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Short Communications

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ON THE SOLVABILITY OF A MULTIPOINT BOUNDARY VALUE PROBLEM FOR SYSTEMS OF NONLINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Necessary and sufficient conditions and effective sufficient conditions are given for the existence of solutions of the multipoint boundary value problem for a system of nonlinear generalized ordinary differential equations.

[illegible]

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Let $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$; for $m \in \{1, 2\}$ and $i, k \in \{1, \dots, n\}$, $a_{mik} : [-a, a] \to R$ be nondecreasing functions continuous at the points $-a$ and a;

$$a_{ik}(t) \equiv a_{1ik}(t) - a_{2ik}(t),$$
$$A = (a_{ik})_{i,k=1}^n, \quad A_m = (a_{mik})_{i,k=1}^n \ \ (m = 1, 2);$$

$f = (f_k)_{k=1}^n : [-a, a] \times R^n \to R^n$ be a vector-function belonging to the Carathéodory class corresponding to the matrix-function A, and $\varphi_i : \mathrm{BV}_s([-a, a], R^n) \to R$ $(i = 1, \dots, n)$ be continuous functionals which are nonlinear in general.

For the system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot f(t, x(t)), \tag{1}$$

where $x = (x_i)_{i=1}^n$, consider the multipoint boundary value problem

$$x_i(-\sigma_i a) = \varphi_i(x_1, \dots, x_n) \ \ (i = 1, \dots, n). \tag{2}$$

In this paper necessary and sufficient conditions as well effective sufficient conditions are given for the existence of solutions of the boundary value

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problem (1), (2). Analogous results are contained in [1]–[4] for multipoint boundary value problems for systems of ordinary differential equations.

The theory of generalized ordinary differential equations enables one to investigate ordinary differential, impulsive and difference equations from a common point of view (see [5]–[16]).

Throughout the paper the following notation and definitions will be used.

$R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$; $[a,b]$ $(a, b \in R)$ is a closed segment.

$R^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$$R_+^{n\times m} = \Big\{(x_{ij})_{i,j=1}^{n,m} : \ x_{ij} \geq 0 \ (i = 1, \dots, n; \ j = 1, \dots, m)\Big\}.$$

$R^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $R_+^n = R_+^{n\times 1}$.

$\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$; δ_{ij} is the Kronecker symbol, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ $(i, j = 1, \dots, n)$.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X : [a,b] \to R^{n\times m}$, i.e., the sum of total variations of the latter's components.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a,b] \to R^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \leq a$ and $X(t) = X(b)$ for $t \geq b$, if necessary);

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t);$$
$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [a,b]\big\}.$$

$\mathrm{BV}([a,b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to R^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X) < +\infty$);

$\mathrm{BV}_s([a,b], R^n)$ is the normed space $(\mathrm{BV}([a,b], R^n), \|\cdot\|_s)$;

If B_1 and B_2 are normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if

$$g(\lambda x) = \lambda g(x)$$

for every $\lambda \in R_+$ and $x \in B_1$.

An operator $\varphi : \mathrm{BV}([a,b], R^n) \to R^n$ is called nondecreasing if for every $x, y \in \mathrm{BV}([a,b], R^n)$ such that $x(t) \leq y(t)$ for $t \in [a,b]$ the inequality $\varphi(x)(t) \leq \varphi(y)(t)$ holds for $t \in [a,b]$.

$s_j : \mathrm{BV}([a,b],R) \to \mathrm{BV}([a,b],R)$ $(j=0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1 x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a\le\tau<t} d_2 x(\tau) \quad \text{for} \quad a<t\le b,$$

and

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \quad \text{for} \quad t \in [a,b].$$

If $g : [a,b] \to R$ is a nondecreasing function, $x : [a,b] \to R$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\, dg(\tau) =$$

$$= \int_{]s,t[} x(\tau)\, dS_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau) d_1 g(\tau) + \sum_{s\le\tau<t} x(\tau) d_2 g(\tau),$$

where $\int_{]s,t[} x(\tau)\, ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$ corresponding to the function $S_0(g)$.

If $a=b$, then we assume

$$\int_a^b x(t)\, dg(t) = 0,$$

and if $a>b$, then we assume

$$\int_a^b x(t)\, dg(t) = -\int_b^a x(t)\, dg(t).$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_s^t x(\tau)\, dg_1(\tau) - \int_s^t x(\tau)\, dg_2(\tau) \quad \text{for} \quad s \le t.$$

$L([a,b],R;g)$ is the set of all functions $x : [a,b] \to R$ measurable and integrable with respect to the measures $\mu(g_i)$ $(i=1,2)$, i.e. such that

$$\int_a^b |x(t)|\, dg_i(t) < +\infty \quad (i=1,2).$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset R^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X =$

$(x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b], R; g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \Big(\sum_{k=1}^{n}\int_s^t x_{kj}(\tau)dg_{ik}(\tau)\Big)_{i,j=1}^{l,m} \quad \text{for} \quad a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \quad (j = 0,1,2).$$

If $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, then $K([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

a) the function $f_{kj}(\cdot,x) : [a,b] \to D_2$ is $\mu(g_{ik})$-measurable for every $x \in D_1$;

b) the function $f_{kj}(t,\cdot) : D_1 \to D_2$ is continuous for $\mu(g_{ik})$-almost every $t \in [a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)| :\ x \in D_0\big\} \in L([a,b], R; g_{ik})$$

for every compact $D_0 \subset D_1$.

If $G_j : [a,b] \to R^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G = G_1 - G_2$ and $X : [a,b] \to R^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \ \text{ for } \ s \le t,$$

$$S_k(G) = S_k(G_1) - S_k(G_2) \quad (k = 0,1,2),$$

$$L([a,b], D; G) = \bigcap_{j=1}^{2} L([a,b], D; G_j),$$

$$K([a,b]\times D_1, D_2; G) = \bigcap_{j=1}^{2} K([a,b]\times D_1, D_2; G_j).$$

If $G(t) \equiv \operatorname{diag}(t,\dots,t)$, then we omit G in the notation containing G.

The inequalities between the vectors and between the matrices are understood componentwise.

A vector-function $x \in \mathrm{BV}([-a,a], R^n)$ is said to be a solution of the system (1) if

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot f(\tau, x(\tau)) \quad \text{for} \quad -a \le s \le t \le a.$$

By a solution of the system of generalized ordinary differential inequalities

$$dx(t) \le dA(t)\cdot f(t, x(t)) \ \ (\ge)$$

we mean a vector-function $x \in \mathrm{BV}([-a,a],R^n)$ such that

$$x(t) \le x(s) + \int_s^t dA(\tau)\cdot f(\tau,x(\tau)) \ \ (\ge) \quad \text{for} \quad -a \le s \le t \le a.$$

If $s \in R$ and $\beta \in \mathrm{BV}[a,b],R)$ are such that

$$1+(-1)^j d_j\beta(t) \ne 0 \ \text{ for } \ \ (-1)^j(t-s)<0 \quad (j=1,2),$$

then by $\gamma_\beta(\cdot,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t)=\gamma(t)\,d\beta(t), \quad \gamma(s)=1.$$

It is known (see [6], [8]) that

$$\gamma_\beta(t,s)=\begin{cases} \exp(s_0(\beta)(t)-s_0(\beta)(s)) \prod\limits_{s<\tau\le t}(1-d_1\beta(\tau))^{-1}\times \\ \qquad \times \prod\limits_{s\le\tau<t}(1+d_2\beta(\tau)) \ \text{ for } \ t>s, \\ \exp(s_0(\beta)(t)-s_0(\beta)(s)) \prod\limits_{t<\tau\le s}(1-d_1\beta(\tau))\times \\ \qquad \times \prod\limits_{t\le\tau<s}(1+d_2\beta(\tau))^{-1} \ \text{ for } \ t<s, \\ \qquad\qquad 1 \qquad\qquad \text{for } \ t=s. \end{cases} \tag{3}$$

Definition 1. Let $\sigma_1,\dots,\sigma_n \in \{-1,1\}$. We say that the pair $((c_{il})_{i,l=1}^n;(\varphi_{0i})_{i=1}^n)$ consisting of a matrix-function $(c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],R^{n\times n})$ and a positive homogeneous nondecreasing operator $(\varphi_{0i})_{i=1}^n : \mathrm{BV}_s([a,b],R_+^n) \to R_+^n$ belongs to the set $U^{\sigma_1,\dots,\sigma_n}$ if the functions c_{il} $(i\ne l;\ i,l=1,\dots,n)$ are nondecreasing on $[a,b]$ and continuous at the point $t_i=-\sigma_i a$,

$$d_j c_{ii}(t) \ge 0 \quad \text{for} \quad t\in[-a,a] \quad (j=1,2;\ \ i=1,\dots,n)$$

and the problem

$$\sigma_i dx_i(t) \le \sum_{l=1}^n x_l(t)dc_{il}(t) \ \text{ for } \ t\in[-a,a]\setminus\{-\sigma_i a\} \ \ (i=1,\dots,n),$$
$$(-1)^j d_j x_i(-\sigma_i a) \le x_i(-\sigma_i a) d_j c_{ii}(-\sigma_i a) \quad (j=1,2;\ \ i=1,\dots,n);$$
$$x_i(-\sigma_i a) \le \varphi_{0i}\big(|x_1|,\dots,|x_n|\big) \quad (i=1,\dots,n)$$

has no nontrivial non-negative solution.

The set $U^{\sigma_1,\dots,\sigma_n}$ has been introduced by I. Kiguradze for ordinary differential equations (see [1], [2]).

Theorem 1. *The problem* (1), (2) *is solvable if and only if there exist vector-functions* $\alpha_m=(\alpha_{mi})_{i=1}^n \in BV([-a,a],R^n)$ $(m=1,2)$ *and* *matrix-functions* $(\beta_{mik})_{i,k=1}^n : [-a,a] \to R^{n\times n}$ $(m=1,2)$ *such that* $\beta_{mik} \in L([-a,a],R;a_{jik})$ $(m,j=1,2;\ i,k=1,\dots,n)$,

$$\alpha_{mi}(t) \equiv \alpha_{mi}(-\sigma_i a) + \sum_{k=1}^n \left(\int_{-\sigma_i a}^t \beta_{mik}(\tau)\, da_{1ik}(\tau) - \int_{-\sigma_i a}^t \beta_{3-mik}(\tau)\, da_{2ik}(\tau) \right)$$
$$(m=1,2;\ \ i=1,\dots,n),$$

$$\alpha_1(t) \le \alpha_2(t) \quad \textit{for}\ \ t \in [-a,a], \tag{4}$$

$$(-1)^m \sigma_i \big(f_k(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n) - \beta_{mik}(t) \big) \le 0$$
$$\textit{for}\ \ \mu(a_{1+|m-j|ik})\textit{-almost every}\ \ t\in[-a,a],$$
$$\alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t)\ \ (m,j=1,2;\ \ i,k=1,\dots,n),$$

$$(-1)^m \left(x_i - (-1)^j \sum_{k=1}^n f_k(t,x_1,\dots,x_n) d_j a_{ik}(t) - \alpha_{mi}(t) - (-1)^j d_j \alpha_{mi}(t) \right) \le$$
$$\le 0\ \ \textit{for}\ \ t\in[-a,a], \quad \alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t),$$
$$(-1)^j \sigma_i > 0 \quad (m,j=1,2;\ \ i=1,\dots,n) \tag{5}$$

and the inequalities

$$\alpha_{1i}(-\sigma_i a) \le \varphi_i(x_l,\dots,x_n) \le \alpha_{2i}(-\sigma_i a)\ \ (i=1,\dots,n) \tag{6}$$

are fulfilled on the set $\{(x_l)_{l=1}^n \in \mathrm{BV}([a,b],R^n),\ \alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t)$ *for* $t\in[-a,a]\}$.

Corollary 1. *Let the matrix-function* $A(t) = (a_{ik})_{i,k=1}^n$ *be nondecreasing on* $[-a,a]$. *Then the problem* (1), (2) *is solvable if and only if there exist vector-functions* $\alpha_m = (\alpha_{mi})_{i=1}^n \in \mathrm{BV}([-a,a],R^n)$ $(m=1,2)$ *and matrix-functions* $(\beta_{mik})_{i,k=1}^n : [-a,a]\to R^{n\times n}$ $(m=1,2)$ *such that* $\beta_{mik} \in L([-a,a],R;a_{ik})$ $(m=1,2;\ i,k=1,\dots,n)$,

$$\alpha_{mi}(t) \equiv \alpha_{mi}(-\sigma_i a) + \sum_{l=1}^n \left(\int_{-\sigma_i a}^t \beta_{mik}(\tau) da_{ik}(\tau) \right)$$
$$(m=1,2;\ \ i,k=1,\dots,n),$$

the conditions (4)–(6) *hold, and the inequalities*

$$(-1)^m \sigma_i \big(f_k(t,x_1,\dots,x_{i-1},\alpha_{ji}(t),x_{i+1},\dots,x_n) - \beta_{jik}(t) \big) \le 0$$
$$(j=1,2;\ \ i,k=1,\dots,n)$$

are fulfilled for $\mu(a_{ik})$*-almost every* $t\in[-a,a]$ *and* $\alpha_1(t) \le (x_l)_{l=1}^n \le \alpha_2(t)$.

Theorem 2. *Let the condition*

$$(-1)^{m+1} \sigma_i f_k(t,x_1,\dots,x_n)\,\mathrm{sgn}\, x_i \le \sum_{l=1}^n p_{mikl}(t)|x_l| + q_k(t)$$
$$\textit{for}\ \ \mu(a_{mik})\textit{-almost every}\ \ t\in[-a,a]\ \ (m=1,2;\ \ i,k=1,\dots,n) \tag{7}$$

be fulfilled on R^n, *and let the inequalities*

$$|\varphi_i(x_1,\dots,x_n)| \le \varphi_{0i}\big(|x_1|,\dots,|x_n|\big) + \zeta_i \quad (i=1,\dots,n)$$

be fulfilled on $BV([-a,a],R^n)$, *where* $(p_{mikl})_{k,l=1}^n \in L([-a,a],R^{n\times n};A_m)$ $(m=1,2;\ i=1,\dots,n)$, $q_k=(q_{ki})_{i=1}^n \in L([-a,a],R_+^n;A_m)$ $(m=1,2)$, $\zeta_i \in R_+$ $(i=1,\dots,n)$. *Let, moreover, there exist a matrix-function* $(c_{il})_{i,l=1}^n \in BV([-a,a],R^{n\times n})$ *such that*

$$\big((c_{il})_{i,l=1}^n;(\varphi_{0i})_{i=1}^n\big) \in U^{\sigma_1,\dots,\sigma_n}$$

and

$$\sum_{m=1}^{2}\sum_{k=1}^{n}\int_s^t p_{mikl}(\tau)\,da_{mik}(\tau) \le c_{il}(t)-c_{il}(s)$$

$$\text{for} \quad -a \le s < t \le a \quad (i,l=1,\dots,n).$$

Then the problem (1), (2) *is solvable.*

Corollary 2. *Let there exist* $m,m_1 \in \{1,2\}$ *such that* $m+m_1=3$ *and the conditions* (7) *and*

$$(-1)^{m_1+1}\sigma_i f_k(t,x_1,\dots,x_n)\operatorname{sgn} x_i \le \sum_{l=1}^{n}\eta_{il}|x_l| + q_k(t)$$

$$\text{for } \mu(a_{m_1ik})\text{-almost every } t\in[-a,a] \ (i,k=1,\dots,n)$$

are fulfilled on R^n, *the inequalities*

$$|\varphi_i(x_1,\dots,x_n)| \le \mu_i|x_i(s_i)| + \zeta_i \quad (i=1,\dots,n) \tag{8}$$

be fulfilled on $BV_s([a,b],R^n)$, *and let*

$$0 \le d_j\alpha_i(t) < |\eta_{ii}|^{-1} \ \text{ for } \ (-1)^j(t+\sigma_i a) > 0 \ \ (j=1,2;\ \ i=1,\dots,n) \tag{9}$$

and

$$\mu_i\gamma_i(s_i,-\sigma_i a) < 1 \quad (i=1,\dots,n), \tag{10}$$

where $(p_{mikl})_{k,l=1}^n \in L([-a,a],R_+^{n\times n};A_m)$ $(i=1,\dots,n)$, $\eta_{il}\in R_+$ $(i\ne l;$ $i,l=1,\dots,n)$, $\eta_{ii}<0$ $(i=1,\dots,n)$, $q_k=(q_{ki})_{k=1}^n \in L([-a,a],R_+^n;A_m)$ $(m=1,2)$, $\zeta_i\in R_+$ $(i=1,\dots,n)$, $\mu_i \in R_+$ *and* $s_i\in[-a,a]$, $s_i\ne -\sigma_i a$ $(i=1,\dots,n)$,

$$\alpha_i(t) \equiv \sum_{k=1}^{n} a_{m_1ik}(t) \quad (i=1,\dots,n),$$

$$\gamma_i(t,s) \equiv \gamma_{a_i}(t,s) \quad (i=1,\dots,n),$$

$$a_i(t) \equiv \eta_{ii}\sigma_i\big(\alpha_i(t)-\alpha_i(-\sigma_i a)\big) \quad (i=1,\dots,n),$$

and the functions γ_{a_i} $(i=1,\dots,n)$ *are defined according to* (3). *Let, moreover,*

$$g_{ii} < 1 \quad (i=1,\dots,n)$$

and the real part of every characteristic value of the matrix $(\xi_{il})_{i,l=1}^{n}$ *be negative, where*

$$\xi_{il} = \eta_{il}\big(\delta_{il} + (1-\delta_{il})h_i\big) - \eta_{ii}g_{il} \quad (i,l=1,\dots,n),$$

$$g_{il} = \mu_i\big(1-\mu_i\gamma_i(s_i,-\sigma_i a)\big)^{-1}\gamma_{il}(s_i)+ \\ +\max\big\{\gamma_{il}(-a),\gamma_{il}(a)\big\} \ \ (i,l=1,\dots,n),$$

$$\gamma_{il}(-\sigma_i a)=0, \quad \gamma_{il}(t) = \big|\beta_{il}(t)-\beta_{il}(-\sigma_i a)\big| - (1-\delta_{il})d_j\beta_{il}(-\sigma_i a)$$

$$\textit{for} \ \ (-1)^j(t+\sigma_i a)>0 \ \ (j=1,2; \ \ i,l=1,\dots,n),$$

$$\beta_{il}(t) \equiv \sum_{k=1}^{n}\int_{-a}^{t} p_{mikl}(\tau)da_{mik}(\tau) \ \ (i=1,\dots,n),$$

$$h_i = 1 \ \ \textit{for} \ \ \mu_i \le 1 \ \ \textit{and}$$

$$h_i = 1+(\mu_i-1)(1-\mu_i\gamma_i(s_i,-\sigma_i a))^{-1} \ \ \textit{for} \ \ \mu_i>1 \ \ (i=1,\dots,n).$$

Then the problem (1), (2) *is solvable.*

Remark 1. In Corollary 2 as the matrix-function $C=(c_{il})_{i,l=1}^{n}$ we take

$$c_{il}(-\sigma_i a) = 0 \quad (i,l=1,\dots,n),$$

$$c_{il}(t) = \eta_{il}\big(\alpha_i(t)-\alpha_i(-\sigma_i a)-(-1)^j d_j\alpha_i(-\sigma_i a)\big)+ \\ +\beta_{il}(t)-\beta_{il}(-\sigma_i a)-(-1)^j d_j\beta_{il}(-\sigma_i a)$$

$$\text{for} \ \ (-1)^j(t+\sigma_i a)>0 \ \ (j=1,2; \ \ i,l=1,\dots,n).$$

If the matrix-function $A=(a_{ik})_{i,k=1}^{n} : [-a,a]\to R^{n\times n}$ is nondecreasing, then Corollary 2 has the following form.

Corollary 3. *Let the matrix-function* $A=(a_{ik})_{i,k=1}^{n} : [-a,a]\to R^{n\times n}$ *be nondecreasing, the conditions* (8)–(10) *hold, the condition*

$$\sigma_i f_k(t,x_1,\dots,x_n)\operatorname{sgn} x_i \le \sum_{l=1}^{n}\eta_{il}|x_l| + q_k(t)$$

$$\textit{for} \ \ \mu_(a_{ik})\textit{-almost every} \ \ t\in[-a,a] \ \ (i,k=1,\dots,n) \tag{11}$$

be fulfilled on R^n *and let the real part of every characteristic value of the matrix* $\big(\eta_{il}(\delta_{il}+(1-\delta_{il})h_i)\big)_{i,l=1}^{n}$ *be negative, where*

$$\alpha_i(t) \equiv \sum_{k=1}^{n} a_{ik}(t) \ \ (i=1,\dots,n),$$

and the functions $\gamma_i(t,s)$ $(i=1,\dots,n)$ *and* $a_i(t)$ $(i=1,\dots,n)$ *and the numbers* h_i $(i=1,\dots,n)$ *are defined as in Corollary* 2. *Then the problem* (1), (2) *is solvable.*

Corollary 4. *Let the matrix-function* $A=(a_{ik})_{i,k=1}^{n} : [-a,a]\to R^{n\times n}$ *be nondecreasing and continuous from the left, the conditions* (8), (10), (11) *and*

$$0 \leq d_2\alpha_i(t) < |\eta_{ii}|^{-1} \ \text{for} \ t \in]-a, a[\ \ (i = 1, \dots, n)$$

hold and let the real part of every characteristic value of the matrix $\big(\eta_{il}(\delta_{il} + (1-\delta_{il})h_i)\big)_{i,l=1}^{n}$ *be negative, where the functions* $\alpha_i(t)$ $(i = 1, \dots, n)$, $\gamma_i(t, s)$ $(i = 1, \dots, n)$ *and* $a_i(t)$ $(i = 1, \dots, n)$ *and the numbers* h_i $(i = 1, \dots, n)$ *are defined as in Corollary* 3. *Then the problem* (1), (2) *is solvable.*

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