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EFFECTIVE SOLUTION OF THE BASIC BOUNDARY VALUE PROBLEM FOR HOMOGENEOUS EQUATIONS OF STATICS OF THE THEORY OF ELASTIC MIXTURE IN A CIRCULAR DOMAIN AND IN AN INFINITE DOMAIN WITH A CIRCULAR HOLE

Abstract. In the paper we consider the boundary value problem for homogeneous equations of statics of the theory of elastic mixtures in a circular domain, and in an infinite domain with a circular hole, when projections of the displacement vector on the normal and of the stress vector on the tangent are prescribed on the boundary of the domain. The arbitrary analytic vector φ appearing in the general representation of the displacement vector is sought as a double layer potential whose density is a linear combination of the normal and tangent unit vectors. Having chosen the displacement vector in a special form, we define the projection of the density on the normal by the function given on the boundary. To find the projection of the stress vector on the tangent, we obtain a singular integral equation with the Hilbert kernel. Using the formula of transposition of singular integrals with the Hilbert kernel, we obtain expressions for the projection on the tangent of the above-mentioned density. Assuming that the function is Hölder continuous, the projection of the displacement vector on the normal and its derivative are likewise Hölder continuous. Under these conditions the obtained expressions for the displacement and stress vectors are continuous up to the boundary. The theorem on the uniqueness of solution is proved, when the boundary is a circumference. The projections of the displacement vector on the normal and tangent are written explicitly. Using these projections, the displacement vector is written in the form of the integral Poisson type formula.

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რეზიუმე. განიხილება დრეკად ნარევთა თეორიის სტატიკის ამოცანა წრიულ არეში და უსასრულო არეში წრიული ხვრელით, როცა არის საზღვარზე მოცემულია გადაადგილების ვექტორის ნორმალზე გეგმილი და ძაბვის ვექტორის მხებზე გეგმილი. გადაადგილების ვექტორის ზოგად წარმოდგენაში მონაწილე ნებისმიერ φ ანალიზურ ვექტორს ვეძებთ ორმაგი ფენის პოტენციალის სახით, რომლის სიმკვრივე არის ნორმალის და მხების ორტების წრფივი კომბინაცია. გადაადგილების ვექტორის გამოსახულების შერჩევით პოტენციალის სიმკვრივის ნორმალზე გეგმილი განისაზღვრება საზღვარზე მოცემული ფუნქციით. პოტენციალის სიმკვრივის მხებზე გეგმილის განსასაზღვრად ვღებულობთ სინგულარულ ინტეგრალურ განტოლებას ჰილბერტის გულით. ვსარგებლობთ რა ჰილბერტის გულიან სინგულარულ ინტეგრალთა გადასმის ფორმულით, ვღებულობთ ხსენებული სიმკვრივის მხებზე გეგმილის გამოსახულებას. მოვითხოვთ, რომ ხსენებული ფუნქცია იყოს ჰელდერის აზრით უწყვეტი. მაშინ გადაადგილების ვექტორის ნორმალზე გეგმილი და მისი წარმოებული აგრეთვე არიან ჰელდერის აზრით უწყვეტნი. ამ პირობებში განსახილველი ამოცანების შესაბამისი გადაადგილების და ძაბვის ვექტორები უწყვეტი იქნებიან საზღვრის ჩათვლით.

მტკიცდება განსახილველი ამოცანის ამონახსნის ერთადერთობის თეორემა, რომელსაც ადგილი აქვს, თუ საზღვარი წრეწირია.

ჩვენს მიერ ცხადი სახით იწერება გადაადგილების ვექტორის ნირმალზე და მხებზე გეგმილები, რომელთა გამოყენებით გადაადგილების ვექტორის გამოსახულება იწერება ინტეგრალური პუასონის ტიპის ფორმულის სახით.

1. Statement of the Third Boundary Value Problem and the Uniqueness Theorem

The basic homogeneous equations of statics of the theory of elastic mixture in the two dimensional case have the form [1]

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0,\\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \end{aligned} \tag{1.1}$$

where

$$\begin{gathered} a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_1 - \lambda_5 - \rho^{-1}\alpha_2\rho_2, \quad a_2 = \mu_2 - \lambda_5,\\ c = \mu_3 + \lambda_5, \quad b_2 = \mu_2 + \lambda_1 + \lambda_5 + \rho^{-1}\alpha_2\rho_2,\\ d = \mu_3 + \lambda_3 - \lambda_5 - \rho^{-1}\alpha_2\rho_2 \equiv \mu_3 + \lambda_4 - \lambda_5 + \rho^{-1}\alpha_2\rho_2,\\ \rho = \rho_1 + \rho_2, \quad \alpha_3 = \lambda_3 - \lambda_4. \end{gathered} \tag{1.2}$$

Here ρ_1 and ρ_2 are partial densities, and μ_1, μ_2, μ_3, λ_1, λ_2, λ_3, λ_4, λ_5 are constants characterizing physical properties of the elastic mixture and satisfying certain inequalities [2], $u' = (u_1, u_2)$ and $u'' = (u_3, u_4)$ are partial displacements.

If we introduce the variables

$$z = x_1 + ix_2, \quad \bar z = x_1 - ix_2,$$

i.e.

$$x_1 = \frac{z+\bar z}{2}, \quad x_2 = \frac{z-\bar z}{2i},$$

then after simple transformations (1.1) can be rewritten as [3]

$$\frac{\partial^2 U}{\partial z\partial \bar z} + K\,\frac{\partial^2 \bar U}{\partial \bar z^2} = 0, \tag{1.3}$$

where

$$U = \begin{pmatrix} u_1 + iu_2\\ u_3 + iu_4\end{pmatrix} = m\varphi(z) - Kmz\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{1.4}$$

$$\begin{gathered} m = \begin{bmatrix} m_1, & m_2\\ m_2, & m_3\end{bmatrix}, \quad m_1 = e_1 + \frac{e_4}{2}, \quad m_2 = e_2 + \frac{e_5}{2}, \quad m_3 = e_3 + \frac{e_6}{2},\\ e_1 = \frac{a_2}{d_2}, \quad e_2 = -\frac{c}{d_2}, \quad e_3 = \frac{a_1}{d_2}, \quad e_1 + e_4 = \frac{a_2+b_2}{d_1},\\ e_2 + e_5 = -\frac{c+d}{d_1}, \quad e_3 + e_6 = \frac{a_1+b_1}{d_1},\\ K = \begin{bmatrix} K_1, & K_3\\ K_2, & K_4\end{bmatrix}, \quad Km = -\frac{e}{2}, \quad e = \begin{bmatrix} e_4, & e_5\\ e_5, & e_6\end{bmatrix},\\ m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3, & -m_2\\ -m_2, & m_1\end{bmatrix}, \quad \Delta_0 = m_1m_3 - m_2^2 > 0, \end{gathered} \tag{1.5}$$

$$\begin{aligned} &\delta_0 K_1 = 2(a_2b_1 - cd) + b_1b_2 - d^2, \quad \delta_0 K_2 = 2(da_1 - cb_1),\\ &\delta_0 K_3 = 2(da_2 - cb_2), \quad \delta_0 K_4 = 2(a_1b_2 - cd) + b_1b_2 - d^2, \end{aligned}$$

$$\delta_0 = (2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2 = 4\Delta_0 d_1 d_2,$$
$$d_1 = (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0, \quad d_2 = a_1 a_2 - c^2 > 0,$$

and the stress vector is

$$TU = \begin{pmatrix} (TU)_2 - i(TU)_1 \\ (TU)_4 - i(TU)_3 \end{pmatrix} = \frac{\partial}{\partial s(x)} \left(-2\varphi(z) + 2\mu U\right), \tag{1.6}$$

where

$$\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}, \tag{1.7}$$

n_1 and n_2 are the projections of the unit vector of the normal onto the axes x_1 and x_2. From this definition, the unit vector of the tangent $s(x) = (-n_2, n_1)$, $(TU)_k$ is the projection of the stress vector on the axis x_k, $k = \overline{1,4}$,

$$\mu = \begin{bmatrix} \mu_1, & \mu_3 \\ \mu_3, & \mu_2 \end{bmatrix}, \quad \det \mu = \Delta_1 = \mu_1 \mu_2 - \mu_3^2 > 0. \tag{1.8}$$

We formulate the third boundary value problem as follows: find a regular solution [4] of the equation (1.3) in a circular domain which on the boundary of the circular domain (i.e., the circumference of radius R) satisfies the following conditions:

$$(nU)^+ = f(t), \quad (s\,TU)^+ = F(t), \tag{1.9}$$

where f and F are given complex functions on the circumference satisfying certain conditions. The sign "+" in (1.9) stands for the limit value from inside. If D is an infinite domain, i.e. we have an infinite domain with a circular hole, then instead of (1.9) we have the conditions

$$(nU)^- = f(t), \quad (s\,TU)^- = F(t), \tag{1.10}$$

where the sign "$-$" denotes the limit value from outside. In the case of an infinite domain, in addition to the conditions of regularity it is necessary to impose the requirements at infinity:

$$U = O(1), \quad \frac{\partial U}{\partial x_k} = O(\rho^{-2}), \ \ k = 1, 2, \quad \rho = \sqrt{x_1^2 + x_2^2}\,. \tag{1.11}$$

If the point x lies on the circumference, then $x = (R, \varphi)$, while if the point z is on the circumference, then $z = Re^{it}$, and $\zeta = Re^{i\tau}$.

The following formulas are valid [3]:

$$\int_{D^+} E(U, U)\, dy_1\, dy_2 = \int_S U\, TU\, ds \equiv \operatorname{Im} \int_S U\, T\bar{U}\, ds, \tag{1.12}$$

$$\int_{D^-} E(U, U)\, dy_1\, dy_2 = -\int_S U\, TU\, ds \equiv -\operatorname{Im} \int_S U\, T\bar{U}\, ds, \tag{1.13}$$

where D^+ is a circular domain of radius R, and D^- is an infinite domain with a circular hole,

$$\operatorname{Im} U\,T\overline{U} = \sum_{k=1}^{4} u_k(Tu)_k = nU_n + s(T\overline{U})_s, \quad U_n = (nU), \quad (TU)_s = (sTU).$$

For the third boundary value problem we prove the following

Theorem. A regular solution of the equation (1.3) in the domains D^+ and D^- satisfying the homogeneous conditions of the third boundary value problem is identically equal to zero if S is not a centerless parabolic line or a pair of straight lines.

Proof. We use the formula (1.12). If in (1.12) $f = F = 0$, then since $E(U,U)$ is the doubled potential energy (which is positively defined), we have

$$u_1 = c_1 - \varepsilon x_2, \quad u_2 = c_2 + \varepsilon x_1, \quad u_3 = c_3 - \varepsilon x_2, \quad u_4 = c_4 + \varepsilon x_1,$$

where c_k $(k = \overline{1,4})$ are arbitrary real constants, ε is also an arbitrary, real, different from zero constant. Compose nU. Then on the boundary we have

$$(u_1 + iu_2)n_1 + (u_3 + iu_4)n_2 = 0, \tag{1.14}$$

where $n_1 = \frac{dx_2}{ds}$, $n_2 = -\frac{dx_1}{ds}$. Further, we insert these expressions in (1.14) and equate to zero the real and imaginary parts. We obtain

$$(c_1 - \varepsilon x_2)\,\frac{dx_2}{ds} - (c_3 - \varepsilon x_2)\,\frac{dx_1}{ds} = 0,$$
$$(c_2 + \varepsilon x_1)\,\frac{dx_2}{ds} - (c_4 + \varepsilon x_1)\,\frac{dx_1}{ds} = 0.$$

Adding these expressions, we find that

$$\frac{d}{ds}\left[-\frac{\varepsilon}{2}\,(x_1^2 + x_2^2 - 2x_1x_2) + (c_1 + c_2)x_2 - (c_3 + c_4)x_1\right] = 0, \tag{1.15}$$

that is,

$$\frac{\varepsilon}{2}\,(x_1^2 + x_2^2 - 2x_1x_2) - (c_1 + c_2)x_2 + (c_3 + c_4)x_1 - c = 0, \tag{1.16}$$

where c is a new real constant.

Next, compose from (1.16) the discriminant D_1 of the equation (1.16) and the discriminant D_2 of higher terms. In our case, using the well-known formulas from the analytic geometry, we have

$$D_1 = \varepsilon \begin{vmatrix} 1, & -1 \\ -1, & 1 \end{vmatrix} = 0, \quad D_2 = \begin{vmatrix} 1, & -1, & A \\ -1, & 1, & B \\ A, & B, & -\frac{2c}{\varepsilon} \end{vmatrix}.$$

Here, $A = \frac{1}{\varepsilon}(c_3 + c_4)$, $B = \frac{1}{\varepsilon}(c_1 + c_2)$.

Since $D_1 = 0$, the line will be centerless, of parabolic type. If $D_2 = 0$, we have $D_2 = -(A+B)^2 = 0$, i.e. $c_1 + c_2 + c_3 + c_4 = 0$. In this case the line is a pair of straight lines. Thus the theorem is proved. □

In our case, i.e. when D^+ is a circular domain, or D^- is an infinite domain with a circular hole, there takes place the uniqueness of the solution.

2. Solution of the Third Boundary Value Problem in a Circular Domain

The analytic vector φ, appearing in (1.4) is sought in the form

$$\varphi(z) = \frac{m^{-1}}{2\pi i} \int_S \frac{\partial \ln \sigma}{\partial s(y)} (ng + sh)\, ds, \tag{2.1}$$

where g and h are scalar complex periodic functions with the period 2π; $\sigma = z - \zeta$, z and ζ are the affixes of the points x and y, n and s are the unit vectors of the normal and of the tangent, respectively, m^{-1} is the matrix inverse to m, and $\Delta_0 > 0$.

From (2.1) we have

$$\overline{\varphi'(z)} = -\frac{m^{-1}}{2\pi i} \int_S \frac{\partial}{\partial s(y)} \frac{1}{\bar\sigma} (n\bar g + s\bar h)\, ds. \tag{2.2}$$

Inserting (2.1) and (2.2) into (1.4), we obtain

$$U(x) = \frac{1}{2\pi i} \int_S \frac{\partial \ln \sigma}{\partial s(y)} (ng + sh)\, ds + \frac{K}{2\pi i} \int_S \frac{z}{\bar\sigma} (n\bar g + s\bar h)\, ds + \overline{\psi(z)}. \tag{2.3}$$

If we take $\overline{\psi(z)}$ in the form

$$\overline{\psi(z)} = -\frac{1}{2\pi i} \int_S \frac{\partial \ln \bar\sigma}{\partial s(y)} (ng + sh)\, ds - \frac{K}{2\pi i} \int_S \frac{\zeta}{\bar\sigma} (n\bar g + s\bar h)\, ds,$$

then we can write $U(x)$ as follows:

$$U(x) = \frac{1}{\pi} \int_S \frac{\partial \theta}{\partial s(y)} (ng + sh)\, ds + \frac{K}{2\pi i} \int_S \frac{\partial}{\partial s(y)} \frac{\sigma}{\bar\sigma} (n\bar g + s\bar h)\, ds, \tag{2.4}$$

where $\theta = \operatorname{arctg} \frac{x_2 - y_2}{x_1 - y_1}$. The other values appearing in (2.4) have been defined above.

Instead of $U(x)$ we consider the expression

$$U(x) = \frac{1}{\pi} \int_S \left(\frac{\partial \theta}{\partial s(y)} - \frac{1}{2R} \right) (ng + sh)\, ds + \\ + \frac{K}{2\pi i} \int_S \left(\frac{\partial}{\partial s(y)} e^{2i\theta} + \frac{iz}{\bar\zeta R} \right). \tag{2.5}$$

It is evident that if $U(x)$ from (2.4) is a solution of the equation (1.3), then $U(x)$ defined from (2.5) is likewise a solution of (1.3), since the difference between these vectors is a linear function.

If $x \in S$, then $z = Re^{it}$, while $\zeta = Re^{i\tau}$, $2\theta = \pi + t + \tau$ and $e^{2i\theta} = -e^{-i(t+\tau)}$. We now pass to the limit in (2.5) as x tends to the boundary point. We have

$$U^{+}(t) = ng + sh, \tag{2.6}$$

whence $nU^{+} = g = f(t)$.

Here f is the given complex function with the period 2π and having certain smoothness. Thus the function h remains unknown; it will be defined below.

For the projection on the tangent of the stress vector we have

$$(TUs(t))^{+} = \frac{\partial}{\partial s(x)} \left(-2\varphi^{+}(t) + 2\mu U^{+}(t)\right) s(t). \tag{2.7}$$

It follows from (2.1) that

$$\varphi^{+}(t) = \frac{m^{-1}}{2}(ng + sh) + \frac{m^{-1}}{4\pi}\int_0^{2\pi}(ng + sh)\,d\tau + \\ + \frac{m^{-1}i}{2\pi}\int_0^{2\pi} \operatorname{ctg}\frac{\tau - t}{2}(ng + sh)\,d\tau. \tag{2.8}$$

Noticing that $\frac{\partial}{\partial s(x)} = \frac{1}{R}\frac{\partial}{\partial\varphi}$, we have

$$\frac{\partial}{\partial t}\operatorname{ctg}\frac{\tau - t}{2}(ng + sh)d\tau = -\frac{\partial}{\partial\tau}\operatorname{ctg}\frac{\tau - t}{2}(ng + sh)\frac{1}{2}$$

and

$$-2\frac{\partial\varphi^{+}}{\partial\varphi} = -m^{-1}\frac{d}{d\varphi}(ng + sh) + \frac{m^{-1}i}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t - \varphi}{2}\frac{d}{d\tau}(ng + sh)\,d\tau\, s(\varphi)\,dt.$$

If we take into account (2.8) and the fact that $U^{+} = (ng + sh)$, we will get

$$(2\mu - m^{-1})\frac{d}{d\varphi}(ng + sh)s(\varphi) + \\ + \frac{m^{-1}i}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t - \varphi}{2}\frac{d}{dt}(ng + sh)\,dt \cdot s(\varphi) = F(\varphi)\cdot R. \tag{2.9}$$

Multiplying (2.9) by the matrix m, we obtain

$$(A' - E)\frac{d}{d\varphi}(ng + sh)s(\varphi) + \\ + \frac{i}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t - \varphi}{2}\frac{d}{dt}(ng + sh)s(\varphi)\,dt = mF(\varphi)\cdot R, \tag{2.10}$$

where A' is the transposed matrix of A, i.e. $A' = 2m\mu$.

Note that

$$[s(\varphi)-s(t)]\ \mathrm{ctg}\,\frac{t-\varphi}{2}=n(\varphi)+n(t).$$

Using this formula, we tr4ansform the expression (2.10) to the form

$$(A'-E)(g+h)+\frac{i}{2\pi}\int_0^{2\pi}\mathrm{ctg}\,\frac{t-\varphi}{2}\,(g+h)\,dt+ \\ +\frac{i}{2\pi}\int_0^{2\pi}\frac{d}{dt}(ng+sh)\,(n(\varphi)+n(t))\,dt=mRF(\varphi). \tag{2.11}$$

Taking into account that

$$\int_0^{2\pi}\frac{d}{dt}\,(ng+sh)\,dt=0,$$

$$\frac{i}{2\pi}\int_0^{2\pi}\frac{d}{dt}\,(ng+sh)n(t)\,dt=-\frac{i}{2\pi}\int_0^{2\pi}(ng+sh)s(t)\,dt=-\frac{-i}{2\pi}\int_0^{2\pi}h\,dt,$$

we obtain

$$(A'-E)(g+h)+\frac{i}{2\pi}\int_0^{2\pi}\mathrm{ctg}\,\frac{t-\varphi}{2}\,(g+h)dt= \\ =mRF(\varphi)+\frac{i}{2\pi}\int_0^{2\pi}h\,dt, \tag{2.12}$$

whence

$$\frac{A'-E}{2\pi}\int_0^{2\pi}\mathrm{ctg}\,\frac{\tau-t}{2}\,(g+h)dt-\frac{i}{4\pi^2}\int_0^{2\pi}\mathrm{ctg}\,\frac{\tau-t}{2}\,d\tau\int_0^{2\pi}\mathrm{ctg}\,\frac{t-\varphi}{2}\,(g+h)\,dt= \\ =\frac{mR}{2\pi}\int_0^{2\pi}\mathrm{ctg}\,\frac{\tau-t}{2}\,F(t)\,dt. \tag{2.13}$$

We now apply the formula of transposition of singular integrals with the Hilbert kernel (see [5, p. 144]):

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{ctg}\,\frac{t-\varphi}{2}\,dt\int_0^{2\pi}\mathrm{ctg}\,\frac{\tau-t}{2}\,(g+h)\,d\tau=-u(\varphi)+\frac{1}{2\pi}\int_0^{2\pi}u(\tau)\,d\tau, \tag{2.14}$$

where $u=g+h=f(\varphi)+h$.

The equalities (2.13) and (2.14) result in

$$i\,\frac{A'-E}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{\tau-\varphi}{2}\,(g+h)\,d\tau-(g+h)=$$
$$=-\frac{imR}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,F(t)\,dt-\frac{1}{2\pi}\int\limits_0^{2\pi}f(t)\,dt-\frac{1}{2\pi}\int\limits_0^{2\pi}h\,dt, \tag{2.15}$$

and from the equalities (2.13) and (2.15) we easily get

$$\big[(A'-E)^2-E\big]\,[h+f(t)]=$$
$$=-\frac{imR}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,F(t)\,dt-\frac{1}{2\pi}\int\limits_0^{2\pi}f(t)\,dt-\frac{1}{2\pi}\int\limits_0^{2\pi}h\,dt. \tag{2.16}$$

Since $\det\big[(A'-E)^2-E\big]\neq 0$, it follows that $\int\limits_0^{2\pi}h\,dt$ is defined uniquely, and from (2.16) we find that

$$h=-f(t)-(A'-2E)(A')^{-1}\Bigg[\frac{imR}{2\pi}\int\limits_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,F(t)\,dt+$$
$$+\int\limits_0^{2\pi}f(t)\,dt-\frac{1}{2\pi}\int\limits_0^{2\pi}h\,dt\Bigg], \tag{2.17}$$

where

$$A'=2m\mu=\begin{bmatrix}A_1, & A_3\\ A_2, & A_4\end{bmatrix},\quad \det A'>0,\quad \Delta_2=\det(A'-2E)>0$$

and

$$\begin{aligned}
A_1&=\frac{d_1+d_2+a_1b_2-cd}{d_1}+\lambda_5\left(\frac{a_2+c}{d_2}+\frac{a_2+b_2+c+d}{d_1}\right),\\
A_2&=\frac{cb_1-da_1}{d_1}-\lambda_5\left(\frac{a_1+c}{d_2}+\frac{a_1+b_1+c+d}{d_1}\right),\\
A_3&=\frac{cb_2-da_2}{d_1}-\lambda_5\left(\frac{a_2+c}{d_2}+\frac{a_2+b_2+c+d}{d_1}\right),\\
A_4&=\frac{d_1+d_2+a_2b_1-cd}{d_1}+\lambda_5\left(\frac{a_1+c}{d_2}+\frac{a_1+b_1+c+d}{d_1}\right);
\end{aligned} \tag{2.18}$$

here d_1 and d_2 are given by the formula (1.5),

$$\begin{aligned}
\Delta_2d_1d_2&=[\Delta_1-2\lambda_5(a_1+a_2+2c)]\,(b_1b_2-d^2)-2\lambda_5d_2(b_1+b_2+2d)\equiv\\
&\equiv[\Delta_1-2\lambda_5(a_1+a_2+2c)]\,\big[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2\big]-\\
&\quad-\lambda_5(b_1+b_2+2d)\Delta_1>0.
\end{aligned} \tag{2.19}$$

Since $g=f(t)=(nU)^+$, $h=(sU)^+$ is defined from (2.17). Obviously, the function h appearing in (2.17) is Hölder continuous just as f' and F.

Thus we have found the projections on the normal of the displacement vector and on the tangent of the displacement. Using the expressions of the above-mentioned functions and substituting them into the expression for the displacement vector, we obtain the expression for the displacement vector in the form of a Poisson type formula. This formula allows one to find formulas for the stress vector which will likewise be of the Poisson type.

Thus the solution of the third boundary value problem in a circular domain will be finally found by a Poisson type formula.

3. Solution of the Third Boundary Value Problem for an Infinite Domain with a Circular Hole

We seek for a solution in the form

$$\varphi(z)=\frac{m^{-1}}{2\pi i}\int\limits_S \frac{\partial \ln\sigma}{\partial s(y)}\,(ng+sh)\,ds, \tag{3.1}$$

where the values appearing in this expression have been determined in Section 2. The functions g and h will be defined below.

From (3.1) we have

$$\overline{\varphi'(z)}=-\frac{m^{-1}}{2\pi i}\int\limits_S \frac{\partial}{\partial s(y)}\,\frac{1}{\bar\sigma}\,(n\bar g+s\bar h)\,ds. \tag{3.2}$$

Inserting (3.1) and (3.2) in (1.4), we obtain

$$U(x)=\frac{1}{2\pi i}\int\limits_S \frac{\partial \ln\sigma}{\partial s(y)}\,(ng+sh)\,ds+\frac{K}{2\pi i}\int\limits_S \frac{z}{\bar\sigma}\,(n\bar g+s\bar h)\,ds+\overline{\psi(z)}. \tag{3.3}$$

Choose $\overline{\psi(z)}$ as follows:

$$\overline{\psi(z)}=-\frac{1}{2\pi i}\int\limits_S \frac{\partial \ln\bar\sigma}{\partial s(y)}\,(ng+sh)\,ds-\frac{K}{2\pi i}\int\limits_S \frac{\zeta}{\bar\sigma}\,(n\bar g+s\bar h)\,ds.$$

Then the displacement vector $U(x)$ from (3.3) takes the form

$$U(x)=\frac{1}{\pi}\int\limits_S \frac{\partial\theta}{\partial s}\,(ng+sh)\,ds+\frac{K}{2\pi i}\int\limits_S \frac{\partial}{\partial s(y)}\,\frac{\sigma}{\bar\sigma}\,(n\bar g+s\bar h)\,ds, \tag{3.4}$$

where K is defined by virtue of (2.4).

Instead of (3.4) we consider $U(x)$:

$$U(x)=\frac{1}{\pi}\int\limits_S\left(\frac{\partial\theta}{\partial s(y)}-\frac{1}{2R}\right)(ng+sh)\,ds+ \\ +\frac{K}{2\pi i}\int\limits_S\left(\frac{\partial}{\partial s(y)}\,e^{2i\theta}+\frac{i\zeta}{\bar z}\right). \tag{3.5}$$

Obviously, if $U(x)$ defined by (3.4) is a solution of the equation (1.3), then (3.5) will likewise be a solution of (1.3).

Assume that to $x \in S$ there corresponds an angle φ which in no way is connected with the analytic vector φ. Let $z = Re^{it}$ and $\zeta = Re^{i\tau}$. Then $2\theta = \pi + t + \tau$. Passing to the limit as x tends to the point of the boundary S, we obtain

$$U^+(t) = -(ng + sh), \tag{3.6}$$

where $nU^+ = -g = f(t)$, and $f(t)$ is a complex function with the period 2π possesing certain smoothness. Thus h remains still unknown and will be defined later on.

Using the formula (1.6) for the projection on the tangent of the stress vector, we have

$$TUs(t) = \frac{\partial}{\partial s(t)} \left(-2\varphi(t) + 2\mu U(t)\right) s(t), \tag{3.7}$$

whence

$$(TUs(t))^+ = \frac{\partial}{\partial s(t)} \big(-2\varphi^+(t) - 2(ng + sh)\big)\, s(t) = F(t). \tag{3.8}$$

Since $\frac{\partial}{\partial s(t)} = \frac{1}{R}\frac{d}{dt}$, we can rewrite (3.8) as follows:

$$\left[-2\,\frac{d\varphi+}{dt} - 2\mu\,\frac{d(ng+sh)}{dt}\right] s(t) = F(t)R. \tag{3.9}$$

Taking into account our calculations performed in Section 2, we obtain

$$\varphi^+(t) = -\frac{m^{-1}}{2}\,(ng+sh) + \frac{m^{-1}}{4}\int_0^{2\pi} (ng+sh)\,d\varphi -$$
$$-\frac{m^{-1}i}{2\pi}\int_0^{2\pi} \operatorname{ctg}\frac{\tau - t}{2}\,\frac{d}{d\tau}\,(ng+sh)\,dt\,s(t) \tag{3.10}$$

and (3.9) takes the form

$$m^{-1}(-A' + E)\,\frac{d}{dt}\,(ng+sh)s(t) -$$
$$-\frac{m^{-1}i}{2\pi}\int_0^{2\pi} \operatorname{ctg}\frac{t-\varphi}{2}\,\frac{d}{dt}\,(ng+sh)\,dt\cdot s(t) = F(t)\cdot R. \tag{3.11}$$

If we multiply the left-hand side of (3.11) by the matrix m, $\det m = \Delta_0 > 0$, we will find that

$$-(A' - E)\,\frac{d}{dt}\,(ng+sh)s(t) -$$
$$-\frac{i}{2\pi}\int_0^{2\pi} \operatorname{ctg}\frac{t-\varphi}{2}\,\frac{d}{d\tau}\,(ng+sh)\,d\tau\cdot s(t) = mRF(t). \tag{3.12}$$

It follows from (3.11) that

$$[s(\varphi)-s(t)]\ \mathrm{ctg}\,\frac{t-\varphi}{2}=-n(\varphi)+n(t).$$

Taking also into account the formulas

$$\frac{d}{\partial\varphi}\,(ng+sh)s(\varphi)=g+h\quad\text{and}\quad\int\limits_0^{2\pi}n(\varphi)s(\varphi)\,dt=0,$$

from (3.12) we obtain

$$-\,(A'-E)\,(ng+sh)+\frac{i}{2\pi}\int\limits_0^{2\pi}\mathrm{ctg}\,\frac{t-\varphi}{2}\,\frac{d}{dt}\,(ng+sh)s(t)\,dt=$$

$$=mRF(t)-\frac{i}{2\pi}\int\limits_0^{2\pi}f(t)\,dt,\quad(3.13)$$

whence

$$\frac{-(A'-E)}{2\pi}\int\limits_0^{2\pi}\mathrm{ctg}\,\frac{\tau-t}{2}\,(g+h)\,d\tau-i\,(g+h)+\frac{i}{2\pi}\int\limits_0^{2\pi}(g+h)\,d\tau=$$

$$=\frac{mR}{2\pi}\int\limits_0^{2\pi}F(\tau)\ \mathrm{ctg}\,\frac{\tau-t}{2}\,dt.\quad(3.14)$$

Multiplying (3.13) by $-(A'-E)$ and (3.14) by $-i$, after summation we have

$$\big[(A'-E)^2-E\big]\,(g+h)-\frac{i}{2\pi}\int\limits_0^{2\pi}(g+h)\,dt=-\frac{imR}{2\pi}\int\limits_0^{2\pi}F(\tau)\ \mathrm{ctg}\,\frac{\tau-t}{2}\,d\tau-$$

$$-\frac{(A'-E)RmF(x)}{2\pi}-\frac{i(A'-E)}{2\pi}\int\limits_0^{2\pi}f(t)\,dt,\quad(3.15)$$

whence

$$h=f(t)-(A'-2E)(A')^{-1}\Bigg[\frac{i}{2\pi}\int\limits_0^{2\pi}(-f+h)\,dt-\frac{imR}{2\pi}\int\limits_0^{2\pi}F(\tau)\ \mathrm{ctg}\,\frac{\tau-t}{2}\,d\tau-$$

$$-\frac{i(A'-E)RmF(t)}{2\pi}+\frac{i(A'-E)}{2\pi}\int\limits_0^{2\pi}f(t)\,dt\Bigg].\quad(3.16)$$

The formula (3.16) allows one to determine $h_0=\frac{1}{2\pi}\int\limits_0^{2\pi}h\,dt$.

Consequently, $h = (sU)^+$ is determined from the formula (3.16). Having found g and h, we can obtain from (3.5) Poisson type formulas for the displacement vector. It can be easily seen that for the validity of the formula (3.16), the functions f and F must be Hölder continuous.

Thus the solution of the third boundary value problem for an infinite domain with a circular hole is found in its final form.

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