Mem. Differential Equations Math. Phys. 41 (2007), 163–165

Sulkhan Mukhigulashvili

ON A PRIORI ESTIMATES OF SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL INEQUALITIES OF HIGHER ORDER WITH BOUNDARY CONDITIONS OF PERIODIC TYPE

(Reported on April 2, 2007)

In the present paper, we consider nonlinear functional differential inequalities appearing in the theory of boundary value problems (see [1]–[6] and the references therein) and formulate new results on a priori estimates of their solutions satisfying the boundary conditions of periodic type.

Let n be a natural number, $\omega > 0$, $C^{n-1}([0,\omega])$ be the space of $(n-1)$ times continuously differentiable functions with the norm

$$\|u\|_{C^{n-1}} = \max\Big\{\sum_{i=1}^{n} |u^{(i-1)}(t)| : \ 0 \le t \le \omega\Big\},$$

and $L([0,\omega])$ be the space of Lebesgue integrable functions $v : [0,\omega] \to \mathbb{R}$ with the norm

$$\|v\|_L = \int_a^b |v(t)|\,dt.$$

On the interval $[0,\omega]$, let us consider the nonlinear differential inequality

$$\Big|u^{(n)}(t) - p(u)(t)u(\tau(t))\Big| \le q(t) \tag{1}$$

with the boundary conditions of periodic type

$$\sum_{i=1}^{n} \big|u^{(i-1)}(0) - u^{(i-1)}(\omega)\big| \le c_0, \tag{2}$$

where $p : C^{n-1}([0,\omega]) \to L([0,\omega])$ is an operator, $q \in L([0,\omega])$ is a nonnegative function, c_0 is a nonnegative number, and $\tau : [0,\omega] \to [0,\omega]$ is a measurable function.

The function $u : [0,\omega] \to \mathbb{R}$ is said to be a **solution of the differential inequality (1)** if it is absolutely continuous together with its derivatives

2000 *Mathematics Subject Classification.* 34K10, 34K13.
Key words and phrases. Nonlinear functional differential inequality of higher order, systems of nonlinear functional differential inequalities, periodic boundary conditions, a priori estimate.

up to the order $n-1$ inclusive and almost everywhere on $[0,\omega]$ satisfies the inequality (1).

A solution of the differential inequality (1) satisfying the condition (2) is called a **solution of the problem (1), (2)**.

For an arbitrary $v \in L([0,\omega])$ and a measurable function $w : [0,\omega] \to [0,\omega]$ we assume

$$\ell(v,w) = \bigg(\int_0^\omega |v(t)|\,|w(t)-t|\,dt\bigg)^{1/2}.$$

Theorem 1. *Let $n=2m$ and let there exist $k\in\{0,1\}$ and nonnegative functions $p_i : L([0,\omega])$ $(i=0,1)$ such that*

$$p_0(t) \le (-1)^{m+k} p(x)(t) \le p_1(t) \ \text{ for } \ t\in[0,\omega] \ \text{ and } \ x\in C([0,\omega]).$$

Moreover, let

$$\int_0^\omega p_0(t)\,dt > 0 \tag{3}$$

and

$$(1-k)p_1(t) + \frac{2\pi}{\omega}\,|p_1(t)|^{1/2}\ell(p_1,\tau) < \Big(\frac{2\pi}{\omega}\Big)^n \ \text{ for } \ t\in[0,\omega].$$

Then there exists a positive constant ρ, independent of p, q, and c_0, such that an arbitrary solution of the problem (1), (2) admits the estimate

$$\|u\|_{C^{n-1}} \le \rho(c_0 + \|q\|_L). \tag{4}$$

Corollary 1. *Let $n=2m$ and there exist nonnegative functions $p_i \in L([0,\omega])$ $(i=0,1)$ such that*

$$p_0(t) \le (-1)^m p(x)(t) \le p_1(t) \ \text{ for } \ t\in[0,\omega] \ \text{ and } \ x\in C^{n-1}([0,\omega]),$$

and the inequality (3) holds. Let, moreover,

$$|p_1(t)| + \Big(\frac{2\pi}{\omega}\Big)^{m+1}\ell(p_1,\tau) < \Big(\frac{2\pi}{\omega}\Big)^n \ \text{ for } \ t\in[0,\omega].$$

Then there exists a positive constant ρ, independent of p, q, and c_0, such that an arbitrary solution of the problem (1), (2) admits the estimate (4).

Corollary 2. *Let $n=2m$ and there exist nonnegative functions $p_i \in L([0,\omega])$ $(i=0,1)$ such that*

$$p_0(t) \le (-1)^{m-1} p(x)(t) \le p_1(t) \ \text{ for } \ t\in[0,\omega] \ \text{ and } \ x\in C^{n-1}([0,\omega])$$

and the inequality (3) holds. Let, moreover,

$$|p_1(t)|^{1/2}\ell(p_1,\tau) < \Big(\frac{2\pi}{\omega}\Big)^{n-1} \ \text{ for } \ t\in[0,\omega]. \tag{5}$$

Then there exists a positive constant ρ, independent of p, q, and c_0, such that every solution of the problem (1), (2) admits the estimate (4).

Theorem 2. *Let $n = 2m+1$ and there exist $k \in \{0, 1\}$ and nonnegative functions $p_i \in L([0,\omega])$ $(i = 0, 1)$ such that*

$$p_0(t) \leq (-1)^k p(x)(t) \leq p_1(t) \quad \text{for } t \in [0,\omega] \quad \text{and } x \in C^{n-1}([0,\omega])$$

and the inequalities (3) *and* (5) *hold. Then there exists a positive constant ρ, independent of p, q, and c_0, such that every solution of the problem* (1), (2) *admits the estimate* (4).

Acknowledgement

This work is supported by the Georgian National Science Foundation (Grant No. GNSF/ST06/3-002).

References

1. P. W. Bates and J. R. Ward, Jr., Periodic solutions of higher order systems. *Pacific J. Math.* **84**(1979), No. 2, 275–282.
2. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh.* **30**(1987), 3–103; English transl.: *J. Sov. Math.* **43**(1988), No. 2, 2259–2339.
3. I. Kiguradze, On periodic solutions of nth order ordinary differential equations. *Nonlinear Anal.* **40**(2000), No. 1–8, 309–321.
4. I. Kiguradze and T. Kusano, On periodic solutions of higher order nonautonomous ordinary differential equations. *Differentsial'nye Uravneniya* **35**(1999), No. 1, 72–78; English transl.: *Differ. Equations* **35**(1999), No. 1, 71–77.
5. I. Kiguradze and T. Kusano, On conditions for the existence and uniqueness of periodic solutions of nonautonomous differential equations. (Russian) *Differentsial'nye Uravneniya* **36**(2000), No. 10, 1301–1306; English transl.: *Differ. Equations* **36**(2000), No. 10, 1436–1442.
6. I. Kiguradze and T. Kusano, On periodic solutions of even-order ordinary differential equations. *Ann. Mat. Pura Appl.* **180**(2001), No. 3, 285–301.
7. I. Kiguradze and B. Půža, Conti-Opial type theorems for systems of functional differential equations. (Russian) *Differentsial'nye Uravneniya* **33**(1997), No. 2, 185–194; English transl.: *Differ. Equations* **33**(1997), No. 2, 184–193.
8. A. Lasota and Z. Opial, Sur les solutions périodiques des équations différentielles ordinaires. *Anales Polonici Mathematici* **16**(1964), No. 1, 69–94.

(Received 9.04.2007)

Author's addresses:

Current address:
Mathematical Institute
Academy of Sciences of the Czech Republic
Žižkova 22, 616 62 Brno, Czech Republic
E-mail: mukhig@ipm.cz

Permanent address:
A. Razmadze Mathematical Institute
1, M. Aleksidze Str., Tbilisi 0193, Georgia
E-mail: smukhig@rmi.acnet.ge