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QUASI-INTEGRALS OF THREE-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH SKEW-SYMMETRIC COEFFICIENT MATRICES

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We consider the linear system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^3, \quad t \geq 0, \tag{1_A}$$

with a continuous piecewise differentiable skew-symmetric matrix $A(\cdot) \equiv (a_{ij})_{i,j=1}^3$ for all $t \geq 0$. Such systems coincide with kinematic equations of the rigid body mechanics, in particular, they are applied in robotics [1] in modelling automatized production based on automatic holonomic systems for parametric construction of programmed motions of executive devices in a three dimensional physical space. Four-dimensional systems with skew-symmetric coefficient matrix are also applied in the gyroscope theory [2].

Following [3,4], for the elements $a_{ij}(t)$ of the skew-symmetric matrix $A(t)$ we define: the function vector

$$a(t) \equiv \big(a_{23}(t), -a_{13}(t), a_{12}(t)\big) \in \mathbb{R}^3, \quad t \geq 0,$$

the scalar functions

$$C(\eta) \equiv \cos \int_0^\eta \|a(\tau)\| d\tau, \quad S(\eta) \equiv \sin \int_0^\eta \|a(\tau)\| d\tau, \quad t \geq 0,$$

and the vector function of two-variables

$$v(t,\eta) \equiv \begin{pmatrix} -(a_{12}^2(t) + a_{13}^2(t))C(\eta) \\ -a_{13}(t)a_{23}(t)C(\eta) + a_{12}(t)\|a(t)\|S(\eta) \\ a_{12}(t)a_{23}(t)C(\eta) + a_{13}(t)\|a(t)\|S(\eta) \end{pmatrix}, \quad t,\, \eta \in [0, +\infty).$$

In the above-mentioned works, for the quasi-integrals

$$L_1(x(t),t) \equiv (x(t), a(t)) - (x(0), a(0)),$$
$$L_2(x(t),t) \equiv (x(t), v(t,t)) - (x(0), v(0,0)),$$

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of the non-stationary system (1_A) on its solutions $x(\cdot) : [0, +\infty) \to \mathbb{R}^3$, which are ordinary integrals in the stationary case and identically vanish, the estimates

$$\bigl|L_1(x(t), t)\bigr| \le \|x(0)\| \int\limits_0^t \|\dot{a}(\tau)\|\, d\tau, \quad t \ge 0, \tag{2_1}$$

$$\bigl|L_2(x(t), t)\bigr| \le c_2 \|x(0)\| \int\limits_0^t \|a(\tau)\|\, \|\dot{a}(\tau)\|\, d\tau, \quad t \ge 0, \tag{2_2}$$

are obtained with the constant $c_2 = 2\sqrt{3}$. In those papers it is also proved that the first estimate may turn into equality (be efficient), whereas for the second one this was established for $c_2 = 1$.

The authors of the present paper improved the estimate (2_2) up to the one with $c_2 = 2$ and proved that the latter is unimprovable. It should be noted that the efficiency of both estimates (the estimate (2_1) and the estimate (2_2) with the constant $c_2 = 2$) is realized for different three-dimensional systems (1_A). In this connection, we have the following two problems on the simultaneous efficiency of the estimates (2_1) and (2_2) with $c_2 = 2$: 1) efficiency of these estimates for the common system (1_A) but, probably, for its different its solutions; 2) simultaneous efficiency of both estimates for one nontrivial solution $x(t)$ of the same system (1_A).

The aim of this paper is to prove that the estimates (2_1) and (2_2) with $c_2 = 2$ cannot be efficient simultaneously for one nontrivial solution $x(t)$ of the system (1_A) at the same moment of time $t = t_0 > 0$ such that $\dot{a}(\tau) \not\equiv 0$ for $\tau \in [0, t_0]$.

The following theorem establishes this.

Theorem. *Let $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^3 \setminus \{0\}$ be an arbitrary solution of any three-dimensional system (1_A), and h be a fixed constant, $h \in (0.9; 1]$. If for some $t_0 > 0$ the estimate*

$$\bigl|L_1(x(t_0), t_0)\bigr| \ge h \|x(0)\| \int\limits_0^{t_0} \|\dot{a}(\tau)\|\, d\tau \tag{3}$$

is fulfilled, then the inequality

$$\bigl|L_2(x(t_0), t_0)\bigr| \le$$

$$\le 2\Bigl[1 - (2 - \sqrt{2})\, \frac{(h - 0.8)(h - 0.9)}{2 + h}\Bigr] \|x(0)\| \int\limits_0^{t_0} \|\dot{a}(\tau)\|\, \|a(\tau)\|\, d\tau \tag{4}$$

is valid.

Proof. The statement of the theorem is evident if $\dot{a}(\tau) \equiv 0$ for all $\tau \in [0, t_0]$. Thus let us consider the opposite case. We introduce the vectors

$$e_1 := (1, 0, 0) \in \mathbb{R}^3, \quad w(t) := e_1 \times a(t), \quad f(t) := \|a(t)\| e_1, \quad t \ge 0.$$

Then the vector function $v(t,\eta)$ satisfies the equality

$$v(t,\eta) = (a(t)\times w(t))C(\eta) - \|a(t)\|w(t)S(\eta), \quad t,\eta \geq 0. \tag{5}$$

According to Lemma 2 in [1], the equality

$$L_2(x(t),t) = \int_0^t \Big(x(\tau), \frac{\partial v(\tau,t)}{\partial \tau}\Big)\, d\tau, \quad t \geq 0, \tag{6}$$

is valid. We now estimate the absolute value of the scalar product under the integral sign in (6) by using the inequality $|\|a(\tau)\|'| \leq \|\dot a(\tau)\|$ (see [3]) and the pairwise orthogonality of the vectors $f(\tau)$ and $w(\tau)$, as well as e_1 and $e_1 \times \dot a(\tau)$:

$$\begin{aligned}
&\Big|\Big(x(\tau), \frac{\partial v(\tau,t)}{\partial \tau}\Big)\Big| = \\
&= \Big|\Big(x(\tau), \Big\{C(t)\big[w(\tau)\times a(\tau)\big] - S(t)\big[f(\tau)\times a(\tau)\big]\Big\}'_\tau\Big)\Big| = \\
&= \Big|\Big(x(\tau), \Big\{\big[C(t)w(\tau) - S(t)f(\tau)\big]\times a(\tau)\Big\}'_\tau\Big)\Big| \leq \\
&\leq \Big|\Big(x(\tau), \big[C(t)w(\tau) - S(t)f(\tau)\big]'_\tau \times a(\tau)\Big)\Big| + \\
&+ \Big|\Big(x(\tau), \big[C(t)w(\tau) - S(t)f(\tau)\big]\times \dot a(\tau)\Big)\Big| \leq \\
&\leq \|x(0)\|\,\|a(\tau)\|\,\Big\|C(t)\big(e_1\times\dot a(\tau)\big) - S(t)\|a(\tau)\|' e_1\Big\| + \\
&+ \big\|x(\tau)\times\dot a(\tau)\big\|\Big\{C^2(t)\|w(\tau)\|^2 + S^2(t)\|f(\tau)\|^2\Big\}^{1/2} \leq
\end{aligned}$$

(here the use is made of the equality $\|f(\tau)\| = \|a(\tau)\|$ and the estimate $\|w(\tau)\| \leq \|a(\tau)\|$ for all $\tau \geq 0$)

$$\leq \|x(0)\|\,\|a(\tau)\|\left[\|\dot a(\tau)\| + \Big\|\frac{x(\tau)}{\|x(\tau)\|}\times\dot a(\tau)\Big\|\right], \quad 0 \leq \tau \leq t.$$

Thus, by virtue of the above inequality, from the equality (6) we obtain the following estimate for all $t \geq 0$:

$$\big|L_2(x(t),t)\big| \leq \|x(0)\| \int_0^t \|a(\tau)\|\left[\big\|\dot a(\tau)\big\| + \Big\|\frac{x(\tau)}{\|x(\tau)\|}\times\dot a(\tau)\Big\|\right] d\tau. \tag{7}$$

Suppose now that the estimate (3) is fulfilled for some $t = t_0 > 0$. Let

$$s(\tau) := \Big|\sin\angle\big\{x(\tau),\dot a(\tau)\big\}\Big|, \quad c(\tau) := \sqrt{1 - s^2(\tau)}, \quad I_0 := \int_0^{t_0} \big\|\dot a(\tau)\big\|\, d\tau.$$

Define also the set $T_0 \equiv \big\{\tau \in [0,t_0] :\ s(\tau) \leq 1/\sqrt{2}\big\}$ and its complement $CT_0 \equiv [0,t_0]\setminus T_0$ in $[0,t_0]$. Since every solution of the system (1_A) satisfies for all $t \geq 0$ the equality $\|x(\tau)\| \equiv \|x(0)\|$, without loss of generality we can assume that in the estimates (3) and (4) the equality $\|x(\tau)\| \equiv 1$, $\tau \ge 0$, is identically fulfilled.

Lemma 2 in [3] implies the estimates

$$hI_0 \le \big|L_1(x(t_0), t_0)\big| = \left|\int\limits_0^{t_0} \big(x(\tau), \dot a(\tau)\big)\, d\tau\right| \le$$

$$\le \int\limits_0^{t_0} \big|\big(x(\tau), \dot a(\tau)\big)\big|\, d\tau \le \int\limits_0^{t_0} \big\|\dot a(\tau)\big\|\, \big|\cos\angle\big(x(\tau), \dot a(\tau)\big)\big|\, d\tau \le$$

$$\le \int\limits_{T_0} \big\|\dot a(\tau)\big\|\, d\tau + \int\limits_{CT_0} \big\|\dot a(\tau)\big\|\, |c(\tau)|\, d\tau \le$$

(we now use the evident equality $|\cos\alpha| \le 1 - 2^{-1}\sin^2\alpha$)

$$\le \max_{\tau\in CT_0} \big\{1 - 2^{-1}s^2(\tau)\big\} \int\limits_{CT_0} \big\|\dot a(\tau)\big\|\, d\tau + \int\limits_{T_0} \big\|\dot a(\tau)\big\|\, d\tau.$$

Since the estimate $s(\tau) \ge 1/\sqrt{2}$ holds for all $\tau \in CT_0$, we have

$$hI_0 \le I_0 - \frac{1}{4} \int\limits_{CT_0} \big\|\dot a(\tau)\big\|\, d\tau,$$

whence $\int\limits_{CT_0} \|\dot a(\tau)\|\, d\tau \le 4(1-h)I_0$. The last estimate yields

$$\int\limits_{T_0} \big\|\dot a(\tau)\big\|\, dt \ge (4h-3)I_0. \tag{8}$$

Consider now the case $\|a(t_0)\| \ge \|a(0)\|$ (the opposite case can be treated analogously). Under this assumption, the estimate (3) implies that

$$\|a(t_0)\| = \max\big\{\|a(t_0)\|, \|a(0)\|\big\} \ge \max\Big\{\big|(x(t_0), a(t_0))\big|, \big|(x(0), a(0))\big|\Big\} \ge$$

$$\ge 2^{-1}\Big|(x(t_0), a(t_0)) - (x(0), a(0))\Big| \ge hI_0/2. \tag{9}$$

Next we define the set

$$T \equiv \left\{t \in [0, t_0] : \int\limits_t^{t_0} \big\|\dot a(\tau)\big\|\, d\tau \le 0.4 I_0\right\}$$

for which the equality $\int\limits_T \|\dot a(\tau)\|\, d\tau = 0.4I_0$ is obviously fulfilled. Using the estimate (8), we obtain the inequalities

$$I_0 \ge \int\limits_{T_0\cup T} \big\|\dot a(\tau)\big\|\, d\tau = \int\limits_{T_0} \ldots\, d\tau + \int\limits_{T} \ldots\, d\tau - \int\limits_{T_0\cap T} \ldots\, d\tau \ge$$

$$\geq (4h - 2.6)I_0 - \int\limits_{T_0 \cap T} \big\|\dot{a}(\tau)\big\|\, d\tau.$$

These inequalities result in the estimate $\int\limits_{T_0 \cap T} \|\dot{a}(\tau)\|\, d\tau \geq (4h - 3.6)I_0$.

Moreover, the inequality

$$\min_{t\in T} \|a(t)\| \geq \|a(t_0)\| - \max_{t\in T} \int\limits_{t}^{t_0} \big\|\dot{a}(\tau)\big\|\, d\tau = \|a(t_0)\| - 0.4I_0$$

implies the estimates

$$\int\limits_{T_0} \|a(\tau)\|\, \big\|\dot{a}(\tau)\big\|\, d\tau \geq$$

$$\geq \min_{\tau\in T} \|a(\tau)\| \int\limits_{T_0\cap T} \big\|\dot{a}(\tau)\big\|\, d\tau \geq (4h - 3.6)\big(\|a(t_0)\| - 0.4I_0\big)I_0.$$

Thus, by virtue of the inequality (11), the following estimates are valid:

$$J_0 \equiv \int\limits_0^{t_0} s(\tau)\|a(\tau)\|\, \big\|\dot{a}(\tau)\big\|\, d\tau \leq$$

$$\leq \int\limits_{CT_0} \|a(\tau)\|\, \big\|\dot{a}(\tau)\big\|\, d\tau + \max_{\tau\in T_0} s(\tau) \int\limits_{T_0} \|a(\tau)\|\big\|\dot{a}(\tau)\big\|\, d\tau \leq$$

$$\leq \int\limits_0^{t_0} \|a(\tau)\|\, \big\|\dot{a}(\tau)\big\|\, d\tau - \frac{\sqrt{2}-1}{\sqrt{2}} \int\limits_{T_0} \|a(\tau)\|\, \big\|\dot{a}(\tau)\big\|\, d\tau \leq$$

$$\leq \int\limits_0^{t_0} \|a(\tau)\|\, \big\|\dot{a}(\tau)\big\|\, d\tau - 2(2-\sqrt{2})(h-0.9)\big(\|a(t_0)\| - 0.4I_0\big)I_0. \quad (10)$$

Moreover, the inequalities

$$J_1 \equiv \int\limits_0^{t_0} \|a(\tau)\|\, \big\|\dot{a}(\tau)\big\|\, d\tau \leq$$

$$\leq \max_{\tau\in[0,t_0]} \|a(\tau)\| \int\limits_{t}^{t_0} \big\|\dot{a}(\tau)\big\|\, d\tau \leq \big(\|a(t_0)\| + I_0\big)I_0 \quad (11)$$

are also fulfilled. Obviously, the inequality (9) is equivalent to the estimate

$$\|a(t_0)\| - 0.4I_0 \geq b(\|a(t_0)\| + I_0\big),$$

where $b \equiv (h - 0.8)/(2 + h)$.

Using (7), (10) and (11), we get the relations

$$\big|L_2(x(t_0), t_0)\big| \leq (J_1 + J_0) \leq$$

$$\leq 2J_1 - 2(2-\sqrt{2})(h-0.9)\big(\|a(t_0)\| - 0.4I_0\big)I_0 \leq$$
$$\leq 2J_1 - 2b(2-\sqrt{2})(h-0.9)\big(\|a(t_0)\| + I_0\big)I_0 \leq$$
$$\leq 2\Big[1 - (2-\sqrt{2})\,\frac{(h-0.8)(h-0.9)}{2+h}\Big]J_1.$$

By virtue of Lemma 2 in [3], the latter inequalities imply the desired inequality (4).

Thus the theorem is proved. □

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