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NONLOCAL SINGULAR BOUNDARY VALUE PROBLEMS FOR EVEN-ORDER DIFFERENTIAL EQUATIONS

Abstract. Differential equations of the type $x^{(2n)} = f(t, x, \dots, x^{(2n-1)})$ are considered. Here a positive function f satisfies local Carathéodory conditions on a subset of $[0, T] \times \mathbb{R}^{2n}$ and f may be singular at the value 0 of all its phase variables. The paper presents conditions guaranteeing the existence of a solution of the above differential equation satisfying nonlocal boundary conditions whose special case are the $(2p, 2n-2p)$ right focal boundary conditions $x^{(j)}(0) = 0$ for $0 \le j \le 2p-1$ and $x^{(j)}(T) = 0$ for $2p \le j \le 2n-1$, where $p \in \mathbb{N}$, $1 \le p \le n-1$.

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[illegible] $x^{(2n)} = f(t, x, \dots, x^{(2n-1)})$ [illegible] f [illegible] $[0, T] \times \mathbb{R}^{2n}$-[illegible] 0 [illegible] $(2p, 2n-2p)$ [illegible] $x^{(j)}(0) = 0$, [illegible] $0 \le j \le 2p-1$ [illegible] $x^{(j)}(T) = 0$, [illegible] $2p \le j \le 2n-1$, [illegible] $p \in \mathbb{N}$, $1 \le p \le n-1$.

1. Introduction

Let T be a positive number and $\mathbb{X} = (0,\infty) \times (\mathbb{R} \setminus \{0\}) \subset \mathbb{R}^2$. Let $\mathcal{A}$ denote the set of functionals $\phi : C^0[0,T] \to \mathbb{R}$ which are

(i) continuous, $\phi(0) = 0$ and
(ii) increasing, that is, $x, y \in C^0[0,T]$, $x < y$ on $[0,T] \Rightarrow \phi(x) < \phi(y)$.

Consider the differential equation

$$x^{(2n)}(t) = f\big(t, x(t), \dots, x^{(2n-1)}(t)\big), \tag{1.1}$$

where $n > 1$, a positive function f satisfies local Carathéodory conditions on $[0,T] \times \mathbb{X}^n$ ($f \in Car([0,T] \times \mathbb{X}^n)$) and f may be singular at the value 0 of all its phase variables.

Let $p \in \mathbb{N}$, $1 \le p \le n-1$. In literature the equation (1.1) together with the boundary conditions

$$\left.\begin{aligned} x^{(i)}(0) &= 0, \quad 0 \le i \le 2p-1 \\ x^{(i)}(T) &= 0, \quad 2p \le i \le 2n-1 \end{aligned}\right\} \tag{1.2}$$

is called the $(2p, 2n-2p)$ *right focal boundary value problem.*

In the papers [2]–[5], [8], [10]–[12] and references therein the authors discussed the $(p, n-p)$ focal problem for regular differential equations ([8], [12]) or differential equations with singularities in the phase variables ([2]–[5], [10], [11]) or differential equations with singularities in the time variables ([1], [9]). The papers [3], [4] and [12] discuss the existence of one and multiple solutions.

The boundary conditions (1.2) can be written in the equivalent form

$$\left.\begin{aligned} &x^{(2i_0-1)}(0) = 0, \quad x^{(2k_0-1)}(T) = 0, \\ &\text{where } i_0 \in \{1,\dots,p\}, \; k_0 \in \{p+1,\dots,n\}, \\ &\min\Big\{\sum_{j=0}^{2p-1} |x^{(j)}(t)| : \; 0 \le t \le T\Big\} = 0, \\ &\min\Big\{\sum_{j=2p}^{2n-1} |x^{(j)}(t)| : \; 0 \le t \le T\Big\} = 0. \end{aligned}\right\}$$

Let α, $\beta \in [0,T]$. Then the boundary conditions

$$\left.\begin{aligned} x^{(i)}(\alpha) &= 0, \quad 0 \le i \le 2p-1 \\ x^{(i)}(\beta) &= 0, \quad 2p \le i \le 2n-1 \end{aligned}\right\} \tag{1.3}$$

are a natural generalization of the focal $(2p, 2n-2p)$ boundary conditions (1.2). If $\alpha = \beta$, we obtain the initial conditions. There are two main ways for determining α and β in (1.3). Namely, either α, β are given in advance or α, β depend on solutions of the considered problem and their derivatives. The second way is used in this paper. We discuss the nonlocal boundary conditions

$$\left.\begin{array}{c}\phi_1(x^{(2i_0-1)})=0,\quad \phi_2(x^{(2k_0-1)})=0\\ \text{where }\ i_0\in\{1,\dots,p\},\ \ k_0\in\{p+1,\dots,n\}\ \text{ and }\ \phi_1,\phi_2\in\mathcal{A},\end{array}\right\}\tag{1.4}$$

$$\begin{aligned}&\min\Big\{\sum_{j=0}^{2p-1}|x^{(j)}(t)|:\ 0\le t\le T\Big\}=0,\\ &\min\Big\{\sum_{j=2p}^{2n-1}|x^{(j)}(t)|:\ 0\le t\le T\Big\}=0.\end{aligned}\tag{1.5}$$

A function $x\in AC^{2n-1}[0,T]$ (the set of functions having absolutely continuous $(2n-1)$st derivatives on $[0,T]$) is said to be *a solution of the problem* (1.1), (1.4), (1.5) if x satisfies the boundary conditions (1.4), (1.5) and (1.1) holds a.e. on $[0,T]$.

The aim of this paper is to give conditions on the function f in (1.1) which guarantee the solvability of the problem (1.1), (1.4), (1.5) for each $p\in\{1,\dots,n-1\}$, $i_0\in\{1,\dots,p\}$, $k_0\in\{p+1,\dots,n\}$ and ϕ_1, $\phi_2\in\mathcal{A}$.

We note that our boundary conditions are nonlocal and that all solutions to the problem (1.1), (1.4), (1.5) and their derivatives 'pass through' the singular points of f at some inner points α, β in $(0,T)$ depending on ϕ_1, $\phi_2\in\mathcal{A}$ and i_0,k_0 (of course if α, $\beta\in(0,T)$). Our existence result for the problem (1.1), (1.4), (1.5) is obtained by combination of regularization and sequential techniques. Existence results for auxiliary regular problems are proved by *a priori* bounds for their solutions and the topological transversality principle (see [6], [7]). In limit processes, a combination of the Fatou theorem with the Lebesgue dominated convergence theorem is used.

Notice that if x is a solution of the problem (1.1), (1.4), (1.5), then (1.4) yields $x^{(2i_0-1)}(\alpha)=0$ and $x^{(2k_0-1)}(\beta)=0$ for some unique α, $\beta\in[0,T]$ (see Lemma 3.4) and (1.5) shows that x satisfies (1.3). Also from f being positive on $[0,T]\times\mathbb{X}^n$ we deduce that any solution x of the problem (1.1), (1.4), (1.5) satisfies

$$\min\big\{x^{(2j)}(t):\ 0\le t\le T\big\}=0\ \text{ for }\ 0\le j\le n-1.$$

We observe that the boundary conditions (1.2) are a special case of (1.4), (1.5) with ϕ_1, $\phi_2\in\mathcal{A}$ defined by $\phi_1(x)=x(0)$ and $\phi_2(x)=x(T)$ for $x\in C^0[0,T]$.

Throughout the paper we will use the following assumptions:

(H_1) $f\in Car([0,T]\times\mathbb{X}^n)$ and there exists a positive constant a such that

$$a\le f(t,x_0,\dots,x_{2n-1})$$

for a.e. $t\in[0,T]$ and all $(x_0,\dots,x_{2n-1})\in\mathbb{X}^n$;

(H_2) For a.e. $t\in[0,T]$ and all $(x_0,\dots,x_{2n-1})\in\mathbb{X}^n$,

$$f(t,x_0,\dots,x_{2n-1})\le\sum_{j=0}^{2n-1}h_j(|x_j|)+\omega\Big(t,\sum_{j=0}^{2n-1}|x_j|\Big),$$

where $h_j \in C^0(0,\infty)$ is positive and nonincreasing, $\omega \in Car([0,T] \times (0,\infty))$ is positive and nondecreasing in the second variable,

$$\int_0^1 h_j(s^{2n-j})\, ds < \infty \ \text{ for } \ 0 \le j \le 2n-2, \qquad \lim_{u\to\infty} h_{2n-1}(u) = c > 0 \tag{1.6}$$

and

$$\limsup_{u\to\infty} \Big(\int_0^u \frac{ds}{h_{2n-1}(s)} \Big)^{-1} \int_0^T \omega(t, Qu)\, dt < c \tag{1.7}$$

with

$$Q = \begin{cases} \dfrac{T^{2n}-1}{T-1} & \text{if } T \ne 1 \\ 2n & \text{if } T = 1 \end{cases}. \tag{1.8}$$

Remark 1.1. From the properties of the function h_{2n-1} given in (H_2) it follows that $\int_0^b \frac{1}{h_{2n-1}(s)}\, ds < \infty$ for all $b > 0$ and

$$\lim_{\to\infty} \frac{1}{u} \int_0^u \frac{ds}{h_{2n-1}(s)} = \frac{1}{c}\,.$$

Throughout the paper $\|x\| = \max\{|x(t)| : \ 0 \le t \le T\}$, $\|x\|_L = \int_0^T |x(t)|\, dt$ and $\|x\|_\infty = \operatorname{ess\,max}\{|x(t)| : \ 0 \le t \le T\}$ stand for the norm in $C^0[0,T]$, $L_1[0,T]$ and the set $L_\infty[0,T]$ of measurable and essentially bounded functions on $[0,T]$, respectively.

The paper is organized as follows. In Section 2 we introduce a family of auxiliary regular differential equations. Section 3 is devoted to the study of auxiliary regular problems. We first present results (Lemmas 3.1–3.6) which are used in the next part of this section. Then we establish *a priori* bounds for solutions of auxiliary problems (Lemma 3.7) and prove their existence (Lemma 3.8). We also show that the sequence of $(2n-1)$st derivatives of solutions to auxiliary problems is equicontinuous on $[0,T]$ (Lemma 3.9). Section 4 contains the main existence results for the problem (1.1), (1.4), (1.5) (Theorem 4.1). An example illustrates our theory (Example 4.2).

2. Auxiliary Regular Problems

Let the assumption (H_1) be satisfied. For $m \in \mathbb{N}$, define $\mathbb{R}_m$ and $f_m \in Car([0,T] \times \mathbb{R}^{2n})$ by the formulas

$$\mathbb{R}_m = \Big(-\infty, -\frac{1}{m} \Big] \cup \Big[\frac{1}{m}, \infty \Big),$$

$$f_m(t, x_0, x_1, x_2, \dots, x_{2n-1}) =$$

$$
= \begin{cases}
f(t,x_0,x_1,x_2,\dots,x_{2n-1}) \\
\quad \text{for } (x_0,x_1,x_2,\dots,x_{2n-1}) \in \Big(\Big[\dfrac{1}{m},\infty\Big) \times \mathbb{R}_m\Big)^n, \ t\in[0,T], \\
f\Big(t,\dfrac{1}{m},x_1,\dfrac{1}{m},\dots,x_{2n-1}\Big) \ \text{for } t\in[0,T], \ x_1,x_3,\dots,x_{2n-1}\in\mathbb{R}_m, \\
\quad x_0,x_2,\dots,x_{2n-2} \in \Big(-\infty,\dfrac{1}{m}\Big), \\
\dfrac{m}{2}\Big[f_m\Big(t,x_0,\dfrac{1}{m},x_2,\dots,x_{2n-1}\Big)\Big(x_1+\dfrac{1}{m}\Big) - \\
\quad -f_m\Big(t,x_0,-\dfrac{1}{m},x_2,\dots,x_{2n-1}\Big)\Big(x_1-\dfrac{1}{m}\Big)\Big] \\
\quad \text{for } (t,x_0,x_2,\dots,x_{2n-1}) \in [0,T]\times\mathbb{R}\times(\mathbb{R}\times\mathbb{R}_m)^{n-1}, \\
\quad x_1 \in \Big(-\dfrac{1}{m},\dfrac{1}{m}\Big), \\
\qquad \vdots \\
\dfrac{m}{2}\Big[f_m\Big(t,x_0,\dots,x_{2i-2},\dfrac{1}{m},x_{2i},\dots,x_{2n-1}\Big)\Big(x_{2i-1}+\dfrac{1}{m}\Big) - \\
\quad -f_m\Big(t,x_0,\dots,x_{2i-2},-\dfrac{1}{m},x_{2i},\dots,x_{2n-1}\Big)\Big(x_{2i-1}-\dfrac{1}{m}\Big)\Big] \\
\quad \text{for } (t,x_0,\dots,x_{2i-2},x_{2i},\dots,x_{2n-1}) \in [0,T]\times\mathbb{R}^{2i-1}\times(\mathbb{R}\times\mathbb{R}_m)^{n-i}, \\
\quad x_{2i-1} \in \Big(-\dfrac{1}{m},\dfrac{1}{m}\Big), \\
\qquad \vdots \\
\dfrac{m}{2}\Big[f_m\Big(t,x_0,x_1,\dots,x_{2n-2},\dfrac{1}{m}\Big)\Big(x_{2n-1}+\dfrac{1}{m}\Big) - \\
\quad -f_m\Big(t,x_0,x_1,\dots,x_{2n-2},-\dfrac{1}{m}\Big)\Big(x_{2n-1}-\dfrac{1}{m}\Big)\Big] \\
\quad \text{for } (t,x_0,x_1,\dots,x_{2n-2}) \in [0,T]\times\mathbb{R}^{2n-1}, \ x_{2n-1} \in \Big(-\dfrac{1}{m},\dfrac{1}{m}\Big).
\end{cases}
$$

Then

$$a \le f_m(t,x_0,\dots,x_{2n-1}) \tag{2.1}$$

for a.e. $t\in[0,T]$ and all $(x_0,\dots,x_{2n-1})\in\mathbb{R}^{2n}$, $m\in\mathbb{N}$.

Consider the family of the regular differential equations

$$x^{(2n)}(t) = (1-\lambda)a + \lambda f_m\big(t,x(t),\dots,x^{(2n-1)}(t)\big) \tag*{$(2.2)^\lambda_m$}$$

depending on the parameters $\lambda\in[0,1]$ and $m\in\mathbb{N}$. Then (see (2.1))

$$a \le (1-\lambda)a + \lambda f_m(t,x_0,\dots,x_{2n-1}) \tag{2.3}$$

for a.e. $t\in[0,T]$ and all $(x_0,\dots,x_{2n-1})\in\mathbb{R}^{2n}$, $\lambda\in[0,1]$, $m\in\mathbb{N}$. The assumption (H_2) implies that

$$(1-\lambda)a+\lambda f_m(t,x_0,\dots,x_{2n-1}) \le \sum_{j=0}^{2n-1} h_j(|x_j|) + \omega\Big(t, 2n+\sum_{j=0}^{2n-1}|x_j|\Big) \tag{2.4}$$

for a.e. $t\in[0,T]$ and all $(x_0,\dots,x_{2n-1})\in(\mathbb{R}\setminus\{0\})^{2n}$, $\lambda\in[0,1]$, $m\in\mathbb{N}$.

3. Auxiliary Results

Let the assumption (H_1) be satisfied. For $m \in \mathbb{N}$ and $\lambda \in [0,1]$, define the operator $\mathcal{K}_{m,\lambda} : C^{2n-1}[0,T] \to L_1[0,T]$ by the formula

$$(\mathcal{K}_{m,\lambda}x)(t) = (1-\lambda)a + \lambda f_m\big(t, x(t), \dots, x^{(2n-1)}(t)\big). \tag{3.1}$$

The following five lemmas are needed in the second part of this section.

Lemma 3.1. *Let (H_1) hold. Let $\phi_2 \in \mathcal{A}$, $m \in \mathbb{N}$ and $k \in \{p+1, \dots, n\}$. Then for each $x \in C^{2n-1}[0,T]$ and $\lambda \in [0,1]$, there exists a unique solution $\beta_0 = \beta_0(x,\lambda) \in [0,T]$ of the equation*

$$S_k(\beta; x, \lambda) = 0, \tag{3.2}$$

where

$$S_k(\beta; x, \lambda) = \phi_2\bigg(\frac{1}{(2n-2k)!}\int_\beta^t (t-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s)\,ds\bigg). \tag{3.3}$$

In addition, β_0 is a continuous function of x and λ.

Proof. Choose $x \in C^{2n-1}[0,T]$ and $\lambda \in [0,1]$. By (2.3), $(\mathcal{K}_{m,\lambda}x)(t) \ge a$ for a.e. $t \in [0,T]$ and consequently

$$\int_0^t (t-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s)\,ds \ge 0, \quad \int_T^t (t-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s)\,ds \le 0$$

for $t \in [0,T]$. Hence $S_k(0; x, \lambda) \ge 0$ and $S_k(T; x, \lambda) \le 0$ and since $S_k(\cdot\,; x, \lambda)$ is a continuous function on $[0,T]$, there exists a solution $\beta_0 \in [0,T]$ of (3.2). In order to prove the uniqueness of β_0, assume that $S_k(\beta_1; x, \lambda) = 0$ for some $\beta_1 \in [0,T]$, $\beta_1 \ne \beta_0$. If

$$\int_{\beta_1}^{t_0} (t_0-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s)\,ds = \int_{\beta_0}^{t_0} (t_0-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s)\,ds$$

for some $t_0 \in [0,T]$, then

$$\int_{\beta_1}^{\beta_0} (t_0-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s)\,ds = 0,$$

contrary to $(t_0-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s) \ge (t_0-s)^{2(n-k)}a$ for a.e. $s \in [0,T]$. Hence

$$\int_{\beta_1}^{t} (t-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s)\,ds - \int_{\beta_0}^{t} (t-s)^{2(n-k)}(\mathcal{K}_{m,\lambda}x)(s)\,ds \ne 0$$

for $t \in [0,T]$, and then $S_k(\beta_1; x, \lambda) \ne S_k(\beta_0; x, \lambda)$, contrary to our assumption $S_k(\beta_1; x, \lambda) = 0$.

Let now $\{(x_j, \lambda_j)\} \subset C^{2n-1}[0,T] \times [0,1]$ be convergent, $\lim_{j\to\infty}(x_j, \lambda_j) = (x_0, \lambda_0)$. Let $\beta_j \in [0,T]$ and $\beta_0 \in [0,T]$ be the unique solution of $S_k(\beta; x_j, \lambda_j) = 0$ and $S_k(\beta; x_0, \lambda_0) = 0$, respectively. If $\{\beta_{j_n}\}$ is a convergent subsequence of $\{\beta_j\}$, $\lim_{n\to\infty} \beta_{j_n} = \Lambda$, then from the continuity of ϕ_2, $f_m \in Car([0,T] \times \mathbb{R}^{2n})$ and the Lebesgue dominated convergence theorem we get $0 = \lim_{n\to\infty} S_k(\beta_{j_n}, x_{j_n}, \lambda_{j_n}) = S_k(\Lambda; x_0, \lambda_0)$. Consequently $\Lambda = \beta_0$. We have proved that any convergent subsequence of $\{\beta_j\}$ has the same limit β_0. Therefore $\lim_{j\to\infty} \beta_j = \beta_0$, which shows that the solution of (3.2) depends continuously on x and λ. □

Lemma 3.2. *Let (H_1) hold. Let $\phi_1 \in \mathcal{A}$, $m \in \mathbb{N}$, $i \in \{1, \dots, p\}$ and $k \in \{p+1, \dots, n\}$. Then for each $x \in C^{2n-1}[0,T]$ and $\lambda \in [0,1]$, there exists a unique solution $\alpha_0 = \alpha_0(x, \lambda) \in [0,T]$ of the equation*

$$V_i(\alpha; x, \lambda) = 0, \tag{3.4}$$

where

$$V_i(\alpha; x, \lambda) = \phi_1(\mathcal{L}(\alpha; x, \lambda)), \tag{3.5}$$

$$\mathcal{L}(\alpha; x, \lambda)(t) = \frac{1}{(2(n-p)-1)!(2p-2i)!} \times$$

$$\times \int_\alpha^t (t-s)^{2(p-i)} \int_{\beta_0}^s (s-v)^{2(n-p)-1} (\mathcal{K}_{m,\lambda} x)(v)\, dv\, ds,$$

and $\beta_0 = \beta_0(x, \lambda) \in [0,T]$ is the unique solution of (3.2)*. In addition, α_0 is a continuous function of x and λ.*

Proof. Choose $x \in C^{2n-1}[0,T]$ and $\lambda \in [0,1]$. (H_1) and (2.1) show that $V_i(\cdot\,; x, \lambda)$ is continuous on $[0,T]$ and $\mathcal{L}(0; x, \lambda)(t) \ge 0$, $\mathcal{L}(T; x, \lambda)(t) \le 0$ for $t \in [0,T]$. Hence $V_i(0; x, \lambda) \ge 0$, $V_i(T; x, \lambda) \le 0$, and therefore $V_i(\alpha_0; x, \lambda) = 0$ for an $\alpha_0 \in [0,T]$. Essentially the same reasoning as in the proof of Lemma 3.1 implies that $V_i(\cdot\,; x, \lambda)$ is injective on $[0,T]$, and consequently α_0 is the unique solution of (3.4).

It remains to show that $\alpha_0 = \alpha_0(x, \lambda)$ depends continuously on x and λ. Let $\{(x_j, \lambda_j)\} \subset C^{2n-1}[0,T] \times [0,1]$ be convergent, $\lim_{j\to\infty}(x_j, \lambda_j) = (x_0, \lambda_0)$. Let α_j be the (unique) solution of $V_i(\alpha; x_j, \lambda_j) = 0$. By Lemma 3.1,

$$\lim_{j\to\infty} \beta_0(x_j, \lambda_j) = \beta_0(x_0, \lambda_0).$$

Using the Lebesgue dominated convergence theorem, we see that for any convergent subsequence $\{\alpha_{j_n}\}$ of $\{\alpha_j\}$, $\lim_{n\to\infty} \alpha_{j_n} = \Lambda$, we have

$$0 = \lim_{n\to\infty} V_i(\alpha_{j_n}, x_{j_n}, \lambda_{j_n}) = V_i(\Lambda; x_0, \lambda_0).$$

Hence $\Lambda = \alpha_0(x_0, \lambda_0)$ which shows that any convergent subsequence of $\{\alpha_j\}$ has the same limit equal to $\alpha_0(x_0, \lambda_0)$. Therefore $\{\alpha_0(x_j, \lambda_j)\}$ is convergent and $\lim\limits_{j\to\infty} \alpha_0(x_j,\lambda_j) = \alpha_0(x_0,\lambda_0)$. We have proved that α_0 is a continuous function of x and λ. □

Lemma 3.3. *Let $\phi \in \mathcal{A}$ and $\phi(x) = 0$ for some $x \in C^0[0,T]$. Then there exists $\xi \in [0,T]$ such that $x(\xi) = 0$.*

Proof. If not, $x > 0$ or $x < 0$ on $[0,T]$. Then $\phi(x) > \phi(0) = 0$ or $\phi(x) < \phi(0) = 0$, contrary to $\phi(x) = 0$. □

Lemma 3.4. *Let (H_1) hold. Let x be a solution of the problem $(2.2)^\lambda_m$, (1.4), (1.5). Then $x^{(2j-1)}$ is increasing on $[0,T]$ for $1 \le j \le n$ and (1.3) is true, where α is the unique zero of $x^{(2i_0-1)}$ and β is the unique zero of $x^{(2k_0-1)}$. In addition, $x^{(2n-2j)} > 0$ on $[0,T] \setminus \{\beta\}$ for $1 \le j \le n-p$ and $x^{(2n-2j)} > 0$ on $[0,T] \setminus \{\alpha\}$ for $n-p+1 \le j \le n$.*

Proof. Let x be a solution of the problem $(2.2)^\lambda_m$, (1.4), (1.5). Lemma 3.3 and (1.4) show that $x^{(2i_0-1)}(\alpha) = 0$ and $x^{(2k_0-1)}(\beta) = 0$ for some $\alpha, \beta \in [0,T]$ and then from (1.5) we see that (1.3) is true. Since $x^{(2n)}(t) \ge a$ for a.e. $t \in [0,T]$ due to (2.3), $x^{(2n-1)}$ is increasing on $[0,T]$ and consequently $x^{(2n-1)} < 0$ on $[0,\beta)$ (if $\beta > 0$) and $x^{(2n-1)} > 0$ on $(\beta, T]$ (if $\beta < T$). Hence β is determined uniquely and $x^{(2n-2)}(\beta) = 0$ implies $x^{(2n-2)} > 0$ on $[0,T] \setminus \{\beta\}$. By this procedure we can verify that $x^{(2j-1)}$ is increasing on $[0,T]$ for $1 \le j \le n$. Consequently, α is the unique zero of $x^{(2i_0-1)}$. Further, $x^{(2n-2j)} > 0$ on $[0,T] \setminus \{\beta\}$ for $1 \le j \le n-p$ and $x^{(2n-2j)} > 0$ on $[0,T] \setminus \{\alpha\}$ for $n-p+1 \le j \le n$. □

Lemma 3.5. *Let (H_1) hold. Then x is a solution of the problem $(2.2)^\lambda_m$, (1.4), (1.5) if and only if x is a fixed point of the operator $\mathcal{S} : C^{2n-1}[0,T] \to C^{2n-1}[0,T]$ defined by the formula*

$$(\mathcal{S}x)(t) = \frac{1}{(2(n-p)-1)!(2p-1)!} \times \\ \times \int_{\alpha_0}^{t} (t-s)^{2p-1} \int_{\beta_0}^{s} (s-v)^{2(n-p)-1} (\mathcal{K}_{m,\lambda}x)(v)\, dv\, ds, \tag{3.6}$$

where $\beta_0 \in [0,T]$ is the unique solution of $S_{k_0}(\beta; x, \lambda) = 0$ with S_{k_0} given in (3.3), and $\alpha_0 \in [0,T]$ is the unique solution of $V_{i_0}(\alpha; x, \lambda) = 0$ with V_{i_0} given in (3.5).

Proof. Let x be a fixed point of the operator $\mathcal{S}$. By direct calculations we can verify that x is a solution of $(2.2)^\lambda_m$, $x^{(j)}(\alpha_0) = 0$ for $0 \le j \le 2p-1$ and $x^{(j)}(\beta_0) = 0$ for $2p \le j \le 2n-1$. From the definition of β_0 and α_0 it follows that $\phi_1(x^{(2i_0-1)}) = 0$ and $\phi_2(x^{(2k_0-1)}) = 0$. Hence x is a solution of the problem $(2.2)^\lambda_m$, (1.4), (1.5).

Let x be a solution of the problem $(2.2)^\lambda_m$, (1.4), (1.5). Then Lemma 3.4 shows that x satisfies (1.3) with α_* and β_* instead of α and β, where α_* and β_* are the unique zeros of $x^{(2i_0-1)}$ and $x^{(2k_0-1)}$, respectively. Hence x is a solution of the problem $(2.2)_m^\lambda$, (1.3). Integrating the equality $x^{(2n)}(t) = (\mathcal{K}_{m,\lambda}x)(t)$ for a.e. $t \in [0,T]$ and using (1.3), we obtain

$$x(t) = \frac{1}{(2(n-p)-1)!(2p-1)!} \times$$
$$\times \int\limits_{\alpha_*}^{t} (t-s)^{2p-1} \int\limits_{\beta_*}^{s} (s-v)^{2(n-p)-1} (\mathcal{K}_{m,\lambda}x)(v)\, dv\, ds$$

for $t \in [0,T]$. Now from (1.4) and Lemmas 3.1 and 3.2 we deduce that α_* and β_* are the unique solutions of the equation $V_{i_0}(\alpha; x, \lambda) = 0$ and $S_{k_0}(\beta; x, \lambda) = 0$, respectively. Hence $\alpha_* = \alpha_0$ and $\beta_* = \beta_0$, and consequently x is a fixed point of the operator $\mathcal{S}$. □

The following result is used in the proofs of Lemmas 3.7 and 3.9 and Theorem 4.1.

Lemma 3.6. *Let* (H_1) *hold. Let* x *be a solution of the problem* $(2.2)_m^\lambda$, (1.4), (1.5). *Then*

$$|x^{(j)}(t)| \ge \frac{a}{(2n-j)!}\, |t-\beta_0|^{2n-j}, \quad t \in [0,T], \quad 2p \le j \le 2n-1, \tag{3.7}$$

and

$$|x^{(j)}(t)| \ge \begin{cases} \dfrac{a}{(2n-j)!}\, |t-\widetilde{\alpha}_0|^{2n-j} & \text{for } t \in \Big[0, \dfrac{\widetilde{\alpha}_0+\widetilde{\beta}_0}{2}\Big] \\ \dfrac{a}{(2n-j)!}\, |t-\widetilde{\beta}_0|^{2n-j} & \text{for } t \in \Big[\dfrac{\widetilde{\alpha}_0+\widetilde{\beta}_0}{2}, T\Big] \end{cases} \tag{3.8}$$

for $0 \le j \le 2p-1$, *where* α_0 *and* β_0 *are the unique zeros of* $x^{(2i_0-1)}$ *and* $x^{(2k_0-1)}$, *respectively, and* $\widetilde{\alpha}_0 = \min\{\alpha_0, \beta_0\}$, $\widetilde{\beta}_0 = \max\{\alpha_0, \beta_0\}$.

Proof. By Lemma 3.5, x is a fixed point of the operator $\mathcal{S}$ defined in (3.6), and therefore

$$x(t) = \frac{1}{(2(n-p)-1)!(2p-1)!} \times$$
$$\times \int\limits_{\alpha_0}^{t} (t-s)^{2p-1} \int\limits_{\beta_0}^{s} (s-v)^{2(n-p)-1} (\mathcal{K}_{m,\lambda}x)(v)\, dv\, ds$$

for $t \in [0,T]$. Since (see (2.3)) $(\mathcal{K}_{m,\lambda}x)(t) \ge a$ for a.e. $t \in [0,T]$, we have

$$|x^{(j)}(t)| = \Bigg| \int\limits_{\beta_0}^{t} \frac{(t-s)^{2n-j-1}}{(2n-j-1)!} (\mathcal{K}_{m,\lambda}x)(s)\, ds \Bigg| \ge$$
$$\ge \frac{a}{(2n-j-1)!} \Bigg| \int\limits_{\beta_0}^{t} (t-s)^{2n-j-1}\, ds \Bigg| = \frac{a}{(2n-j)!}\, |t-\beta_0|^{2n-j}$$

for $t \in [0, T]$ and $2p \le j \le 2n-1$, which proves (3.7).

It remains to verify (3.8). Assume for example that $\alpha_0 \le \beta_0$ (the case where $\alpha_0 > \beta_0$ is treated similarly). Since (see (3.7) and Lemma 3.4)

$$x^{(2p)}(t) \ge \frac{a}{(2n-2p)!}\,(t-\beta_0)^{2(n-p)}\,, \quad t \in [0, T],$$

and $x^{(j)}(\alpha_0) = 0$ for $0 \le j \le 2p-1$, we have

$$|x^{(2p-1)}(t)| = \left|\int_{\alpha_0}^{t} x^{(2p)}(s)\,ds\right| \ge \frac{a}{(2n-2p)!}\left|\int_{\alpha_0}^{t}(s-\beta_0)^{2(n-p)}\,ds\right| \ge$$

$$\ge \begin{cases} \dfrac{a}{(2n-2p+1)!}\,|t-\alpha_0|^{2(n-p)+1} & \text{for } t \in \left[0, \dfrac{\alpha_0+\beta_0}{2}\right] \\ \dfrac{a}{(2n-2p+1)!}\,|t-\beta_0|^{2(n-p)+1} & \text{for } t \in \left[\dfrac{\alpha_0+\beta_0}{2}, T\right] \end{cases}.$$

Then

$$|x^{(2p-2)}(t)| = \left|\int_{\alpha_0}^{t} x^{(2p-1)}(s)\,ds\right| \ge$$

$$\ge \begin{cases} \dfrac{a}{(2n-2p+2)!}\,|t-\alpha_0|^{2(n-p+1)} & \text{for } t \in \left[0, \dfrac{\alpha_0+\beta_0}{2}\right] \\ \dfrac{a}{(2n-2p+2)!}\,|t-\beta_0|^{2(n-p+1)} & \text{for } t \in \left[\dfrac{\alpha_0+\beta_0}{2}, T\right] \end{cases}.$$

Applying the above procedure repeatedly, we can verify the validity of (3.8) for all $0 \le j \le 2p-1$. □

We are now in a position to give *a priori* bounds for solutions of the problem $(2.2)^{\lambda}_{m}$, (1.4), (1.5).

Lemma 3.7. *Let the assumptions* (H_1) *and* (H_2) *be satisfied. Let* x *be a solution of the problem* $(2.2)^{\lambda}_{m}$, (1.4), (1.5). *Then there exists a positive constant* K *independent of* m, λ, p, i_0, k_0, ϕ_1 *and* ϕ_2 *such that*

$$\|x^{(j)}\| < K \ \textit{ for } \ 0 \le j \le 2n-1. \tag{3.9}$$

Proof. By Lemma 3.4, there exist a unique zero α of $x^{(2i_0-1)}$ and a unique zero β of $x^{(2k_0-1)}$, and x satisfies (1.3). Hence

$$\|x^{(j)}\| \le T^{2n-j-1}\|x^{(2n-1)}\|, \quad 0 \le j \le 2n-1, \tag{3.10}$$

and therefore

$$\sum_{j=0}^{2n-1}\|x^{(j)}\| \le Q\|x^{(2n-1)}\|, \tag{3.11}$$

where Q is given in (1.8). From Lemma 3.6 it follows that

$$|x^{(j)}(t)| \ge \frac{a}{(2n-j)!}\,|t-\beta|^{2n-j}, \quad t \in [0, T], \quad 2p \le j \le 2n-1,$$

and

$$|x^{(j)}(t)| \geq \begin{cases} \dfrac{a}{(2n-j)!}\,|t-\widetilde{\alpha}|^{2n-j} & \text{for } t \in \Big[0, \dfrac{\widetilde{\alpha}+\widetilde{\beta}}{2}\Big] \\ \dfrac{a}{(2n-j)!}\,|t-\widetilde{\beta}|^{2n-j} & \text{for } t \in \Big[\dfrac{\widetilde{\alpha}+\widetilde{\beta}}{2}, T\Big] \end{cases}$$

for $0 \leq j \leq 2p-1$, where $\widetilde{\alpha} = \min\{\alpha, \beta\}$ and $\widetilde{\beta} = \max\{\alpha, \beta\}$. Set

$$I_j = \sqrt[2n-j]{\frac{a}{(2n-j)!}} \quad \text{for } 0 \leq j \leq 2n-2. \tag{3.12}$$

Since the function h_j is positive and nonincreasing on $(0, \infty)$ by (H_2), we have

$$\begin{aligned} \int_0^T h_j(|x^{(j)}(t)|)\,dt &\leq \int_0^T h_j\Big(\frac{a}{(2n-j)!}\,|t-\beta|^{2n-j}\Big)\,dt \leq \\ &\leq \frac{1}{I_j}\Big(\int_0^{I_j\beta} h_j(s^{2n-j})\,ds + \int_0^{I_j(T-\beta)} h_j(s^{2n-j})\,ds\Big) < \\ &< \frac{2}{I_j}\int_0^{I_jT} h_j(s^{2n-j})\,ds \end{aligned} \tag{3.13}$$

for $2p \leq j \leq 2n-2$ and

$$\begin{aligned} &\int_0^T h_j(|x^{(j)}(t)|)\,dt \leq \\ &\leq \int_0^T h_j\Big(\frac{a}{(2n-j)!}\,|t-\alpha|^{2n-j}\Big)\,dt + \int_0^T h_j\Big(\frac{a}{(2n-j)!}\,|t-\beta|^{2n-j}\Big)\,dt < \\ &< \frac{4}{I_j}\int_0^{I_jT} h_j(s^{2n-j})\,ds \end{aligned} \tag{3.14}$$

for $0 \leq j \leq 2p-1$. Next, by (1.6) and (2.4) we get

$$\begin{aligned} &(0 <)\ \frac{x^{(2n)}(t)}{h_{2n-1}(|x^{(2n-1)}(t)|)} \leq \\ &\leq 1 + \frac{1}{c}\Big(\sum_{j=0}^{2n-2} h_j(|x^{(j)}(t)|) + \omega\Big(t, 2n + \sum_{j=0}^{2n-1} |x^{(j)}(t)|\Big) \end{aligned} \tag{3.15}$$

for a.e. $t \in [0, T]$. Besides, $x^{(2n)} \geq a$ a.e. on $[0, T]$ and $x^{(2n-1)}(\beta) = 0$ imply

$$\|x^{(2n-1)}\| = \max\big\{|x^{(2n-1}(0)|, x^{(2n-1)}(T)\big\}. \tag{3.16}$$

Since

$$\int_0^\beta \frac{x^{(2n)}(t)}{h_{2n-1}(|x^{(2n-1)}(t)|)}\,dt = \int_0^\beta \frac{x^{(2n)}(t)}{h_{2n-1}(-x^{(2n-1)}(t))}\,dt = \int_0^{-x^{(2n-1)}(0)} \frac{ds}{h_{2n-1}(s)},$$

$$\int_\beta^T \frac{x^{(2n)}(t)}{h_{2n-1}(|x^{(2n-1)}(t)|)}\,dt = \int_\beta^T \frac{x^{(2n)}(t)}{h_{2n-1}(x^{(2n-1)}(t))}\,dt = \int_0^{x^{(2n-1)}(T)} \frac{ds}{h_{2n-1}(s)},$$

we have (see (3.16))

$$\int_0^{\|x^{(2n-1)}\|} \frac{ds}{h_{2n-1}(s)} \le \int_0^T \frac{x^{(2n)}(t)}{h_{2n-1}(|x^{(2n-1)}(t)|)}\,dt. \tag{3.17}$$

Integrating (3.15) over $[0,T]$ and combining (3.11), (3.13), (3.14) and the fact that ω is nondecreasing in the second variable, we get

$$\int_0^T \frac{x^{(2n)}(t)}{h_{2n-1}(|x^{(2n-1)}(t)|)}\,dt < T + \frac{1}{c}\left(A + \int_0^T \omega\big(t, 2n + Q\|x^{(2n-1)}\|\big)\,dt\right), \tag{3.18}$$

where

$$A = 2\sum_{j=2p}^{2n-2} \frac{1}{I_j}\int_0^{I_jT} h_j(s^{2n-j})\,ds + 4\sum_{j=0}^{2p-1} \frac{1}{I_j}\int_0^{I_jT} h_j(s^{2n-j})\,ds.$$

Hence (see (3.17) and (3.18))

$$\int_0^{\|x^{(2n-1)}\|} \frac{ds}{h_{2n-1}(s)} < T + \frac{1}{c}\left(A + \int_0^T \omega\big(t, 2n + Q\|x^{(2n-1)}\|\big)\,dt\right). \tag{3.19}$$

From (1.7) and Remark 1.1 it follows that there exists a positive constant S such that

$$\int_0^u \frac{ds}{h_{2n-1}(s)} > T + \frac{1}{c}\left(A + \int_0^T \omega(t, 2n + Qu)\,dt\right)$$

for all $u \ge S$. Therefore (3.19) shows that $\|x^{(2n-1)}\| < S$ and, by (3.10), we see that (3.9) is true with $K = S\max\{1, T^{2n-1}\}$. □

We now present an existence result for the problem $(2.2)^1_m$, (1.4), (1.5).

Lemma 3.8. *Let* (H_1) *and* (H_2) *hold. Then for each* $m \in \mathbb{N}$, $p \in \{1,\dots,n-1\}$, $i_0 \in \{1,\dots,p\}$, $k_0 \in \{p+1,\dots,n\}$ *and* $\phi_1, \phi_2 \in \mathcal{A}$, *the problem* $(2.2)^1_m$, (1.4), (1.5) *has a solution* x *satisfying* (3.9), *where* K *is the positive constant in Lemma* 3.7.

Proof. Let K be the positive constant in Lemma 3.7 and put

$$\Omega = \big\{x \in C^{2n-1}[0,T] : \ \|x^{(j)}\| < K \text{ for } 0 \le j \le 2n-1\big\}.$$

Choose $m \in \mathbb{N}$, $p \in \{1,\dots,n-1\}$, $i_0 \in \{1,\dots,p\}$, $k_0 \in \{p+1,\dots,n\}$ and $\phi_1, \phi_2 \in \mathcal{A}$. Define the operator $\mathcal{F} : C^{2n-1}[0,T] \times [0,1] \to C^{2n-1}[0,T]$ by the formula

$$\mathcal{F}(x,\lambda)(t) = \frac{1}{(2(n-p)-1)!(2p-1)!} \times$$
$$\times \int_{\alpha_0(x,\lambda)}^{t} (t-s)^{2p-1} \int_{\beta_0(x,\lambda)}^{s} (s-v)^{2(n-p)-1} (\mathcal{K}_{m,\lambda}x)(v)\, dv\, ds,$$

where $\alpha_0 = \alpha_0(x,\lambda)$ and $\beta_0 = \beta_0(x,\lambda)$ are the unique solutions of the equation $V_{i_0}(\alpha; x,\lambda) = 0$ with V_{i_0} given in (3.5) (see Lemma 3.2) and the equation $S_{k_0}(\beta; x,\lambda) = 0$ with S_{k_0} given in (3.3) (see Lemma 3.1), respectively, and $\mathcal{K}_{m,\lambda}$ is given in (3.1). Lemma 3.5 shows that x is a solution of the problem $(2.2)_m^\lambda$, (1.4), (1.5) if and only if x is a fixed point of the operator $\mathcal{F}(\cdot\,,\lambda)$. Hence our lemma will be proved if the operator $\mathcal{F}(\cdot\,,1)$ has a fixed point in Ω. In order to prove the existence of a fixed point of $\mathcal{F}(\cdot\,,1)$, we use the topological transversality principle. Let $\mathcal{F}_* = \mathcal{F}|_{\overline{\Omega}\times[0,1]}$ denote the restriction of $\mathcal{F}$ on the set $\overline{\Omega} \times [0,1]$. It suffices to verify that

(i) $\mathcal{F}_*(\cdot\,,0)$ is a constant operator on $\overline{\Omega}$ and $\mathcal{F}_*(x,0) \in \Omega$ for $x \in \overline{\Omega}$,
(ii) $\mathcal{F}_*$ is a compact operator and
(iii) $\mathcal{F}_*(x,\lambda) \ne x$ for all $(x,\lambda) \in \partial\Omega \times [0,1]$.

Since $(\mathcal{K}_{m,0}x)(t) = a$ for $t \in [0,T]$, we have

$$\mathcal{F}_*(x,0)(t) = \frac{a}{(2(n-p)-1)!(2p-1)!} \times$$
$$\times \int_{\alpha_0(x,0)}^{t} (t-s)^{2p-1} \int_{\beta_0(x,0)}^{s} (s-v)^{2(n-p)-1}\, dv\, ds =$$
$$= \frac{a}{(2n-2p)!(2p-1)!} \int_{\alpha_0(x,0)}^{t} (t-s)^{2p-1}(s-\beta_0(x,0))^{2(n-p)}\, ds,$$

where $\beta_0 = \beta_0(x,0)$ is the unique solution of the equation

$$\phi_2\Big(\frac{a}{(2(n-k_0)+1)!}\,(\beta-t)^{2(n-k_0)+1}\Big) = 0$$

and $\alpha_0 = \alpha_0(x,0)$ is the unique solution of the equation

$$\phi_1\bigg(\frac{a}{(2n-2p)!(2p-2i_0)!} \int_{\alpha}^{t} (t-s)^{2(p-i_0)}(s-\beta_0)^{2(n-p)}\, ds\bigg) = 0.$$

From the above two equation we see that β_0 and α_0 are independent of x and therefore $\mathcal{F}_*(\cdot\,,0)$ is a constant operator. In addition, $(\mathcal{F}_*(x,0))^{(j)}(\alpha_0) =$

0 for $0 \le j \le 2p-1$, $(\mathcal{F}_*(x,0))^{(j)}(\beta_0) = 0$ for $2p \le j \le 2n-1$ and $(\mathcal{F}_*(x,0))^{(2n)}(t) = a$ for $t \in [0,T]$. Hence $\mathcal{F}_*(x,0)(t)$ is a solution of the problem $(2.2)^0_m$, (1.4), (1.5) and consequently $\mathcal{F}_*(x,0) \in \Omega$ for $x \in \overline{\Omega}$ due to Lemma 3.7, which proves (i).

For (ii), we first note that $f_m \in Car([0,T] \times \mathbb{R}^{2n})$, and therefore there exists $\gamma \in L_1[0,T]$ such that

$$a \le (1-\lambda)a + \lambda f_m(t, x_0, \dots, x_{2n-1}) \le \gamma(t) \tag{3.20}$$

for a.e. $t \in [0,T]$ and all $\lambda \in [0,1]$, $|x_j| \le K$ $(0 \le j \le 2n-1)$. Let $\{(x_k, \lambda_k)\} \subset \overline{\Omega} \times [0,1]$ be a convergent sequence, $\lim\limits_{k\to\infty}(x_k,\lambda_k) = (x_0,\lambda_0)$. Then

$$\lim_{m\to\infty}(\mathcal{K}_{m,\lambda_k} x_k)(t) = (\mathcal{K}_{m,\lambda_0} x_0)(t)$$

for a.e. $t \in [0,T]$, $a \le (\mathcal{K}_{m,\lambda_k} x_k)(t) \le \gamma(t)$ for a.e. $t \in [0,T]$ and all $k \in \mathbb{N}$, and (see Lemmas 3.1 and 3.2) $\lim\limits_{k\to\infty} \beta_0(x_k,\lambda_k) = \beta_0(x_0,\lambda_0)$ and $\lim\limits_{k\to\infty} \alpha_0(x_k,\lambda_k) = \alpha_0(x_0,\lambda_0)$. Hence $\mathcal{F}_*$ is a continuous operator by the Lebesgue dominated convergence theorem. Let $\{(x_i,\lambda_i)\} \subset \overline{\Omega} \times [0,1]$. Then (see (3.20))

$$\left|(\mathcal{F}_*(x_i,\lambda_i))^{(2n)}(t)\right| \le \gamma(t)$$

for a.e. $t \in [0,T]$ and all $i \in \mathbb{N}$, and since

$$(\mathcal{F}_*(x_i,\lambda_i))^{(j)}(\alpha_0(x_i,\lambda_i)) = 0 \;\text{ for }\; 0 \le j \le 2p-1$$

and

$$(\mathcal{F}_*(x_i,\lambda_i))^{(j)}(\beta_0(x_i,\lambda_i)) = 0 \;\text{ for }\; 2p \le j \le 2n-1,$$

we see that $\{\mathcal{F}_*(x_i,\lambda_i)\}$ is bounded in $C^{2n-1}[0,T]$ and also that $\{(\mathcal{F}_*(x_i,\lambda_i))^{(2n-1)}\}$ is equicontinuous on $[0,T]$. Hence by the Arzelà–Ascoli theorem there exists a convergent subsequence of $\{\mathcal{F}_*(x_i,\lambda_i)\}$ in $C^{2n-1}[0,T]$. We have proved that $\mathcal{F}_*$ is a compact operator.

Finally, assume that $\mathcal{F}_*(x_*,\lambda_*) = x_*$ for some $(x_*,\lambda_*) \in \overline{\Omega} \times [0,1]$. Then x_* is a solution of the problem $(2.2)^{\lambda_*}_m$, (1.4), (1.5) and so $x_* \in \Omega$ by Lemma 3.7. Hence $\mathcal{F}_*(x,\lambda) \ne x$ for each $(x,\lambda) \in \partial\Omega \times [0,1]$, which proves the property (iii). □

The next result is needed in the proof of Theorem 4.1.

Lemma 3.9. *Let the assumptions* (H_1) *and* (H_2) *be satisfied. Let* x_m *be a solution of the problem* $(2.2)^1_m$, (1.4), (1.5) *for* $m \in \mathbb{N}$. *Then* $\{x_m^{(2n-1)}\}$ *is equicontinuous on* $[0,T]$.

Proof. By Lemma 3.8 we have

$$\|x_m^{(j)}\| < K \;\text{ for }\; m \in \mathbb{N}, \;\; 0 \le j \le 2n-1, \tag{3.21}$$

where K is a positive constant. Hence (see (3.15))

$$(0 <)\; \frac{x_m^{(2n)}(t)}{h_{2n-1}(|x_m^{(2n-1)}(t)|)} \le$$

$$\leq 1 + \frac{1}{c}\Big(\sum_{j=0}^{2n-2} h_j(|x_m^{(j)}(t)|) + \omega(t, 2n(K+1))\Big) \tag{3.22}$$

for a.e. $t \in [0,T]$ and all $m \in \mathbb{N}$. Let α_m and β_m be the unique zeros of $x_m^{(2i_0-1)}$ and $x_m^{(2k_0-1)}$, respectively. Then Lemma 3.6 shows that

$$|x_m^{(j)}(t)| \geq$$
$$\geq \frac{a}{(2n-j)!}\,|t-\beta_m|^{2n-j}, \quad t\in[0,T], \quad 2p\leq j\leq 2n-1, \quad m\in\mathbb{N}, \tag{3.23}$$

and

$$|x_m^{(j)}(t)| \geq \begin{cases} \dfrac{a}{(2n-j)!}\,|t-\widetilde{\alpha}_m|^{2n-j} & \text{for } t \in \Big[0, \dfrac{\widetilde{\alpha}_m + \widetilde{\beta}_m}{2}\Big] \\ \dfrac{a}{(2n-j)!}\,|t-\widetilde{\beta}_m|^{2n-j} & \text{for } t \in \Big[\dfrac{\widetilde{\alpha}_m + \widetilde{\beta}_m}{2}, T\Big] \end{cases} \tag{3.24}$$

for $0 \leq j \leq 2p-1$, where $\widetilde{\alpha}_m = \min\{\alpha_m, \beta_m\}$ and $\widetilde{\beta}_m = \max\{\alpha_m, \beta_m\}$. Set

$$H(u) = \begin{cases} \displaystyle\int_0^u \frac{ds}{h_{2n-1}(s)} & \text{for } u \in [0,\infty) \\ \displaystyle -\int_0^{-u} \frac{ds}{h_{2n-1}(s)} & \text{for } u \in (-\infty, 0) \end{cases}.$$

Then $H \in C^0[0,T]$ is an increasing and odd function. Since $x_m^{(2n-1)} < 0$ on $[0,\beta_m)$ (if $\beta_m \in (0,T]$) and $x_m^{(2n-1)} > 0$ on $(\beta_m, T]$ (if $\beta_m \in [0,T)$), we have

$$\int_{t_1}^{t_2} \frac{x_m^{(2n)}(t)}{h_{2n-1}(|x_m^{(2n-1)}(t)|)}\,dt =$$

$$= \begin{cases} \displaystyle\int_{-x_m^{(2n-1)}(t_2)}^{-x_m^{(2n-1)}(t_1)} \frac{ds}{h_{2n-1}(s)} & \text{if } 0 \leq t_1 < t_2 \leq \beta_m \\ \displaystyle\int_0^{-x_m^{(2n-1)}(t_1)} \frac{ds}{h_{2n-1}(s)} + \int_0^{x_m^{(2n-1)}(t_2)} \frac{ds}{h_{2n-1}(s)} & \text{if } 0 \leq t_1 < \beta_m < t_2 \leq T \\ \displaystyle\int_{x_m^{(2n-1)}(t_1)}^{x_m^{(2n-1)}(t_2)} \frac{ds}{h_{2n-1}(s)} & \text{if } \beta_m \leq t_1 < t_2 \leq T \end{cases} \cdot$$

Consequently,

$$\int_{t_1}^{t_2} \frac{x_m^{(2n)}(t)}{h_{2n-1}(|x_m^{(2n-1)}(t)|)}\,dt = H\big(x_m^{(2n-1)}(t_2)\big) - H\big(x_m^{(2n-1)}(t_1)\big)$$

for $0 \le t_1 < t_2 \le T$ and $m \in \mathbb{N}$. Integrating (3.22) over $[t_1, t_2] \subset [0, T]$ yields

$$H\big(x_m^{(2n-1)}(t_2)\big) - H\big(x_m^{(2n-1)}(t_1)\big) \le$$
$$\le t_2 - t_1 + \frac{1}{c}\left(\sum_{j=0}^{2n-2}\int_{t_1}^{t_2} h_j(|x_m^{(j)}(t)|)\,dt + \int_{t_1}^{t_2}\omega(t, 2n(K+1))\,dt\right). \quad (3.25)$$

Since $\omega(\cdot\,, 2n(K+1)) \in L_1[0,T]$, (3.25) shows that $\{H(x_m^{(2n-1)})\}$ is equicontinuous on $[0,T]$ if

$$\left\{\int_0^t h_j(|x_m^{(j)}(s)|)\,ds\right\}$$

is equicontinuous on $[0,T]$ for $j = 0, 1, \dots, 2n-2$. To prove this property of

$$\left\{\int_0^t h_j(|x_m^{(j)}(s)|)\,ds\right\},$$

let $0 \le t_1 < t_2 \le T$ and let the constant I_j be given in (3.12). If $2p \le j \le 2n-2$, then (see (3.23))

$$\int_{t_1}^{t_2} h_j(|x_m^{(j)}(t)|)\,dt \le \int_{t_1}^{t_2} h_j\Big(\frac{a}{(2n-j)!}\,|t-\beta_m|^{2n-j}\Big)\,dt =$$
$$= \begin{cases} \dfrac{1}{I_j}\displaystyle\int_{I_j(\beta_m - t_2)}^{I_j(\beta_m - t_1)} h_j(s^{2n-j})\,ds \ \text{ if } \ 0 \le t_1 < t_2 \le \beta_m \\ \dfrac{1}{I_j}\Bigg(\displaystyle\int_0^{I_j(\beta_m - t_1)} h_j(s^{2n-j})\,ds + \int_0^{I_j(t_2 - \beta_m)} h_j(s^{2n-j})\,ds\Bigg) \\ \qquad \text{if } \ 0 \le t_1 < \beta_m < t_2 \le T \\ \dfrac{1}{I_j}\displaystyle\int_{I_j(t_1 - \beta_m)}^{I_j(t_2 - \beta_m)} h_j(s^{2n-j})\,ds \ \text{ if } \ \beta_m \le t_1 < t_2 \le T \end{cases}.$$

If $0 \le j \le 2p-1$, then (see (3.24))

$$\int_{t_1}^{t_2} h_j\big(|x_m^{(j)}(t)|\big)\,dt =$$

$$
= \begin{cases}
\dfrac{1}{I_j} \displaystyle\int_{I_j(\widetilde{\alpha}_m - t_2)}^{I_j(\widetilde{\alpha}_m - t_1)} h_j(s^{2n-j})\, ds \ \text{ if } \ 0 \le t_1 < t_2 \le \widetilde{\alpha}_m \\
\dfrac{1}{I_j} \Bigg(\displaystyle\int_0^{I_j(\widetilde{\alpha}_m - t_1)} h_j(s^{2n-j})\, ds + \int_0^{I_j(t_2 - \widetilde{\alpha}_m)} h_j(s^{2n-j})\, ds \Bigg) \\
\qquad \text{if } \ 0 \le t_1 < \widetilde{\alpha}_m < t_2 \le \dfrac{\widetilde{\alpha}_m + \widetilde{\beta}_m}{2} \\
\dfrac{1}{I_j} \displaystyle\int_{I_j(t_1 - \widetilde{\alpha}_m)}^{I_j(t_2 - \widetilde{\alpha}_m)} h_j(s^{2n-j})\, ds \ \text{ if } \ \widetilde{\alpha}_m \le t_1 < t_2 \le \dfrac{\widetilde{\alpha}_m + \widetilde{\beta}_m}{2} \\
\dfrac{1}{I_j} \Bigg(\displaystyle\int_0^{I_j(t_1 - \widetilde{\alpha}_m)} h_j(s^{2n-j})\, ds + \int_0^{I_j(t_2 - \widetilde{\beta}_m)} h_j(s^{2n-j})\, ds \Bigg) \\
\qquad \text{if } \ \widetilde{\alpha}_m \le t_1 < \dfrac{\widetilde{\alpha}_m + \widetilde{\beta}_m}{2} < t_2 \le T \\
\dfrac{1}{I_j} \displaystyle\int_{I_j(\widetilde{\beta}_m - t_2)}^{I_j(\widetilde{\beta}_m - t_1)} h_j(s^{2n-j})\, ds \ \text{ if } \ \dfrac{\widetilde{\alpha}_m + \widetilde{\beta}_m}{2} \le t_1 < t_2 \le \widetilde{\beta}_m \\
\dfrac{1}{I_j} \Bigg(\displaystyle\int_0^{I_j(\widetilde{\beta}_m - t_1)} h_j(s^{2n-j})\, ds + \int_0^{I_j(t_2 - \widetilde{\beta}_m)} h_j(s^{2n-j})\, ds \Bigg) \\
\qquad \text{if } \ \dfrac{\widetilde{\alpha}_m + \widetilde{\beta}_m}{2} \le t_1 < \widetilde{\beta}_m < t_2 \le T \\
\dfrac{1}{I_j} \displaystyle\int_{I_j(t_1 - \widetilde{\beta}_m)}^{I_j(t_2 - \widetilde{\beta}_m)} h_j(s^{2n-j})\, ds \ \text{ if } \ \widetilde{\beta}_m \le t_1 < t_2 \le T
\end{cases} .
$$

Summarizing, we have

$$
\left.
\begin{array}{c}
\displaystyle\int_{t_1}^{t_2} h_j(|x_m^{(j)}(t)|)\, dt \le \frac{2}{I_j} \int_{\nu_1}^{\nu_2} h_j(s^{2n-j})\, ds \\
\text{for } \ 0 \le j \le 2n-2, \ \ m \in \mathbb{N}, \\
\text{where } \ 0 \le \nu_1 < \nu_2 \le I_j T, \ \ \nu_2 - \nu_1 \le I_j(t_2 - t_1).
\end{array}
\right\} \tag{3.26}
$$

Since $h_j(s^{2n-j}) \in L_1(I_j T)$ for $j = 0, 1, \dots, 2n-2$ by (H_2) (see Remark 1.1), (3.26) shows that $\{\int_0^t h_j(|x_m^{(j)}(s)|)\, ds\}$ is equicontinuous on $[0, T]$ for $0 \le j \le 2n-2$. We have proved that $\{H(x_m^{(2n-1)})\}$ is equicontinuous on $[0, T]$, and from H being continuous and increasing on $\mathbb{R}$ we see that $\{x_m^{(2n-1)}\}$ is equicontinuous on $[0, T]$ as well. $\square$

4. An Existence Result and an Example

We now state our main result.

Theorem 4.1. *Let* (H_1) *and* (H_2) *hold. Then, for each* $p \in \{1, \dots, n-1\}$, $i_0 \in \{1, \dots, p\}$, $k_0 \in \{p+1, \dots, n\}$ *and* ϕ_1, $\phi_2 \in \mathcal{A}$, *there exist a solution* x *of the problem and* α, $\beta \in [0,T]$ *such that* $x^{(2j)} > 0$ *on* $[0,T] \setminus \{\alpha\}$ *for* $0 \le j \le p-1$ *and* $x^{(2j)} > 0$ *on* $[0,T] \setminus \{\beta\}$ *for* $p \le j \le n-1$.

Proof. Choose $p \in \{1, \dots, n-1\}$, $i_0 \in \{1, \dots, p\}$, $k_0 \in \{p+1, \dots, n\}$ and $\phi_1, \phi_2 \in \mathcal{A}$. By Lemma 3.8, for each $m \in \mathbb{N}$ there exists a solution x_m of the problem $(2.2)^1_m$, (1.4), (1.5) such that (3.21) is true, where K is a positive constant and $\{x_m^{(2n-1)}\}$ is equicontinuous due to Lemma 3.9. In addition (see Lemma 3.5),

$$x_m(t) = \frac{1}{(2(n-p)-1)!(2p-1)!} \times$$
$$\times \int\limits_{\alpha_m}^{t} (t-s)^{2p-1)} \int\limits_{\beta_m}^{s} (s-v)^{2(n-p)-1} (\mathcal{K}_{m,1} x_m)(v)\, dv\, ds$$

for $t \in [0,T]$ and $m \in \mathbb{N}$, where β_m and α_m are the unique solutions in $[0,T]$ of the equation $S_{k_0}(\beta; x_m, 1) = 0$ and $V_{i_0}(\alpha; x_m, 1) = 0$, respectively. Here S_{k_0} and V_{i_0} are defined in (3.3) and (3.5). Besides, the inequalities (3.23) and (3.24) are true, where $\widetilde{\alpha}_m = \min\{\alpha_m, \beta_m\}$, $\widetilde{\beta}_m = \min\{\alpha_m, \beta_m\}$. Hence (see Lemma 3.4)

$$\left.\begin{aligned} x_m^{(j)}(\alpha_m) &= 0 \ \text{ for } \ 0 \le j \le 2p-1, \\ x_m^{(j)}(\beta_m) &= 0 \ \text{ for } \ 2p \le j \le 2n-1, \\ x_m^{(2n-2j)} &> 0 \ \ texton \ \ [0,T] \setminus \{\beta_m\} \ \text{ for } \ 1 \le j \le n-p, \\ x_m^{(2n-2j)} &> 0 \ \text{ on } \ [0,T] \setminus \{\alpha_m\} \ \text{ for } \ n-p+1 \le j < n. \end{aligned}\right\} \tag{4.1}$$

By the Arzelà–Ascoli theorem and the compactness principle, passing if necessary to subsequences, we may assume that $\{x_m\}$ converges in $C^{2n-1}[0,T]$ and $\{\alpha_m\}$. $\{\beta_m\}$ in $\mathbb{R}$. Let $\lim\limits_{m\to\infty} x_m = x$, $\lim\limits_{m\to\infty} \alpha_m = \alpha_*$ and $\lim\limits_{m\to\infty} \beta_m = \beta_*$. Then $x \in C^{2n-1}[0,T]$, $\phi_1(x^{(2i_0-1)}) = 0$, $\phi_2(x^{(2k_0-1)}) = 0$,

$$|x^{(j)}(t)| \ge \frac{a}{(2n-j)!} |t-\beta_*|^{2n-j} \ \text{ for } \ t \in [0,T], \ \ 2p \le j \le 2n-1,$$

and

$$|x^{(j)}(t)| \ge \begin{cases} \dfrac{a}{(2n-j)!}\, |t-\widetilde{\alpha}_*|^{2n-j} & \text{for } \ t \in \left[0, \dfrac{\widetilde{\alpha}_* + \widetilde{\beta}_*}{2}\right] \\ \dfrac{a}{(2n-j)!}\, |t-\widetilde{\beta}_*|^{2n-j} & \text{for } \ t \in \left[\dfrac{\widetilde{\alpha}_* + \widetilde{\beta}_*}{2}, T\right] \end{cases} \tag{4.2}$$

for $0 \le j \le 2p-1$, where $\widetilde{\alpha}_* = \min\{\alpha_*, \beta_*\}$, $\widetilde{\beta}_* = max\{\alpha_*, \beta_*\}$. Therefore, β_* is the unique zero of $x^{(j)}$ for $2p \le j \le 2n-1$ and from (4.1) and (4.2) we deduce that α_* is the unique zero of $x^{(j)}$ for $0 \le j \le 2p-1$. Besides, $x^{(2n-2j)} > 0$ on $[0,T]\backslash\{\beta_*\}$ for $1 \le j \le n-p$ and $x^{(2n-2j)} > 0$ on $[0,T]\backslash\{\alpha_*\}$ for $n-p+1 \le j < n$. Consequently

$$\lim_{m\to\infty} f_m\big(t, x_m(t), \dots, x_m^{(2n-1)}(t)\big) = \\ = f\big(t, x(t), \dots, x^{(2n-1)}(t)\big) \text{ for a.e. } t \in [0,T],$$

and then from the boundedness of $\{x_m^{(2n-1)}(0)\}$, $\{x_m^{(2n-1)}(T)\}$ and the equality

$$x_m^{(2n-1)}(T) = x_m^{(2n-1)}(0) + \int_0^T f_m\big(t, x_m(t), \dots, x_m^{(2n-1)}(t)\big)\, dt$$

we see that $f(t, x(t), \dots, x^{(2n-1)}(t)) \in L_1[0,T]$ by the Fatou theorem. Without loss of generality we can assume that for example $\alpha_* \le \beta_*$. Consider the intervals $[0, \alpha_*]$ (if $\alpha_* > 0$), $[\alpha_*, \beta_*]$ (if $\alpha_* < \beta_*$) and $[\beta_*, T]$ (if $\beta_* < T$). Let $[\eta, \tau]$ be an arbitrary but fixed from the above intervals. From (2.4) with $\lambda = 1$ and the Lebesgue dominated convergence theorem it follows that letting $m \to \infty$ in

$$x_m^{(2n-1)}(t) = x_m^{(2n-1)}\Big(\frac{\eta+\tau}{2}\Big) + \int_{(\eta+\tau)/2}^{t} f_m\big(s, x_m(s), \dots, x_m^{(2n-1)}(s)\big)\, ds,$$

we get

$$x^{(2n-1)}(t) = x^{(2n-1)}\Big(\frac{\eta+\tau}{2}\Big) + \int_{(\eta+\tau)/2}^{t} f\big(s, x(s), \dots, x^{(2n-1)}(s)\big)\, ds \quad (4.3)$$

for $t \in (\eta, \tau)$. We know that $x \in C^{2n-1}[0,T]$ and $f(t, x(t), \dots, x^{(2n-1)}(t)) \in L_1[0,T]$. Consequently, (4.3) is true even for $t \in [\eta, \tau]$. This shows that

$$x^{(2n-1)}(t) = x^{(2n-1)}(0) + \int_0^t f\big(s, x(s), \dots, x^{(2n-1)}(s)\big)\, ds \text{ for } t \in [0,T].$$

Hence $x \in AC^{2n-1}[0,T]$ and x is a solution of the problem (1.1), (1.4), (1.5). □

Example 4.2. Consider the differential equation

$$x^{(2n)} = q(t) + \sum_{j=0}^{2n-1} \frac{b_j(t)}{|x^{(j)}|^{\gamma_j}} + \sum_{j=0}^{2n-1} c_j(t)|x^{(j)}|^{\delta_j}, \quad (4.4)$$

where $q, b_j \in L_\infty[0,T]$, $c_j \in L_1[0,T]$ are nonnegative for $0 \le j \le 2n-1$, $q(t) \ge a > 0$ for a.e. $t \in [0,T]$ and $\gamma_j \in (0, \frac{1}{2n-j})$ for $0 \le j \le 2n-2$, $\gamma_{2n-1} > 0$, $\delta_j \in (0,1)$ for $0 \le j \le 2n-1$.

The equation (4.4) is a special case of (1.1) with

$$f(t,x_0,\dots,x_{2n-1}) = q(t) + \sum_{j=0}^{2n-1} \frac{b_j(t)}{|x_j|^{\gamma_j}} + \sum_{j=0}^{2n-1} c_j(t)|x_j|^{\delta_j}$$

satisfying (H_1). Put $L = \max\{\|b_j\|_\infty :\ 0 \le j \le 2n-2\}$ and $\delta = \max\{\delta_j : 0 \le j \le 2n-1\} < 1$. Since

$$\sum_{j=0}^{2n-1} c_j(t)|x_j|^{\delta_j} \le \sum_{j=0}^{2n-1} c_j(t) \sum_{j=0}^{2n-1} |x_j|^{\delta_j} \le \sum_{j=0}^{2n-1} c_j(t)\Big(2n + \sum_{j=0}^{2n-1} |x_j|^{\delta}\Big) \le$$
$$\le \sum_{j=0}^{2n-1} c_j(t)\Big(2n + (2n)^{1-\delta}\Big(\sum_{j=0}^{2n-1} |x_j|\Big)^{\delta}\Big),$$

where the inequality $\sum\limits_{j=0}^{2n-1} b_j^{\varrho} \le (2n)^{1-\varrho}\big(\sum_{j=0}^{2n-1} b_j\big)^{\varrho}$ $(b_j \ge 0,\ \varrho \in (0,1])$ is used, we have

$$f(t,x_0,\dots,x_{2n-1}) \le$$
$$\le \|q\|_\infty + L\sum_{j=0}^{2n-1} \frac{1}{|x_j|^{\gamma_j}} + \sum_{j=0}^{2n-1} c_j(t)\Big(2n + (2n)^{1-\delta}\Big(\sum_{j=0}^{2n-1} |x_j|\Big)^{\delta}\Big).$$

Hence

$$f(t,x_0,\dots,x_{2n-1}) \le \sum_{j=0}^{2n-1} h_j(|x_j|) + \omega\Big(t, \sum_{j=0}^{2n-1} |x_j|\Big),$$

where $h_j(u) = Lu^{-\gamma_j}$ for $0 \le j \le 2n-2$, $h_{2n-1} = \|q\|_\infty + Lu^{-\gamma_{2n-1}}$ and $w(t,u) = \sum\limits_{j=0}^{2n-1} c_j(t)(2n + (2n)^{1-\delta}u^{\delta})$. Then

$$\int_0^1 h_j(s^{2n-j})\,ds = \int_0^1 s^{-\frac{\gamma_j}{2n-j}}\,ds < \infty$$

for $0 \le j \le 2n-2$ and

$$\lim_{u\to\infty} h_{2n-1}(u) = \|q\|_\infty.$$

Since

$$\int_0^u \frac{ds}{h_{2n-1}(s)} = \int_0^u \frac{s^{\gamma_{2n-1}}}{\|q\|_\infty s^{\gamma_{2n-1}} + L}\,ds > \frac{1}{\|q\|_\infty + L}\int_1^u ds = \frac{u-1}{\|q\|_\infty + L}$$

for $u \ge 1$ and

$$\int_0^T \omega(t,Qu)\,dt = \big(2n + (2n)^{1-\delta}(Qu)^{\delta}\big)\sum_{j=0}^{2n-1} \|c_j\|_L,$$

where Q is given in (1.8), we have

$$\lim_{u\to\infty}\left(\int_0^u \frac{1}{h_{2n-1}(s)}\,ds\right)^{-1}\int_0^T \omega(t,Qu)\,dt = 0,$$

and therefore f satisfies (H_2). Now Theorem 4.1 guarantees that for each $p \in \{1,\dots,n-1\}$, $i_0 \in \{1,\dots,p\}$, $k_0 \in \{p+1,\dots,n\}$ and $\phi_1, \phi_2 \in \mathcal{A}$ there exists a solution of the problem (4.4), (1.4), (1.5). Hence, since the functionals $\phi_1, \phi_2 : C^0[0,T] \to \mathbb{R}$ defined by

$$\phi_1(x) = \int_0^T (x(s))^3\,ds, \quad \phi_2(x) = x(t_1) + e^{x(t_2)} - 1, \;\; t_1,\, t_2 \in [0,T],$$

belong to $\mathcal{A}$, for each $p \in \{1,\dots,n-1\}$, $i_0 \in \{1,\dots,p\}$ and $k_0 \in \{p+1,\dots,n\}$ there exists a solution x of (4.4) such that

$$\int_0^T (x^{(2i_0-1)}(s))^3\,ds = 0 \;\; x^{(2k_0-1)}(t_1) + e^{x^{(2k_0-1)}(t_2)} = 1$$

and $x^{(2j)} > 0$ on $[0,T] \setminus \{\alpha\}$ for $0 \le j \le p-1$, $x^{(2j)} > 0$ on $[0,T] \setminus \{\beta\}$ for $p \le j \le n-1$, where α and β are the unique zeros of $x^{(2i_0-1)}$ and $x^{(2k_0-1)}$, respectively.

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