5.27), (5.28) is based on the theorem on solution of the following differential problem without the functional dependence. Suppose that $g : \mathcal{H} \times R^n \to R$, $\omega : [-b,b] \to R$ are given functions and consider the nonlinear partial differential equation

$$\partial_t u(t,x) = g\big(t, x, \partial_x u(t,x)\big) \tag{5.29}$$

with the initial condition

$$u(0,x) = \omega(x), \quad x \in [-b,b]. \tag{5.30}$$

To state a theorem on solution of the above problem, we formulate the following assumptions on g and ω.

Assumption H$[g,\omega]$. The functions $g : \mathcal{H} \times R^n \to R$ and $\omega : [-b,b] \to R$ satisfy the conditions:

1) the function g of the variables (t,x,q), $q = (q_1, \ldots, q_n)$, is continuous and bounded on $\mathcal{H} \times R^n$, the derivatives $\partial_x g$, $\partial_q g$ exist and they are continuous on $\mathcal{H} \times R^n$,

2) there are $C_0, L_0 \in R_+$ such that

$$\|\partial_x g(t,x,q)\| \le C_0,$$
$$\big\|\partial_x g(t,x,q) - \partial_x g(t,\overline{x},\overline{q})\big\| \le L_0\big(\|x-\overline{x}\| + \|q-\overline{q}\|\big),$$
$$\big\|\partial_q g(t,x,q) - \partial_q g(t,\overline{x},\overline{q})\big\| \le L_0\big(\|x-\overline{x}\| + \|q-\overline{q}\|\big),$$

where (t,x,q), $(t,\overline{x},\overline{q}) \in \mathcal{H} \times R^n$, and

$$|\partial_{q_j} g(t,x,q)| \le \widetilde{M}_j, \quad 1 \le j \le n, \quad (t,x,q) \in \mathcal{H} \times R^n,$$

3) the function $\omega : [-b,b] \to R$ is of class C^1 and there are A_0, $B_0 \in R_+$ such that

$$\|\partial_x \omega(x)\| \le A_0, \quad \big\|\partial_x \omega(x) - \partial_x \omega(\overline{x})\big\| \le B_0 \|x-\overline{x}\| \text{ on } [-b,b],$$

4) $|g(0,x,q)| \le A_0$ for $(x,q) \in [-b,b] \times R^n$, and the derivative $\partial_t g$ exists on $\mathcal{H} \times R^n$ and

$$|\partial_t g(t,x,q)| \le C_0 \quad \text{on} \quad \mathcal{H} \times R^n.$$

Now we state the auxiliary theorem.

Theorem 5.3. *If Assumption* H$[g,\omega]$ *is satisfied, then there exists a unique solution* $\overline{u}$ *of the problem* (5.29), (5.30) *defined on* $\mathcal{H}_\delta$, *where*

$$\delta = \min\Big\{a, \frac{1}{L_0(1+B_0)}\Big\}.$$

Moreover, the solution $\overline{u}$ *satisfies the conditions*

$$\begin{aligned} &\|\partial_x \overline{u}(t,x)\| \le A_0 + C_0 t, \\ &\big\|\partial_x \overline{u}(t,x) - \partial_x \overline{u}(t,\overline{x})\big\| \le \Gamma(t)\|x-\overline{x}\|, \\ &\big\|\partial_x \overline{u}(t,x) - \partial_x \overline{u}(\overline{t},x)\big\| \le \big(C_0 + \|M\|\Gamma(t)\big)|t-\overline{t}| \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} &|\partial_t \overline{u}(t,x)| \le A_0 + C_0 t, \\ &\big|\partial_t \overline{u}(t,x) - \partial_t \overline{u}(t,\overline{x})\big| \le \big(C_0 + \|M\|\Gamma(t)\big)\|x-\overline{x}\|, \\ &\big|\partial_t \overline{u}(t,x) - \partial_t \overline{u}(\overline{t},x)\big| \le \Big(C_0 + \|M\|\big(C_0 + \|M\|\,\Gamma(t)\big)\Big)|t-\overline{t}|, \end{aligned} \tag{5.32}$$

where (t,x), $(t,\overline{x}), (\overline{t},x) \in \mathcal{H}_\delta$ *and*

$$\Gamma(t) = \frac{L_0(1+B_0)t + B_0}{1 - L_0(1+B_0)t}, \quad t \in [0,\delta].$$

The existence of a solution of (5.29), (5.30) on $\mathcal{H}_\delta$ and the estimates (5.31) are proved in [**35**]. The conditions 1)–3) of Assumption H$[g,\omega]$ are sufficient. If we additionally assume the condition 4) of H$[g,\omega]$, then we prove the estimates (5.32) by applying the theorem on weak partial differential inequalities (for details see [**22**]).

We adopt the following assumptions on F.

Assumption $\mathbf{H}^N[F]$. The function $F : \mathcal{H} \times X_a \times R^n \to R^k$, $F = (F_1, \ldots, F_k)$, of the variables (t, x, w, q), $x = (x_1, \ldots, x_n)$, $w = (w_1, \ldots, w_k)$, $q = (q_1, \ldots, q_n)$, is continuous and it satisfies the conditions:

1) there are $\alpha : R_+ \to R_+$ and $d > 0$ such that

$$\|F(t,x,w,q)\|_\infty \le \alpha(\mu), \ \ t \in [0,a], \ \ (x,w,q) \in S_t \times X_t[\mu] \times R^n,$$

and

$$\big\|F(t,x,w,q) - F(t,x,\overline{w},q)\big\|_\infty \le d\|w - \overline{w}\|_{X_t}$$

for $t \in [0,a]$, $(x,w,q), (x,\overline{w},q) \in S_t \times X_t \times R^n$,

2) for each $P = (t,x,w,q)$, where $t \in [0,a]$, $(x,w,q) \in S_t \times X_t^I \times R^n$, there exist the derivatives

$$\partial_t F(P), \ \big[\partial_{x_j} F_i(P)\big]_{i=1,\ldots,k,\, j=1,\ldots,n} = \partial_x F(P),$$
$$\big[\partial_{q_j} F_i(P)\big]_{i=1,\ldots,k,\, j=1,\ldots,n} = \partial_q F(P)$$

and they are continuous on $\mathcal{H} \times X_a^I \times R^n$,

3) there exist positive constants d_0, d_1 such that

$$\big\|\partial_t F(t,x,w,q)\big\|_\infty \le d_0 + \mu d_1, \ \ \big\|\partial_x F(t,x,w,q)\big\|_\infty \le d_0 + \mu d_1$$

for $t \in [0,a]$, $(x,w,q) \in S_t \times X_t^I[\mu] \times R^n$ and

$$\big\|\partial_{q_j} F(t,x,w,q)\big\|_\infty \le \widetilde{M}_j, \ \ 1 \le j \le n, \ \ t \in [0,a], \ \ (x,w,q) \in S_t \times X_t^I \times R^n,$$

4) there is $\beta : R_+ \to (0, +\infty)$ such that

$$\big\|\partial_x F(t,x,w,q) - \partial_x F(t,\overline{x},w,\overline{q})\big\|_\infty \le \beta(\mu)\big(\|x - \overline{x}\| + \|q - \overline{q}\|\big),$$
$$\big\|\partial_q F(t,x,w,q) - \partial_q F(t,\overline{x},w,\overline{q})\big\|_\infty \le \beta(\mu)\big(\|x - \overline{x}\| + \|q - \overline{q}\|\big),$$

where $t \in [0,a]$, (x,w,q), $(\overline{x},w,\overline{q}) \in S_t \times X_t^{I.L}[\mu] \times R^n$.

Let $\mathcal{J}^N[X]$ denote the set of all initial functions $\varphi : D_0 \to R^k$, $\varphi = (\varphi_1, \ldots, \varphi_k)$, such that $\varphi \in X_0^{I.L}$ and

1) for $x \in [-b,b]$ there exist the derivatives

$$\partial_t \varphi(0,x), \ \big[\partial_{x_j} \varphi_i(0,x)\big]_{i=1,\ldots,k,\, j=1,\ldots,n} = \partial_x \varphi(0,x)$$

and they are continuous on $[-b,b]$,

2) there are b_1, b_2, c_0, c_1, $c_2 \in R_+$ with the properties

$$\|\varphi\|_{X_0^I} \le b_1, \ \ \|\varphi\|_{X_0^{I.L}} \le b_2,$$
$$\|\varphi(0,x)\|_\infty \le c_0, \ \ \|\partial_x \varphi(0,x)\|_\infty \le c_1,$$
$$\big\|\partial_x \varphi(0,x) - \partial_x \varphi(0,\overline{x})\big\|_\infty \le c_2 \|x - \overline{x}\| \ \text{ on } \ [-b,b],$$

3) the consistency condition

$$\partial_t \varphi_i(0,x) = f_i\big(0,x,\varphi,\partial_x \varphi_i(0,x)\big), \ \ 1 \le i \le k,$$

is satisfied for $x \in [-b,b]$.

Put

$$\|\varphi\|_0^* = \max\big\{\|\varphi(0,x)\|_\infty : \ x \in [-b,b]\big\},$$

where $\varphi \in \mathcal{J}^N[X]$.

The main theorem for the problem (5.27), (5.28) is the following.

Theorem 5.4. *If Assumptions* $\mathrm{H}^N[X]$ *and* $\mathrm{H}^N[F]$ *are satisfied, then there exists* $c \in (0,a]$ *such that for each* $\varphi \in \mathcal{J}^N[X]$ *the problem* (5.27), (5.28) *has exactly one classical solution* $\overline{z}$ *defined on* $\mathcal{H}_c$. *Moreover, if* $\varphi, \overline{\varphi} \in \mathcal{J}^N[X]$ *and* w, $v : \mathcal{H}_c \to R^k$ *are the solutions of* (5.27), (5.28) *with the initial functions* φ *and* $\overline{\varphi}$ *respectively, then*

$$\|w - v\|_c \le G_1\|\varphi - \overline{\varphi}\|_0^* + G_2\|\varphi - \overline{\varphi}\|_{X_0} \tag{5.33}$$

for some G_1, $G_2 \in R_+$.

Proof. Assume that $\lambda > K_1\, d$. Let us denote by $C_\lambda(\mathcal{H}_c, R^k)$, $0 < c \le a$, the Banach space of all continuous functions $z : \mathcal{H}_c \to R^k$ with the norm of z given by

$$\|z\|_{[\lambda]} = \max\Big\{ e^{-\lambda t}\,\|z(t,x)\|_\infty : \ (t,x) \in \mathcal{H}_c\Big\}.$$

Fix $\varphi \in \mathcal{J}^N[X]$. Let $W_{\varphi.c}$ be the set of all functions $z : D_0 \cup \mathcal{H}_c \to R^k$ such that z is of class C^1 on $\mathcal{H}_c$ and $z(t,x) = \varphi(t,x)$ for $(t,x) \in D_0$. Denote by $W^*_{\varphi.c}$ the set of all $z_{|\mathcal{H}_c}$ with $z \in W_{\varphi.c}$. Write

$$p_1 = d_1M_1c_0 + d_1M_0b_1, \quad A = 2c_1 + \frac{d_0 + p_1}{(2+a)d_1M_1}\,,$$

$$\mu_1 = M_1(Aa + c_0 + 2A) + M_0b_1,$$

$$B = d_0 + d_1\mu_1 + \|M\|(1 + 2c_2), \quad C = d_0 + d_1\mu_1 + \|M\|B,$$

$$\mu_2 = N_1\big(Aa + c_0 + 2A + 2B + C + 2c_2 + 1\big) + N_0b_2$$

and

$$c = \min\Big\{a, \frac{1}{2\beta(\mu_2)(1+c_2)}, \frac{1}{2(2+a)d_1M_1}\Big\}.$$

Let $\mathcal{W}_{\varphi.c}$ with the above given c be the set of all functions $z \in W^*_{\varphi.c}$ such that

$$\|\partial_x z(t,x)\|_\infty \le A, \quad \|\partial_t z(t,x)\|_\infty \le A, \tag{5.34}$$

$$\big\|\partial_x z(t,x) - \partial_x z(\overline{t},\overline{x})\big\|_\infty \le B|t - \overline{t}| + (2c_2+1)\|x - \overline{x}\|, \tag{5.35}$$

$$\big\|\partial_t z(t,x) - \partial_t z(\overline{t},\overline{x})\big\|_\infty \le C|t - \overline{t}| + B\|x - \overline{x}\| \tag{5.36}$$

on $\mathcal{H}_c$. The set $\mathcal{W}_{\varphi.c}$ is a closed subset of the Banach space $C_\lambda(\mathcal{H}_c, R^k)$. Fix $u \in \mathcal{W}_{\varphi.c}$, $1 \le i \le k$, and consider the initial problem (5.29), (5.30), where

$$g(t,x,q) = F_i(t,x,\widetilde{u},q) \ \text{ on } \mathcal{H}_c \times R^n, \ \ \omega(x) = \varphi(0,x) \ \text{ on } [-b,b], \tag{5.37}$$

and $\widetilde{u}(t,x) = u(t,x)$ on $\mathcal{H}_c$, $\widetilde{u}(t,x) = \varphi(t,x)$ on D_0. We will prove that there exists a unique solution $z_i[u] : \mathcal{H}_c \to R^k$ of the problem (5.29), (5.30) with the above given g, ω and the function $z[u] = (z_1[u], \dots, z_k[u])$ satisfies the conditions (5.34)–(5.36).

Since $u \in \mathcal{W}_{\varphi.c}$, we have

$$\|u\|_t \le Aa + c_0, \quad \|\widetilde{u}\|_{X_t^I} \le \mu_1, \quad \|\widetilde{u}\|_{X_t^{I.L}} \le \mu_2, \ \ t \in [0, c].$$

It follows from Assumption $\mathrm{H}^N[F]$ that

$$|\partial_t g(t,x,q)| \le d_0 + d_1\mu_1, \quad \|\partial_x g(t,x,q)\| \le d_0 + d_1\mu_1$$

and the terms $\|\partial_x g(t,x,q) - \partial_x g(t,\overline{x},\overline{q})\|$, $\|\partial_q g(t,x,q) - \partial_q g(t,\overline{x},\overline{q})\|$ are bounded from above by $\beta(\mu_2)(\|x-\overline{x}\| + \|q - \overline{q}\|)$. Let

$$\widetilde{\Gamma}(t) = \frac{\beta(\mu_2)(1+c_2)t + c_2}{1 - \beta(\mu_2)(1+c_2)t}\,, \ \ t \in [0,c].$$

Theorem 5.3 implies that there is on $\mathcal{H}_c$ the solution $z_i[u]$ of the problem (5.29), (5.30) with g, ω given by (5.37) and

$$\big\|\partial_x z_i[u](t,x) - \partial_x z_i[u](\overline{t},\overline{x})\big\| \le \big(d_0 + d_1\mu_1 + \|M\|\widetilde{\Gamma}(t)\big)|t-\overline{t}| + \widetilde{\Gamma}(t)\|x-\overline{x}\|,$$

$$\big|\partial_t z_i[u](t,x) - \partial_t z_i[u](\overline{t},\overline{x})\big| \le$$

$$\le \Big(d_0 + d_1\mu_1 + \|M\|\big(d_0 + d_1\mu_1 + \|M\|\widetilde{\Gamma}(t)\big)\Big)|t-\overline{t}| +$$

$$+ \big(d_0 + d_1\mu_1 + \|M\|\widetilde{\Gamma}(t)\big)\|x - \overline{x}\|$$

and

$$\big\|\partial_x z_i[u](t,x)\big\| \le c_1 + (d_0 + d_1\mu_1)t, \quad \big|\partial_t z_i[u](t,x)\big| \le c_1 + (d_0 + d_1\mu_1)t$$

on $\mathcal{H}_c$. The above estimates and the condition $\widetilde{\Gamma}(c) \le 2c_2 + 1$ imply that

$$\big\|\partial_x z_i[u](t,x) - \partial_x z_i[u](\overline{t},\overline{x})\big\| \le B|t-\overline{t}| + (2c_2+1)\|x-\overline{x}\|,$$

$$\big|\partial_t z_i[u](t,x) - \partial_t z_i[u](\overline{t},\overline{x})\big| \le C|t-\overline{t}| + B\|x-\overline{x}\|.$$

Since $2(2+a)d_1M_1c \le 1$ and

$$c_1 + (d_0 + d_1\mu_1)t \le c_1 + (d_0 + p_1)c + (2+a)d_1M_1cA \le 2c_1 + 2(d_0+p_1)c \le A,$$

we get

$$\big\|\partial_x z_i[u](t,x)\big\| \le A, \quad \big|\partial_t z_i[u](t,x)\big| \le A \ \text{ on } \ \mathcal{H}_c.$$

Thus $z[u] = (z_1[u], \dots, z_k[u])$ is an element of $\mathcal{W}_{\varphi.c}$ and the operator $u \mapsto z[u]$ maps the set $\mathcal{W}_{\varphi.c}$ into itself. We prove that it is contractive. Let $u, v \in \mathcal{W}_{\varphi.c}$. It follows from Assumptions $\mathrm{H}^N[F]$ and $\mathrm{H}^N[X]$ that

$$\Big|\partial_t\big(z_i[u](t,x) - z_i[v](t,x)\big)\Big| \le$$

$$\le dK_1\|u-v\|_t + \sum_{j=1}^n \widetilde{M}_j\Big|\partial_{x_j}\big(z_i[u](t,x) - z_i[v](t,x)\big)\Big| \le$$

$$\le dK_1 e^{\lambda t}\|u-v\|_{[\lambda]} + \sum_{j=1}^n \widetilde{M}_j\Big|\partial_{x_j}\big(z_i[u](t,x) - z_i[v](t,x)\big)\Big|, \ \ 1 \le i \le k,$$

on $\mathcal{H}_c$. We have also $z[u](0,x) - z[v](0,x) = \mathbf{0}$, $x \in [-b,b]$. By theorems on partial differential inequalities we get

$$\big\|z[u](t,x) - z[v](t,x)\big\|_\infty \le \frac{dK_1}{\lambda}\, e^{\lambda t}\|u-v\|_{[\lambda]} \;\text{ on }\; \mathcal{H}_c.$$

Thus

$$\|z[u] - z[v]\|_{[\lambda]} \le \frac{dK_1}{\lambda}\, \|u-v\|_{[\lambda]}.$$

Since $dK_1 < \lambda$, the Banach fixed point theorem implies that there exists $u^* \in \mathcal{W}_{\varphi.c}$ such that $u^* = z[u^*]$. Let $\overline{z}(t,x) = u^*(t,x)$ on $\mathcal{H}_c$ and $\overline{z}(t,x) = \varphi(t,x)$ on D_0. The function $\overline{z}$ is a solution of the problem (5.27), (5.28).

Let $\varphi, \overline{\varphi} \in \mathcal{J}^N[X]$ and $w, v : \mathcal{H}_c \to R^k$, $w = (w_1, \dots, w_k)$, $v = (v_1, \dots, v_k)$, satisfy (5.27), (5.28) with φ and $\overline{\varphi}$ respectively. Assumptions $\mathrm{H}^N[F]$ and $\mathrm{H}^N[X]$ imply the following differential inequalities

$$\Big|\partial_t\big(w_i(t,x) - v_i(t,x)\big)\Big| \le$$

$$\le de^{\lambda t}\Big(K_1\|w-v\|_{[\lambda]} + K_0\|\varphi-\overline{\varphi}\|_{X_0}\Big) + \sum_{j=1}^n \widetilde{M}_j\Big|\partial_{x_j}\big(w_i(t,x) - v_i(t,x)\big)\Big|,$$

where $1 \le i \le k$, $(t,x) \in \mathcal{H}_c$. Thus

$$\|w-v\|_{[\lambda]} \le \frac{d}{\lambda}\Big(K_1\|w-v\|_{[\lambda]} + K_0\|\varphi-\overline{\varphi}\|_{X_0}\Big) + \|\varphi-\overline{\varphi}\|_0^*.$$

Since $\|w-v\|_c \le e^{\lambda c}\|w-v\|_{[\lambda]}$, we obtain the assertion (5.33) for

$$G_1 = e^{\lambda c}\Big(1 - \frac{dK_1}{\lambda}\Big)^{-1}, \quad G_2 = \frac{dK_0}{\lambda}\, G_1.$$

The proof of Theorem 5.4 is complete. □

Remark 5.2. As a special case of (5.27), (5.28), we obtain the following general problem

$$\partial_t z(t,x) = \widetilde{F}\big(t,x,(Vz)(t,x),\partial_x z(t,x)\big), \tag{5.38}$$

$$z(t,x) = \widetilde{\varphi}(t,x), \;\; (t,x) \in D_0, \tag{5.39}$$

where $\widetilde{F} : \mathcal{H} \times R \times R^n \to R$, $V : X_a \to R$, $\widetilde{\varphi} : D_0 \to R$. A result for Cinquini–Cibrario solutions of (5.38), (5.39) is obtained in [**33**].

Bibliography

1. A. Augustynowicz and Z. Kamont, On Kamke's functions in uniqueness theorems for first order partial differential-functional equations. *Nonlinear Anal.* **14** (1990), No. 10, 837–850.
2. P. Bassanini, On a recent proof concerning a boundary value problem for quasilinear hyperbolic systems in the Schauder canonic form. *Boll. Un. Mat. Ital. A (5)* **14** (1977), No. 2, 325–332.
3. P. Brandi and R. Ceppitelli, Existence, uniqueness and continuous dependence for a hereditary nonlinear functional partial differential equation of the first order. *Ann. Polon. Math.* **47** (1986), No. 2, 121–136.
4. P. Brandi, Z. Kamont, and A. Salvadori, Differential and differential-difference inequalities related to mixed problems for first order partial differential-functional equations. *Atti Sem. Mat. Fis. Univ. Modena* **39** (1991), No. 1, 255–276.
5. P. Brandi and C. Marcelli, Haar inequality in hereditary setting and applications. *Rend. Sem. Mat. Univ. Padova* **96** (1996), 177–194.
6. P. Brandi, A. Salvadori, and Z. Kamont, Existence of generalized solutions of hyperbolic functional differential equations. *Nonlinear Anal.* **50** (2002), No. 7, Ser. A: Theory Methods, 919–940.
7. L. Cesari, A boundary value problem for quasilinear hyperbolic systems in the Schauder canonic form. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **1** (1974), 311–358 (1975).
8. L. Cesari, A boundary value problem for quasi linear hyperbolic systems. *Riv. Mat. Univ. Parma (3)* **3** (1974), 107–131.
9. S. Cinquini, Sopra i sistemi iperbolici di equazioni a derivate parziali (non lineari) in piu variabili indipendenti. (Italian) *Ann. Mat. Pura Appl.* **120** (1979), No. 1, 201–214.
10. M. Cinquini-Cibrario, A class of systems of nonlinear partial differential equations in several independent variables. (Italian) *Ann. Mat. Pura Appl. (4)* **140** (1985), 223–253.
11. C. Corduneanu and V. Lakshmikantham, Equations with unbounded delay: a survey. *Nonlinear Anal.* **4** (1980), No. 5, 831–877.
12. W. Czernous, Generalized solutions of mixed problems for first-order partial functional differential equations. *Ukrain. Mat. Zh.* **58** (2006), No. 6, 803–828; translation in *Ukrainian Math. J.* **58** (2006), No. 6, 904–936
13. T. Człapiński, On the existence of generalized solutions of nonlinear first order partial differential-functional equations in two independent variables. *Czechoslovak Math. J.* **41(116)** (1991), No. 3, 490–506.
14. T. Człapiński, On the mixed problem for quasilinear partial differential-functional equations of the first order. *Z. Anal. Anwendungen* **16** (1997), No. 2, 463–478.
15. T. Człapiński, On the mixed problem for quasilinear partial functional-differential equations with unbounded delay. *Ann. Polon. Math.* **72** (1999), No. 1, 87–98.
16. T. Człapiński and Z. Kamont, Existence of weak solutions of nonlinear functional differential equations with unbounded delay. *Appl. Anal.* **77** (2001), No. 3-4, 249–272.

17. T. Człapiński, Weak solutions of quasilinear functional differential systems in the second canonical form with unbounded delay. *Studia Sci. Math. Hungar.* **41** (2004), No. 2, 127–145.
18. Y. Hino, S. Murakami, and T. Naito, Functional-differential equations with infinite delay. *Lecture Notes in Mathematics,* 1473. *Springer-Verlag, Berlin,* 1991.
19. D. Jaruszewska-Walczak, Existence of solutions of first order partial differential-functional equations. *Boll. Un. Mat. Ital. B (7)* **4** (1990), No. 1, 57–82.
20. D. Jaruszewska-Walczak, Infinite systems of hyperbolic differential-functional inequalities. *Univ. Iagel. Acta Math.* No. 43 (2005), 219–228.
21. Z. Kamont, Existence of solutions of first order partial differential-functional equations. *Comment. Math. Prace Mat.* **25** (1985), No. 2, 249–263.
22. Z. Kamont, Hyperbolic functional differential inequalities and applications. *Mathematics and its Applications,* 486. *Kluwer Academic Publishers, Dordrecht,* 1999.
23. Z. Kamont, Hyperbolic functional-differential equations with unbounded delay. *Z. Anal. Anwendungen* **18** (1999), No. 1, 97–109.
24. Z. Kamont and S. Kozieł, First order partial functional differential equations with unbounded delay. Dedicated to the 100th birthday anniversary of Professor Victor Kupradze. *Georgian Math. J.* **10** (2003), No. 3, 509–530.
25. Z. Kamont and S. Kozieł, Mixed problems for hyperbolic functional differential equations with unbounded delay. *Nonlinear Anal.* **58** (2004), No. 5-6, 489–515.
26. S. Kozieł, Hyperbolic functional differential systems with unbounded delay. *Z. Anal. Anwendungen* **23** (2004), No. 2, 377–405.
27. Z. Kamont and H. Leszczyński, Uniqueness result for the generalized entropy solutions to the Cauchy problem for first-order partial differential-functional equations. *Z. Anal. Anwendungen* **13** (1994), No. 3, 477–491.
28. G. S. Ladde, V. Lakshmikantham, and A. S. Vatsala, Monotone iterative techniques for nonlinear differential equations. *Monographs, Advanced Texts and Surveys in Pure and Applied Mathematics,* 27. *Pitman (Advanced Publishing Program), Boston, MA; distributed by John Wiley & Sons, Inc., New York,* 1985.
29. V. Lakshmikantham and S. Leela, Differential and Integral Inequalities. *Acad. Press, New York–London,* 1969.
30. V. Lakshmikantham, Wen Li-Zhi, and Zhang Bing Gen, Theory of differential equations with unbounded delay. *Mathematics and its Applications,* 298. *Kluwer Academic Publishers Group, Dordrecht,* 1994.
31. H. Leszczyński, On the existence and uniqueness of weak solutions of the Cauchy problem for weakly coupled systems of nonlinear partial differential-functional equations of the first order. *Boll. Un. Mat. Ital. B (7)* **7** (1993), No. 2, 323–354.
32. A. D. Myshkis and A. M. Filimonov, Continuous solutions of quasilinear hyperbolic systems with two independent variables. (Russian) *Differentsial'nye Uravneniya* **17** (1981), No. 3, 488–500, 573.
33. A. Nadolski, Hamilton–Jacobi functional differential equations with unbounded delay. *Ann. Polon. Math.* **82** (2003), No. 2, 105–126.
34. R. Nagel and E. Sinestrari, Nonlinear hyperbolic Volterra integrodifferential equations. *Nonlinear Anal.* **27** (1996), No. 2, 167–186.
35. J. Szarski, Characteristics and Cauchy problem for nonlinear partial differential equations of first order. *University of Kansas, Lawrence, Kansas,* 1959.
36. J. Szarski, Cauchy problem for an infinite system of differential-functional equations with first order partial derivatives. Special issue dedicated to Władysław Orlicz on the occasion of his seventy-fifth birthday. *Comment. Math. Special Issue* **1** (1978), 293–300.
37. K. Topolski, On the uniqueness of viscosity solutions for first order partial differential-functional equations. *Ann. Polon. Math.* **59** (1994), No. 1, 65–75.
38. K. Topolski, Classical methods for viscosity solutions of differential-functional inequalities. *Nonlinear World* **4** (1997), No. 1, 1–17.

39. J. Turo, Mixed problems for quasilinear hyperbolic systems. P*roceedings of the Second World Congress of Nonlinear Analysts,* Part 4 (*Athens*, 1996). *Nonlinear Anal.* **30** (1997), No. 4, 2329–2340.
40. W. Walter, Functional differential equations of the Cauchy-Kowalevsky type. *Aequationes Math.* **28** (1985), No. 1-2, 102–113.
41. T. Wazewski, Sur le probleme de Cauchy relatif à un système d'équations aux dérivées partielles. *Ann. Soc. Polonaise Math.* **15** (1937), 101–127.

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