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ON THE SOLVABILITY OF A MULTIPOINT BOUNDARY VALUE PROBLEM FOR SYSTEMS OF NONLINEAR IMPULSIVE EQUATIONS WITH FINITE AND FIXED POINTS OF IMPULSES ACTIONS

Abstract. Necessary and sufficient conditions and effective sufficient conditions are given for the solvability of the multipoint boundary value problem for a system of nonlinear impulsive equations.

რეზიუმე. არაწრფივ იმპულსურ განტოლებათა სისტემისათვის განხილულია მრავალწერტილოვანი სასაზღვრო ამოცანა. მოყვანილია ამ ამოცანის ამოხსნადობის აუცილებელი და საკმარისი პირობები და ეფექტური საკმარისი პირობები.

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For the nonlinear impulsive system with finite number of impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ for a.e. } \ t \in [a,-a] \setminus \{\tau_k\}_{k=1}^{m_0}, \tag{1}$$

$$x(\tau_k+) - x(\tau_k-) = I_k(x(\tau_k-)) \ \ (k = 1,\dots,m_0), \tag{2}$$

consider the multipoint boundary value problem

$$x_i(-\sigma_i a) = \varphi_i(x_1,\dots,x_n) \quad (i = 1,\dots,n), \tag{3}$$

where $\sigma_i \in \{-1,1\}$ $(i = 1,\dots,n)$, $-a < \tau_1 < \dots < \tau_{m_0} \le a$ (we will assume $\tau_0 = -a$ and $\tau_{m_0+1} = a$ if necessary), $f = (f_k)_{k=1}^n \in K([-a,a] \times R^n, R^n)$, $I_k = (I_{ki})_{i=1}^n : R^n \to R^n$ $(k = 1,\dots,n)$ are continuous operators, and $\varphi_i : \mathrm{BV}_s([-a,a],R^n)$ are continuous functionals which are nonlinear in general.

In this paper necessary and sufficient conditions as well effective sufficient conditions are given for the existence of solutions of the boundary value problem (1), (2), (3). Analogous results are contained in [1]–[4] for the multipoint boundary value problems for systems of ordinary differential equations.

Throughout the paper the following notation and definitions will be used.

$R =]-\infty,+\infty[$, $R_+ = [0,+\infty[$; $[a,b]$ $(a,b \in R)$ is a closed segment.

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$R^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|;$$

$$R_+^{n\times m}=\Big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\geq 0\ (i=1,\dots,n;\ j=1,\dots,m)\Big\}.$$

$R^n=R^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^{n}$; $R_+^n=R_+^{n\times 1}$.

$\operatorname{diag}(\lambda_1,\dots,\lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1,\dots,\lambda_n$; δ_{ij} is the Kronecker symbol, i.e., $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$ $(i,j=1,\dots,n)$.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X:[a,b]\to R^{n\times m}$, i.e., the sum of total variations of the latter's components.

$X(t-)$ and $X(t+)$ are the left and the right limit of the matrix-function $X:[a,b]\to R^{n\times m}$ at the point t (we will assume $X(t)=X(a)$ for $t\leq a$ and $X(t)=X(b)$ for $t\geq b$, if necessary);

$$d_1X(t)=X(t)-X(t-),\quad d_2X(t)=X(t+)-X(t);$$

$$\|X\|_s=\sup\big\{\|X(t)\|:\ t\in[a,b]\big\}.$$

$\mathrm{BV}([a,b],R^{n\times m})$ is the set of all matrix-functions of bounded variation $X:[a,b]\to R^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X)<+\infty$);

$\mathrm{BV}_s([a,b],R^n)$ is the normed space $(\mathrm{BV}([a,b],R^n),\|\cdot\|_s)$;

$\widetilde{C}([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$;

$\widetilde{C}_{loc}([a,b]\setminus\{\tau_k\}_{k=1}^{m},D)$ is the set of all matrix-functions $X:[a,b]\to D$ whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus\{\tau_k\}_{k=1}^{m}$ belong to $\widetilde{C}([c,d],D)$.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$.

An operator $\varphi:\mathrm{BV}([a,b],R^n)\to R^n$ is called nondecreasing if for every $x,y\in\mathrm{BV}([a,b],R^n)$ such that $x(t)\leq y(t)$ for $t\in[a,b]$ the inequality $\varphi(x)(t)\leq\varphi(y)(t)$ holds for $t\in[a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X:[a,b]\to D$.

If $D_1\subset R^n$ and $D_2\subset R^{n\times m}$, then $K([a,b]\times D_1,D_2)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

a) the function $f_{kj}(\cdot,x):[a,b]\to D_2$ is measurable for every $x\in D_1$;

b) the function $f_{kj}(t,\cdot) : D_1 \to D_2$ is continuous for almost every $t \in [a,b]$, and $\sup\big\{|f_{kj}(\cdot,x)| : x \in D_0\big\} \in L([a,b],R;g_{ik})$ for every compact $D_0 \subset D_1$.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x \in \widetilde{C}_{loc}([-a,a]\setminus\{\tau_k\}_{k=1}^{m_0}, R^n) \cap \mathrm{BV}_s([-a,a],R^n)$ satisfying both the system (1) for a.e. $t \in [-a,a]\setminus\{\tau_k\}_{k=1}^{m_0}$ and the relation (2) for every $k \in \{1,\dots,m_0\}$.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see, e.g., [5]–[11], and references therein). But the above-mentioned works do not contain the results analogous to those obtained in [1]–[4] for ordinary differential equations.

Using the theory of so called generalized ordinary differential equations (see, e.g. [5], [6], [12] and references therein), we extend these results to the systems of impulsive equations.

By $\nu(t)$ $(-a < t \le a)$ we denote the number of the points τ_k $(k = 1,\dots,m_0)$ belonging to $[-a,t[$.

To establish the results dealing with the boundary value problems for the impulsive system (1), (2) we use the following concept.

It is easy to show that the vector-function x is a solution of the impulsive system (1), (2) if and only if it is a solution of the following system of generalized ordinary differential equations (see, e.g. [5], [6], [12] and references therein) $dx(t) = dA(t)\cdot f(t,x(t))$, where

$$A(t) \equiv \mathrm{diag}(a_{11}(t),\dots,a_{nn}(t)),$$
$$a_{ii}(t) = \begin{cases} t & \text{for } -a \le t \le \tau_1, \\ t+k & \text{for } \tau_k < t \le \tau_{k+1} \ (k=1,\dots,m_0;\ i=1,\dots,n); \end{cases}$$
$$f(\tau_k,x) \equiv I_k(x) \quad (k=1,\dots,m_0).$$

It is evident that the matrix-function A is continuous from the left, $d_2A(t) = 0$ if $t \in [-a,a]\setminus\{\tau_k\}_{k=1}^{m_0}$ and $d_2A(\tau_k) = 1$ $(k=1,\dots,m_0)$.

Definition 1. Let $\sigma_1,\dots,\sigma_n \in \{-1,1\}$ and $-a < \tau_1 < \dots < \tau_{m_0} \le a$. We say that the triple $(P,\{H_k\}_{k=1}^{m_0},\varphi_0)$ consisting of a matrix-function $P = (p_{il})_{i,l=1}^n \in L([-a,a],R^{n\times n})$, a finite sequence of constant matrices $H_k = (h_{kil})_{i,l=1}^n \in R^{n\times n}$ $(k=1,\dots,m_0)$ and a positive homogeneous nondecreasing continuous operator $\varphi_0 = (\varphi_{0i})_{i=1}^n : \mathrm{BV}_s([-a,a],R_+^n) \to R_+^n$ belongs to the set $U^{\sigma_1,\dots,\sigma_n}(\tau_1,\dots,\tau_{m_0})$ if $p_{il}(t) \ge 0$ for a.e. $t \in [-a,a]$ $(i \ne l;\ i,l=1,\dots,n)$, $h_{kil} \ge 0$ $(i \ne l;\ i,l=1,\dots,n;\ k=1,\dots,m_0)$, and the system

$$\sigma_i x_i'(t) \le \sum_{l=1}^n p_{il}(t)x_l(t) \ \text{ for a. e. } t \in [a,b]\setminus\{\tau_k\}_{k=1}^{m_0} \quad (i=1,\dots,n),$$

$$x_i(\tau_k+) - x_i(\tau_k-) \le \sum_{l=1}^{n} h_{kil} x_l(\tau_k) \ \ (i = 1, \dots, n; \ \ k = 1, \dots, m_0)$$

has no nontrivial nonnegative solution satisfying the condition

$$x_i(-\sigma_i a) \le \varphi_{0i}(x_1, \dots, x_n) \ \ (i = 1, \dots, n).$$

The set $U^{\sigma_1,\dots,\sigma_n}(\tau_1, \dots, \tau_{m_0})$ has been introduced by I. Kiguradze for ordinary differential equations (see [1], [2]).

Theorem 1. *The problem* (1), (2), (3) *is solvable if and only if there exist continuous from the left vector-functions* $\alpha_m = (\alpha_{mi})_{i=1}^n \in \widetilde{C}_{loc}([-a,a] \setminus \{\tau_k\}_{k=1}^{m_0}, R^n) \cap \mathrm{BV}_s([-a,a], R^n)$ $(m = 1, 2)$ *such that the conditions*

$$\alpha_1(t) \le \alpha_2(t) \quad \text{for } \ t \in [-a, a]$$

and

$$(-1)^j \sigma_i \Big(f_i\big(t, x_1, \dots, x_{i-1}, \alpha_{ji}(t), x_{i+1}, \dots, x_n\big) - \alpha'_{ji}(t) \Big) \le 0$$
$$\text{for almost every } \ t \in [-a, a] \setminus \{\tau_k\}_{k=1}^{m_0},$$
$$\alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t) \ \ (j = 1, 2; \ \ i = 1, \dots, n);$$
$$(-1)^m \Big(x_i - I_{ki}(x_1, \dots, x_n) - \alpha_{mi}(\tau_k+) \Big) \le 0$$
$$\text{for } \ \alpha_1(\tau_k) \le (x_l)_{l=1}^n \le \alpha_2(\tau_k) \ \ (m = 1, 2; \ \ i = 1, \dots, n; \ \ k = 1, \dots, m_0)$$

hold, and the inequalities

$$\alpha_{1i}(-\sigma_i a) \le \varphi_i(x_l, \dots, x_n) \le \alpha_{2i}(-\sigma_i a) \ \ (i = 1, \dots, n)$$

are fulfilled on the set

$$\big\{ (x_l)_{l=1}^n \in \widetilde{C}_{loc}([-a, a] \setminus \{\tau_k\}_{k=1}^{m_0}, R^n) \cap \mathrm{BV}_s([-a, a], R^n),$$
$$\alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t) \ \ \text{for } \ t \in [-a, a] \big\}.$$

Theorem 2. *Let the conditions*

$$\sigma_i f_i(t, x_1, \dots, x_n) \operatorname{sgn} x_i \le \sum_{l=1}^{n} p_{il}(t) |x_l| + q_i(t)$$
$$\text{for almost every } \ t \in [-a, a] \setminus \{\tau_k\}_{k=1}^{m_0} \ \ (i = 1, \dots, n)$$

and

$$\sigma_i I_{ki}(x_1, \dots, x_n) \operatorname{sgn} x_i \le \sum_{l=1}^{n} h_{kil} |x_l| + q_i(\tau_k) \ (k = 1, \dots, m_0; \ i = 1, \dots, n) \quad (4)$$

be fulfilled on R^n*, the inequalities*

$$|\varphi_i(x_1, \dots, x_n)| \le \varphi_{0i}\big(|x_1|, \dots, |x_n|\big) + \zeta_i \ \ (i = 1, \dots, n)$$

be fulfilled on the set $\widetilde{C}_{loc}([-a, a] \setminus \{\tau_k\}_{k=1}^{m_0}, R^n) \cap \mathrm{BV}_s([-a, a], R^n)$*, and let*

$$\Big((p_{il})_{i,l=1}^n, \big\{ (h_{kil})_{i,l=1}^n \big\}_{k=1}^{m_0}; (\varphi_{0i})_{i=1}^n \Big) \in U^{\sigma_1,\dots,\sigma_n}(\tau_1, \dots, \tau_{m_0}),$$

where $q_i \in L([-a,a],R_+)$ $(i=1,\dots,n)$, $\zeta_i \in R_+$ $(i=1,\dots,n)$. *Then the problem* (1), (2), (3) *is solvable.*

Corollary 1. *Let the conditions* (4) *and*

$$\sigma_i f_i(t,x_1,\dots,x_n)\operatorname{sgn} x_i \le \sum_{l=1}^{n} \eta_{il}(t)|x_l| + q_i(t)$$

$$\text{for almost every } t \in [-a,a]\setminus\{\tau_k\}_{k=1}^{m_0} \quad (i=1,\dots,n)$$

be fulfilled on R^n, *the inequalities*

$$|\varphi_i(x_1,\dots,x_n)| \le \mu_i |x_i(s_i)| + \zeta_i \quad (i=1,\dots,n)$$

be fulfilled on the set $\widetilde{C}_{loc}([-a,a]\setminus\{\tau_k\}_{k=1}^{m_0},R^n)\cap \mathrm{BV}_s([-a,a],R^n)$, *and let*

$$-1 < \eta_{ii} < 0 \quad (i=1,\dots,n)$$

and

$$\mu_i (1+\eta_{ii})^{\nu(s_i)} \exp(\eta_{ii}(s_i+a)) < 1 \quad (i=1,\dots,n),$$

where $h_{kii} \in R$ $(k=1,\dots,m_0;\ i=1,\dots,n)$, h_{kil} *and* $\eta_{il} \in R_+$ $(k=1,\dots,m_0;\ i\neq l;\ i,l=1,\dots,n)$, μ_i *and* $\zeta_i \in R_+$ $(i=1,\dots,n)$, $s_i \in [-a,a]$ *and* $s_i \neq -\sigma_i a$ $(i=1,\dots,n)$, *and* $q_i \in L([-a,a],R_+)$ $(i=1,\dots,n)$. *Let, moreover, the condition*

$$g_{ii} < 1 \quad (i=1,\dots,n)$$

hold and the real part of every characteristic value of the matrix $(\xi_{il})_{i,l=1}^{n}$ *be negative, where*

$$\xi_{il} = \eta_{il}\big(\delta_{il} + (1-\delta_{il})h_i\big) - \eta_{ii} g_{il} \quad (i,l=1,\dots,n),$$

$$g_{il} = \mu_i (1-\mu_i\gamma_i)^{-1}\gamma_{il}(s_i) + \gamma_{il}(a) \quad (i,l=1,\dots,n),$$

$$\gamma_i = (1+\eta_{ii})^{\nu(s_i)} \exp(\eta_{ii}(s_i+a)) \quad (i=1,\dots,n),$$

$$\gamma_{il}(-a)=0, \quad \gamma_{il}(t) = \Big|\sum_{-a<\tau_k<t} h_{kil}\Big| \ \text{ for } t \in]-a,a] \quad (i,l=1,\dots,n),$$

$$h_i = 1 \ \text{ for } \mu_i \le 1 \ \text{ and}$$

$$h_i = 1 + (\mu_i - 1)(1-\mu_i\gamma_i) \ \text{ for } \mu_i > 1 \quad (i=1,\dots,n).$$

Then the problem (1), (2), (3) *is solvable.*

Remark 1. In the Corollary 1 as matrix-function $C=(c_{il})_{i,l=1}^{n}$ we take

$$c_{il}(t) \equiv \eta_{il} t + \beta_{il}(t) \quad (i,l=1,\dots,n),$$

where

$$\beta_{il}(t) \equiv \sum_{a<\tau_k<t} h_{kil} \quad (i,l=1,\dots,n).$$

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