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ON A PERIODIC BOUNDARY VALUE PROBLEM FOR CYCLIC FEEDBACK TYPE LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. Unimprovable effective sufficient conditions are established for unique solvability of the periodic problem

$$u_i''(t) = \ell_i(u_{i+1})(t) + q_i(t) \quad (i = 1, \dots, n-1),$$
$$u_n''(t) = \ell_n(u_1)(t) + q_n(t),$$
$$u_j(t+\omega) = u_j(t) \quad (j = 1, \dots, n) \quad \text{for} \quad t \in R,$$

where $\omega > 0$, $\ell_i : C_\omega \to L_\omega$ are linear bounded operators, and $q_i \in L_\omega$.

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$$u_n''(t) = \ell_n(u_1)(t) + q_n(t),$$
$$u_j(t+\omega) = u_j(t) \quad (j = 1, \dots, n), \quad \text{როცა} \quad t \in R,$$

პერიოდული ამოცანის ცალსახად ამოხსნადობის არაგაუმჯობესებადი ეფექტური საკმარისი პირობები, სადაც $\omega > 0$, $\ell_i : C_\omega \to L_\omega$ წრფივი შემოსაზღვრული ოპერატორებია და $q_i \in L_\omega$.

1. Statement of Problem and Formulation of Main Results

Consider on R the system

$$u_i''(t) = \ell_i(u_{i+1})(t) + q_i(t) \ \ (i = 1, \dots, n-1),$$
$$u_n''(t) = \ell_n(u_1)(t) + q_n(t), \tag{1.1}$$

with periodic conditions

$$u_j(t+\omega) = u_j(t) \ \ (j = 1, \dots, n) \ \text{ for } \ t \in R, \tag{1.2}$$

where $\omega > 0$, $\ell_i : C_\omega \to L_\omega$ are linear bounded operators and $q_i \in L_\omega$.

By a solution of the problem (1.1), (1.2) we understand a vector valued function $u = \{u_i\}_{i=1}^n$ with $u_i \in \widetilde{C}'([0,\omega])$ $(i = 1, \dots, n)$ which satisfies the system (1.1) almost everywhere on R and the conditions (1.2).

Much investigation has been carried out on the existence and uniqueness of the solution to the periodic boundary value problem for systems of ordinary differential equations and many interesting results have been obtained (see, for instance, [1]– [7] and references therein). However, an analogous problem for functional differential equations, even in the case of linear equations remains little investigated.

In the present paper, we study the problem (1.1), (1.2) under the assumption that ℓ_i $(i = 1, \dots, n)$ are monotone linear operators. We establish new unimprovable integral sufficient conditions for unique solvability of the problem (1.1), (1.2) which generalize the well-known results of A. Lasota and Z. Opial obtained in [8]. These results are new also if (1.1) is the following system of ordinary differential equations

$$u_i''(t) = p_i(t)u_{i+1}(t) + q_i(t) \quad (i = 1, \dots, n-1),$$
$$u_n''(t) = p_n(t)u_1(t) + q_n(t), \tag{1.3}$$

where q_i, $p_i \in L_\omega$.

The method used for the investigation of the problem considered is based on the method developed in our previous papers (see [9], [11]) for functional differential equations.

The following notation is used throughout:

$N(R)$ is the set of all natural (real) numbers;

$R_+ = [0, +\infty[$;

C_ω is the Banach space of ω-periodic continuous functions $u : R \to R$ with the norm $\|u\|_{C_\omega} = \max\{|u(t)| : \ 0 \le t \le \omega\}$;

$\widetilde{C}_\omega$ is the set of ω-periodic absolutely continuous functions $u : R \to R$;

$\widetilde{C}'_\omega$ is the set of ω-periodic functions $u : R \to R$ which are absolutely continuous together with their first derivatives;

L_ω is the Banach space of ω-periodic and Lebesgue integrable on $[0,\omega]$ functions $p : R \to R$ with the norm $\|p\|_{L_\omega} = \int\limits_0^\omega |p(s)|\,ds$;

if $\ell : C_\omega \to L_\omega$ is a linear operator, then $\|\ell\| = \sup\limits_{0 \le \|x\|_{C_\omega} \le 1} \|\ell(x)\|_L$.

Definition 1.1. We will say that a linear operator $\ell : C_\omega \to L_\omega$ is *nonnegative* (*nonpositive*), if for any nonnegative $x \in C_\omega$ the inequality $\ell(x)(t) \geq 0$ ($\ell(x)(t) \leq 0$) for $t \in R$ is satisfied.

We will say that an operator ℓ is *monotone* if it is nonnegative or nonpositive.

Theorem 1.1. *Let $\ell_i : C_\omega \to L_\omega$ $(i = 1, \dots, n)$ be linear monotone operators,*

$$\|\ell_i\| \neq 0 \ \textit{for} \ i = 1, \dots, n, \tag{1.4}$$

and

$$\prod_{i=1}^{n} \|\ell_i\| \leq \Big(\frac{16}{\omega}\Big)^n. \tag{1.5}$$

Then the problem (1.1), (1.2) *has a unique solution.*

The condition (1.5) in Theorem 1.1 is optimal. For the sake of simplicity, we show this for the case where $n = 2$.

Example 1.1. For the case where $n = 2$, the example below shows that the condition (1.5) in Theorem 1.1 is optimal and it cannot be replaced by the condition

$$\|\ell_1\| \cdot \|\ell_2\| \leq \Big(\frac{16}{\omega}\Big)^2 + \varepsilon \tag{1.5$_1$}$$

no matter how small $\varepsilon \in]0, 1]$ is. Let the numbers α, β, functions τ_1, τ_2, u_0, and operators l_1, l_2, be given by the equalities

$$\beta = \Big(\frac{1}{4} - \frac{4}{(16^2+\varepsilon)^{1/2}}\Big)\frac{\pi}{\pi - 2}\,, \quad \alpha = \frac{\pi}{\pi - 4\beta(\pi - 2)} = \frac{(16^2+\varepsilon)^{1/2}}{16}\,,$$

$$u_0(t) = \begin{cases} \alpha t & \text{for } t \in \Big[0, \frac{1}{4} - \beta\Big[, \\ \Big(\frac{1}{4} - \beta\Big)\alpha + \frac{8\alpha\beta}{\pi} \sin\Big(\frac{\pi}{2\beta}\Big(\beta + t - \frac{1}{4}\Big)\Big) & \text{for } t \in \Big[\frac{1}{4} - \beta, \frac{1}{4} + \beta\Big], \\ \Big(\frac{1}{2} - t\Big)\alpha & \text{for } t \in \Big]\frac{1}{4} + \beta, \frac{3}{4} - \beta\Big[, \\ \Big(\beta - \frac{1}{4}\Big)\alpha - \frac{8\alpha\beta}{\pi} \sin\Big(\frac{\pi}{2\beta}\Big(\beta + t - \frac{3}{4}\Big)\Big) & \text{for } t \in \Big[\frac{3}{4} - \beta, \frac{3}{4} + \beta\Big], \\ \alpha(t-1) & \text{for } t \in \Big]\frac{3}{4} + \beta, 1\Big], \end{cases}$$

$$u_0(t) = u_0(t+1) \ \text{ for } \ t \in R,$$

$$\tau_i(t) = \begin{cases} \frac{1}{4} & \text{for } (-1)^{1+i} u_0''(t) \geq 0 \\ \frac{3}{4} & \text{for } (-1)^{1+i} u_0''(t) < 0 \end{cases} \quad (i = 1, 2) \tag{1.6}$$

for $t \in R$, and

$$l_1(y)(t) = |u_0''(t)| y(\tau_1(t)), \quad l_2(y)(t) = -|u_0''(t)| y(\tau_2(t)), \tag{1.7}$$

for any $y \in C_\omega$, $t \in R$. It is evident that

$$u_0\Big(\frac{1}{4}\Big) = 1, \quad u_0\Big(\frac{3}{4}\Big) = -1, \tag{1.8}$$

l_1, $l_2 : C_\omega \to L_\omega$ are linear non negative operators and

$$\int\limits_0^\omega \big|l_i(1)(s)\big|\, ds = 16\alpha \int\limits_{1/4-\beta}^{1/4} \left|\sin'\Big(\frac{\pi}{2\beta}\Big(\beta + s - \frac{1}{4}\Big)\Big)\right| ds = 16\alpha$$

for $i = 1, 2$. Thus all the requirements of Theorem 1.1, except of (1.5), are satisfied and instead of (1.5) the equality

$$\|\ell_1\| \cdot \|\ell_2\| = 16^2\alpha^2 = 16^2 + \varepsilon$$

is fulfilled. On the other hand, from (1.6)-(1.8) we get

$$u_0''(t) = |u_0''(t)|\mathrm{sgn} u_0''(t) = |u_0''(t)|u_0(\tau_1(t)) = l_1(u_0)(t),$$
$$u_0''(t) = |u_0''(t)|\mathrm{sgn} u_0''(t) = -|u_0''(t)|u_0(\tau_2(t)) = l_2(u_0)(t).$$

Thus the vector valued function $(u_1, u_2) : R \to R^2$ with $u_i \equiv u_0$, $i = 1, 2$, is a $\omega = 1$ periodic nontrivial solution of the system (1.1).

Consider on $[0, \omega]$ the system of differential equations with deviating arguments

$$\begin{aligned} u_i''(t) &= p_i(t)u_{i+1}(\tau_{i+1}(t)) + q_i(t) \quad (i = 1, \dots, n-1), \\ u_n''(t) &= p_n(t)u_1(\tau_1(t)) + q_n(t), \end{aligned} \tag{1.9}$$

where q_i, $p_i \in L_\omega$, $\tau_i : R \to R$ are measurable functions such that

$$\tau_i(t + \omega) = \mu_i(t)\omega + \tau_i(t) \;\; (i = 1, \dots, n) \;\text{ for }\; t \in R,$$

and the functions μ_i take only integral values.

Corollary 1.1. *Let*

$$0 \le \sigma_i p_i(t) \not\equiv 0 \;\; (i = 1, \dots, n), \tag{1.10}$$

where $\sigma_i \in \{-1, 1\}$ *and*

$$\prod_{i=1}^n \|p_i\|_L \le \Big(\frac{16}{\omega}\Big)^n. \tag{1.11}$$

Then the problem (1.3), (1.2) ((1.9), (1.2)) *has a unique solution.*

2. Proofs

To prove Theorem 1.1, we need the following two lemmas.

Lemma 2.1. *Let* $\sigma \in \{-1, 1\}$ *and* $\sigma\ell : C_\omega \to L_\omega$ *be a nonnegative linear operator. Then for an arbitrary* $v \in C_\omega$ *the inequalities*

$$-m|\ell(1)(t)| \le \sigma\ell(v)(t) \le M|\ell(1)(t)| \;\text{ for }\; t \in R$$

hold, where $m = \max\{-v(t) :\; 0 \le t \le \omega\}$, $M = \max\{v(t) :\; 0 \le t \le \omega\}$.

Proof. It is clear that $v(t) - M \le 0$, $v(t) + m \ge 0$ on R. Then from the nonnegativity of $\sigma\ell$ we get $\sigma\ell(v-M)(t) \le 0$, $\sigma\ell(v+m)(t) \ge 0$ on R, whence follows the validity of the lemma. □

Let $\omega > 0$. Define the functional $\Delta : C_\omega \to R_+$ by the equality

$$\Delta(x) = \max\big\{x(t) :\ 0 \le t \le \omega\big\} + \max\big\{ - x(t) :\ 0 \le t \le \omega\big\}. \tag{2.1}$$

Then the following lemma is valid:

Lemma 2.2. *Let* $z \in \widetilde{C}'_\omega$, *and*

$$z(t) \ne Const, \quad z(t+\omega) = z(t) \ \ (j = 0, \dots, k) \ \ for \ \ t \in R. \tag{2.2}$$

Then the estimate

$$\Delta(z) < \frac{\omega}{4}\Delta(z') \tag{2.3}$$

is satisfied.

Proof. Define $t_1 \in [0, \omega[\,,\ t_2 \in]t_1, t_1 + \omega[$ and the numbers M_1, m_1 by the equalities

$$z(t_1) = \max\big\{z(t) :\ 0 \le t \le \omega\big\}, \quad z(t_2) = -\max\big\{ - z(t) :\ t_1 \le t \le t_1 + \omega\big\},$$
$$M_1 = \max\big\{z'(t) :\ t_1 \le t \le t_1 + \omega\big\}, \quad m_1 = \max\big\{ - z'(t) :\ t_1 \le t \le t_1 + \omega\big\}.$$

It follows from the definition of t_1, t_2, and the conditions (2.2) that

$$M_1 > 0, \quad m_1 > 0, \tag{2.4}$$

and

$$z'(t_1) = 0, \quad z'(t_1 + \omega) = 0, \quad z'(t_2) = 0.$$

Hence

$$\Delta(z) = -\int_{t_1}^{t_2} z'(s)\, ds, \quad \Delta(z) = \int_{t_2}^{t_1+\omega} z'(s)\, ds. \tag{2.5}$$

In view of the conditions (2.2) we have

$$z'(t) \ne Const \ \text{ for } \ t \in [t_1, t_2] \tag{2.6}$$

and/or $z'(t) \ne Const$ for $t \in [t_2, t_1 + \omega]$. Without loss of generality we can assume that the condition (2.6) is satisfied.

Then from (2.4) and (2.5) we get $\Delta(z) < m_1(t_2 - t_1)$, $\Delta(z) \le M_1(t_1 + \omega - t_2)$, and therefore

$$\Delta^2(z) < m_1 M_1 (t_1 + \omega - t_2)(t_2 - t_1).$$

From the last estimate by virtue of (2.4) and the inequality

$$4\lambda_1\lambda_2 \le (\lambda_1 + \lambda_2)^2 \tag{2.7}$$

we obtain (2.3). □

Consider now on R the homogeneous problem

$$v_i''(t) = \ell_i(v_{i+1})(t) \ \ (i = 1, \dots, n-1), \tag{2.8_i}$$
$$v_n''(t) = \ell_n(v_1)(t), \tag{2.8_n}$$
$$v_j(t+\omega) = v_j(t) \ \ (j = 1, \dots, n) \ \text{ for } \ t \in R. \tag{2.9_j}$$

Lemma 2.3. *Let $\ell_i : C([0,\omega]) \to L([0,\omega])$ $(i = 1,\dots,n)$ be linear monotone operators,*

$$\int_0^\omega \ell_i(1)(s)\,ds \neq 0 \ \ (i = \dots, 1, n), \tag{2.10}$$

and $v(t) = (v_i(t))_{i=1}^n$ be a nontrivial solution to the problem $((2.8_i))_{i=1}^n$, $((2.9_j))_{j=1}^n$. Then the functions v_i and v_i' $(i = 1,\dots,n)$ change their signs on $[0,\omega]$.

Proof. Introduce the notation $v_0(t) \equiv v_n(t)$, $v_{n+i}(t) \equiv v_i(t)$, $\ell_0 \equiv \ell_n$, $\ell_{n+i} \equiv \ell_i$ if $i = 1,\dots,n$, and let $k_0 = \min\{k \in \{1,\dots,n\} : \ v_k \not\equiv 0\}$. Then from (2.8_{k_0-1}), (2.9_{k_0-1}) ((2.8_n), (2.9_n) if $k_0 = 1$), it follows that

$$\int_0^\omega \ell_{k_0-1}(v_{k_0})(s)\,ds = 0.$$

Thus in view of the conditions (2.10), $v_{k_0}(t) \not\equiv 0$ and the monotonity of the operator ℓ_{k_0-1}, it follows that there exists $t_0 \in]0,\omega]$ such that $v_{k_0}(t_0) = 0$. Then in view of the condition (2.9_{k_0}) there exist sets of positive measure $I_{1j}, I_{2j} \subset [0,\omega]$ such that

$$v_j'(t) > 0 \ \text{ for } \ t \in I_{1j}, \quad v_j'(t) < 0 \ \text{ for } \ t \in I_{2j}, \tag{2.11$_j$}$$

with $j = k_0$. From (2.8_{k_0}) and (2.11_{k_0}) in view of the monotonity of the operator ℓ_{k_0} it follows that the function v_{k_0+1} changes its sign. Thus, there exist sets of positive measure $I_{1\,k_0+1}$ and $I_{2\,k_0+1}$ ($I_{1\,1}$ and $I_{2\,1}$ if $k_0 = n$) from $[0,\omega]$ such that the inequalities (2.11_{k_0+1}) ((2.11_1) if $k_0 = n$) are satisfied. Therefore, from (2.8_{k_0+1}) and (2.11_{k_0+1}) ((2.8_1) and (2.11_1) if $k_0 = n$) in view of the monotonity of the operator ℓ_{k_0+1} it follows that the function v_{k_0+2} changes its sign. Reasoning analogously, we can see that the functions v_j and then the functions v_j' too, change their signs for all $j \in \{1,\dots,n\}$. $\square$

Proof of Theorem 1.1. It is well known from the general theory of boundary value problems for functional differential equations that if ℓ_i $(i = 1,\dots,n)$ are monotone operators, then the problem (1.1), (1.2) has the Fredholm property (see [6]). Thus, the problem (1.1), (1.2) is uniquely solvable iff the homogeneous problem $((2.8_i))_{i=1}^n$, $((2.9_j))_{j=1}^n$ has only the trivial solution.

Assume that, on the contrary, the problem $((2.8_i))_{i=1}^n$, $((2.9_j))_{j=1}^n$ has a nontrivial solution $v(t) = (v_i(t))_{i=1}^n$, and let the numbers M_i, m_i, M_i', m_i', and t_{1i}, $t_{2i} \in [0,\omega[$ be given by the equalities

$$M_j = \max\big\{v_i(t) : \ 0 \le t \le \omega\big\}, \ \ m_i = \max\big\{-v_i(t) : \ 0 \le t \le \omega\big\},$$
$$M_j' = \max\big\{v_i'(t) : \ 0 \le t \le \omega\big\}, \ \ m_i' = \max\big\{-v_i'(t) : \ 0 \le t \le \omega\big\},$$

and

$$v_i'(t_{1i}) = M_i', \quad v_i'(t_{2i}) = -m_i'.$$

Then from Lemma 2.3 it follows that $t_{1i} \neq t_{2i}$,

$$M_i' > 0, \quad m_i' > 0 \ \text{ for } \ i = 1,\dots,n. \tag{2.12}$$

Thus if $\alpha_1 = \min\{t_{1i}, t_{2i}\}$, $\alpha_2 = \max\{t_{1i}, t_{2i}\}$ and $T_{1i} = [\alpha_1, \alpha_2]$, $T_{2i} = [0, \alpha_1] \cup [\alpha_2, \omega]$, in view of the definition (2.1) and the condition (2.9_i) we get

$$0 < \Delta(v_i') = (-1)^{k-1} \int\limits_{T_{ki}} \ell_i(v_{i+1})(s)\, ds \operatorname{sgn}(t_{1i} - t_{2i}) \ \ (k = 1, 2). \qquad (2.13_k)$$

If $\operatorname{sgn}(t_{1i} - t_{2i})\ell_i$ is a nonpositive operator, then from (2.13_k) $(k = 1, 2)$ in view of (2.12) by Lemma 2.1 we get the following estimates:

$$0 < \Delta(v_i') \le m_{i+1} \int\limits_{T_{1i}} |\ell_i(1)(s)|\, ds, \quad 0 < \Delta(v_i') \le M_{i+1} \int\limits_{T_{2i}} |\ell_i(1)(s)|\, ds,$$

respectively. By multiplying these estimates and applying the inequality (2.7), we obtain

$$0 < \Delta(v_i') \le \frac{\Delta(v_{i+1})}{4} \int\limits_0^\omega |\ell_i(1)(s)|\, ds, \qquad (2.14_i)$$

where $v_{n+1} \equiv v_1$ if $i = n$. Reasoning analogously, we can see that the estimate (2.14_i) is valid also in the case where the operator $\operatorname{sgn}(t_{1i} - t_{2i})\ell_i$ is nonnegative. By multiplying all the inequalities (2.14_i) $(i = 1, \dots, n)$ and using the inequalities (2.3) with $z \equiv v_j$ $(j = 1, \dots, n)$, we get the contradiction to the condition (1.5). Thus our assumption fails, and $v(t) \equiv 0$. □

Proof of Corollary 1.1. Let $\ell_i(x)(t) = p_i(t)x(\tau_{i+1}(t))$ $(\ell_i(x)(t) = p_i(t)x(t))$ $(i = 1, \dots, n)$. According to (1.10) and (1.11) it is clear that ℓ_i are monotone operators, and the conditions (1.4) and (1.5) of Theorem 1.1 are fulfilled. □

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