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FUZZY SOLUTIONS FOR MULTI-POINT BOUNDARY VALUE PROBLEMS

Abstract. In this paper, a fixed point theorem for absolute retracts is used to investigate the existence of fuzzy solutions for three and four point boundary value problems for second order differential equations.

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1. Introduction

In modelling real systems one is frequently confronted with a differential equation

$$y''(t) = f(t, y(t)), \quad t \in [0, T], \quad y(0) = y_0, \quad y(T) = y_T,$$

where the structure of the equation is known (represented by the vector field f) but the model parameters and the values y_0 and y_T are not known exactly. One method of treating this in certainty is to use a fuzzy set theory formulation of the problem.

This paper is concerned with the existence of fuzzy solutions for three and four-point boundary value problems for second order differential equations. More precisely, in the first part of Section 3 we will consider the following three-point problem

$$y''(t) = f(t, y(t)), \quad t \in J := [0, 1], \tag{1}$$

$$y(0) = \widehat{0} \in E^n, \quad y(\eta) = y(1), \tag{2}$$

where we let E^n be the set of all upper semi-continuous, convex, normal fuzzy numbers with bounded $\alpha-$level, $f : J \times E^n \to E^n$ a continuous function and $\eta \in [0, 1]$.

The second part of this section will be devoted to the following four-point problem

$$y''(t) = f(t, y(t)), \quad t \in J = [0, 1], \tag{3}$$

$$y(0) = y'(\eta), y(1) = y(\tau), \tag{4}$$

where f, η are as in problem (1)–(2), and $\tau \in [0, 1]$.

The study of multi-point boundary value problem for linear second order ordinary differential equations was initiated by Il'in and Moiseev [5], [6]. In the early 1990's Gupta [4] studied the three-point boundary value problem for nonlinear ordinary differential equations and this paper led to much activity in the area.

Kandel and Byatt [7] introduced the concept of fuzzy differential equations and later it was applied in fuzzy processes and fuzzy dynamical systems. For recent results on fuzzy differential equations, see [2], [8], [9], [11], [12], [14] and the references therein. In this paper using some ideas from [12] we initiate the study multi-point boundary value problems for fuzzy differential equations. Our approach relies on a fixed point theorem in absolute retract spaces [3].

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper.

$CC(\mathbb{R}^n)$ denotes the set of all nonempty compact, convex subsets of $\mathbb{R}^n$. Denote by

$$E^n = \{y : \mathbb{R}^n \to [0, 1] \text{ satisfying (i) to (iv) mentioned below}\} :$$

(i) y is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $y(x_0) = 1$;
(ii) y is fuzzy convex, that is for $x, z \in \mathbb{R}^n$ and $0 < \lambda \leq 1$,

$$y(\lambda x + (1-\lambda)z) \geq \min[y(x), y(z)];$$

(iii) y is upper semi-continuous;
(iv) $[y]^0 = \overline{\{x \in \mathbb{R}^n : y(x) > 0\}}$ is compact.

For $0 < \alpha \leq 1$, we denote $[y]^\alpha = \{x \in \mathbb{R}^n : y(x) \geq \alpha\}$. Then from (i) to (iv), it follows that the $\alpha-$level sets $[y]^\alpha \in CC(\mathbb{R}^n)$. If $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, then, according to Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the function defined by

$$g(y, \overline{y})(z) = \sup_{z=g(x,\bar{z})} \min\{y(x), \overline{y}(\bar{z})\}.$$

It is well known that

$$[g(y, \overline{y})]^\alpha = g([y]^\alpha, [\overline{y}]^\alpha) \quad \text{for all} \quad y, \overline{y} \in E^n \quad \text{and} \quad 0 \leq \alpha \leq 1,$$

and g is continuous. For addition and scalar multiplication, we have

$$[y + \overline{y}]^\alpha = [y]^\alpha + [\overline{y}]^\alpha, \quad [ky]^\alpha = k[y]^\alpha.$$

Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|a - b\|\right\}$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Then $(CC(\mathbb{R}^n), H_d)$ is a complete and separable metric space [13]. We define the supremum metric d_∞ on E^n by

$$d_\infty(u, \overline{u}) = \sup_{0<\alpha\leq 1} H_d([u]^\alpha, [\overline{u}]^\alpha) \quad \text{for all} \quad u, \overline{u} \in E^n.$$

(E^n, d_∞) is a complete metric space and for all $u, v, w \in E^n$ and $\lambda \in \mathbb{R}$ we have

$$d_\infty(u + w, v + w) = d_\infty(u, v)$$

and

$$d_\infty(\lambda u, \lambda v) = |\lambda| d_\infty(u, v).$$

We define $\widehat{0} \in E^n$ as $\widehat{0}(x) = 1$ if $x = 0$ and $\widehat{0}(x) = 0$ if $x \neq 0$. It is well known that (E^n, d_∞) can be embedded isometrically as a cone in a Banach space X, i.e. there exists an embedding $j : E^n \to X$ (see also [9]) defined by

$$j(u) = \langle u, \widehat{0}\rangle \quad \text{where} \quad u \in E^n;$$

here $\langle\cdot,\cdot\rangle$ is defined in [13]. Notice also that

$$\|\langle u, v\rangle\|_X = d_\infty(u, v) \quad \text{for} \quad u, v \in E^n,$$

so, in particular,

$$\|ju\|_X = d_\infty(u, \widehat{0}) \quad \text{for} \quad u \in E^n.$$

The supremum metric H_1 on $C(J, E^n)$ is defined by

$$H_1(w, \overline{w}) = \sup_{t \in J} d_\infty(w(t), \overline{w}(t)).$$

It is well known that $C([0,1], E^n)$ is a complete metric space. Now since $j : E^n \to C \subset X$ we can define a map $\bar{J} : C(J, E^n) \to C(J, X)$ by

$$[\bar{J}x](t) = j(x(t)) = jx(t) \quad \text{for} \quad t \in [0,1];$$

here $x \in C(J, E^n)$ (note that if $x \in C(J, E^n)$ and $t_0, t \in [0,1]$, then by definition of j we have

$$\|[\bar{J}x](t) - [\bar{J}x](t_0)\|_{C([0,1],X)} = \sup_{t \in [0,1]} \|jx(t) - jx(t_0)\| = d_\infty(x(t), x(t_0)).$$

Also, it is easy to check that

$$\bar{J} : C(J, E^n) \to \bar{J}(C(J, E^n))$$

is a homeomorphism. To see that $\bar{J}$ is continuous let $x_n, n \in \mathbb{N}$, $x \in C(J, E^n)$ be such that

$$H_1(x_n, x) = \sup_{t \in [0,1]} d_\infty(x_n(t), x(t)) \to 0 \quad \text{as} \quad n \to \infty.$$

Then

$$\begin{aligned} \|\bar{J}x_n - \bar{J}x\|_{C(J,X)} &= \sup_{t \in [0,1]} \|jx_n(t) - jx(t)\| = \\ &= \sup_{t \in [0,1]} d_\infty(x_n(t), x(t)) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

so $\bar{J}$ is continuous. To see that $\bar{J}^{-1}$ is continuous, let $y_n \in J(C([0,1], E^n))$, $y \in J(C([0,1], E^n))$ with $\|y_n - y\|_{C(J,X)} \to 0$ as $n \to \infty$. Then there exist $x_n, x \in C(J, E^n)$with $\bar{J}x_n = y_n$ and $y = \bar{J}x$. Thus

$$\begin{aligned} H_1(\bar{J}^{-1}y_n, \bar{J}^{-1}y) &= \sup_{t \in [0,1]} d_\infty(\bar{J}^{-1}y_n(t), \bar{J}^{-1}y(t)) = \sup_{t \in [0,1]} d_\infty(y_n(t), y(t)) = \\ &= \sup_{t \in [0,1]} \|jy_n(t) - jy(t))\|_X = \sup_{t \in [0,1]} \|y_n(t) - y(t)\|_X = \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

since $y_n(t) = jx_n(t)$ and $y(t) = jx(t)$. Thus $\bar{J}^{-1}$ is continuous.

Definition 2.1. A map $f : J \to E^n$ is strongly measurable if for all $\alpha \in [0,1]$ the multi-valued map $f_\alpha : J \to CC(\mathbb{R}^n)$ defined by $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue measurable, where $CC(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric H_d.

Definition 2.2. A map $f : J \to E^n$ is called integrably bounded if there exists an integrable function h such that $\|y\| \leq h(t)$ for all $y \in f_0(t)$.

Definition 2.3. Let $f : J \to E^n$. The integral of f over J, denoted $\int_0^1 f(t)dt$, is defined by the equation

$$\left(\int_0^1 f(t)dt\right)^{\alpha} = \int_0^1 f_\alpha(t)dt =$$

$$= \left\{ \int_0^1 v(t)dt \mid v : J \to \mathbb{R}^n \text{ is a measurable selection for } f_\alpha \right\}$$

for all $\alpha \in (0, 1]$.

A strongly measurable and integrably bounded map $f : J \to E^n$ is said to be *integrable* over J, if $\int_0^1 f(t)dt \in E^n$.

If $f : J \to E^n$ is measurable and integrable bounded, then f is integrable.

Definition 2.4. A map $f : J \to E^n$ is called differentiable at $t_0 \in J$ if there exists a $f'(t_0) \in E^n$ such that the limits

$$\lim_{h \to 0+} \frac{f(t_0 + h) - f(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0-} \frac{f(t_0) - f(t_0 - h)}{h}$$

exist and are equal to $f'(t_0)$. Here the limit is taken in the metric space (E^n, H_d). At the end points of J, we consider only the one-side derivatives.

If $f : J \to E^n$ is differentiable at $t_0 \in J$, then we say that $f'(t_0)$ is the *fuzzy derivative* of $f(t)$ at the point t_0. For the concepts of fuzzy measurability and fuzzy continuity we refer to [10].

3. Main Results

In this section we are concerned with the existence of fuzzy solutions for the problems (1)–(2) and (3)–(4). Firstly, we shall present an existence result for the problem (1)–(2).

Definition 3.1. A function $y \in C^2((0, 1), E^n)$ is said to be a solution of (1)–(2) if y satisfies the equation $y''(t) = f(t, y(t))$ on $[0, 1]$ and the condition (2).

We state the fixed point result we will need in Sections 3 and 4. Its proof can be found in [3] (in fact a more general version can be found in [1]).

Theorem 3.2. *Let $X \in AR$ and $F : X \to X$ be a continuous and completely continuous map. Then F has a fixed point.*

Remark 3.3. Recall that a space Z is called an absolute retract (written $Z \in AR$) if Z is metrizable and for any metrizable space W and any embedding $h : Z \to W$ the set $h(Z)$ is a retract of W.

Theorem 3.4. *Let $f : [0, 1] \times E^n \to E^n$ be continuous and assume that the following conditions hold:*

(A1) *There exist a continuous non-decreasing function* $\psi : [0, \infty) \longrightarrow (0, \infty)$ *and* $p \in L^1(J, \mathbb{R}_+)$ *such that*

$$d_\infty(f(t, y), \widehat{0}) \leq p(t)\psi(d_\infty(y, \widehat{0})) \quad for \quad t \in J, \quad y \in E^n;$$

(A2) *There exists* $M > 0$ *with*

$$\frac{M}{\psi(M)\left(1 + \frac{2}{1-\eta}\right)\int\limits_0^1 p(s)ds} \geq 1$$

such that for each $t \in J$ *the set*

$$\left\{ \begin{array}{r} \int\limits_0^t (t-s)f(s, y(s))ds + \dfrac{t}{1-\eta}\int\limits_0^\eta (\eta - s)f(s, y(s))ds \\ -\dfrac{t}{1-\eta}\left[\int\limits_0^1 (1-s)f(s, y(s))ds\right], \quad y \in \mathcal{A} \end{array} \right\}$$

is a totally bounded subset of E^n, *where*

$$\mathcal{A} = \{y \in C(J, E^n) : d_\infty(y(t), \widehat{0}) \leq M, \quad t \in J\}.$$

Then the problem (1)–(2) *has at least one fuzzy solution on* J.

Proof. We transform the problem (1)–(2) into a fixed point problem. It is clear that the solutions of the problem (1)–(2) are fixed points of the operator $N : C(J, E^n) \to C(J, E^n)$ defined by

$$N(y)(t) := \int\limits_0^t (t-s)f(s, y(s))ds + \frac{t}{1-\eta}\int\limits_0^\eta (\eta - s)f(s, y(s))ds - $$
$$- \frac{t}{1-\eta}\left[\int\limits_0^1 (1-s)f(s, y(s))ds\right].$$

Let

$$\mathcal{A} \cong \mathcal{B} \equiv \{\bar{J}y \in C(J, E^n) : y \in C(J, E^n) \text{ and } d_\infty(y(t), \widehat{0}) \leq M), \quad t \in J\}.$$

Clearly, $\mathcal{B}$ is a convex subset of the Banach spaces $C(J, X)$, so in particular $\mathcal{B}$ is an absolute retract. As a result, $\mathcal{A}$ is an absolute retract. We will show that N maps $\mathcal{A}$ into $\mathcal{A}$ and is continuous and completely continuous. The proof will be given in several steps.

Step 1: $N : \mathcal{A} \to \mathcal{A}$.

Let $y \in \mathcal{A}$ and $t \in [0, 1]$. From (A1) we have

$$d_\infty(Ny(t), \widehat{0}) = d_\infty\Bigg(\int\limits_0^t (t-s)f(s, y(s))ds + \frac{t}{1-\eta}\int\limits_0^\eta (\eta - s)f(s, y(s))ds -$$

$$
\begin{aligned}
&- \frac{t}{1-\eta}\Big[\int_0^1 (1-s)f(s,y(s))ds\Big],\widehat{0}\Big) \leq \\
&\leq \int_0^t (t-s)d_\infty(f(s,y(s)),\widehat{0})ds+ \\
&+ \frac{t}{1-\eta}\int_0^\eta (\eta-s)d_\infty(f(s,y(s)),\widehat{0})+ \\
&+ \frac{t}{1-\eta}\int_0^1 (1-s)d_\infty(f(s,y(s)),\widehat{0})ds \leq \\
&\leq \int_0^1 p(s)\psi(d_\infty(y(s),\widehat{0}))ds + \frac{1}{1-\eta}\int_0^1 p(s)\psi(d_\infty(y(s),\widehat{0}))ds+ \\
&+ \frac{1}{1-\eta}\int_0^1 p(s)\psi(d_\infty(y(s),\widehat{0}))ds \leq \\
&\leq \psi(M)(1+\frac{2}{1-\eta})\int_0^1 p(s)ds \leq M.
\end{aligned}
$$

Thus $N(\mathcal{A}) \subset \mathcal{A}$.

Step 2: N *is continuous.*

Let $\{y_n\} \in \mathcal{A}$ be a sequence such that $y_n \to y \in \mathcal{A}$ in $C([0,1],E^n)$.

$$
\begin{aligned}
H_1(Ny_n(t),Ny(t)) = H_1\Big(&\int_0^t (t-s)f(s,y_n(s))ds+ \\
&+ \frac{t}{1-\eta}\int_0^\eta (\eta-s)f(s,y_n(s))ds- \\
&- \frac{t}{1-\eta}\Big[\int_0^1 (1-s)f(s,y_n(s))ds\Big], \\
&\int_0^t (t-s)f(s,y(s))ds + \frac{t}{1-\eta}\int_0^\eta (\eta-s)f(s,y(s))ds- \\
&- \frac{t}{1-\eta}\Big[\int_0^1 (1-s)f(s,y(s))ds\Big]\Big) \leq
\end{aligned}
$$

$$\leq \int_0^1 H_1(f(s,y_n(s)),f(s,y(s)))ds+$$
$$+\frac{1}{1-\eta}\int_0^1 H_1(f(s,y_n(s)),f(s,y(s)))ds+$$
$$+\frac{1}{1-\eta}\int_0^1 H_1(f(s,y_n(s)),f(s,y(s)))ds.$$

Hence

$$H_1(Ny_n,Ny) \leq \left(1+\frac{2}{1-\eta}\right)\int_0^1 H_1(f(s,y_n(s)),f(s,y(s)))ds.$$

Let

$$\rho_n(s) = d_\infty(f(s,y_n(s)),f(s,y(s))).$$

Since f is continuous, we have

$$\rho_n(t) \to 0 \quad \text{as} \quad n\to\infty \text{ for } t\in[0,1].$$

From (A1) we have that

$$\begin{aligned}\rho_n(t) &\leq d_\infty(f(t,y_n(t)),\widehat{0}) + d_\infty(\widehat{0},f(t,y(t))) \leq \\ &\leq p(t)[\psi(d_\infty(y_n(t),\widehat{0})) + \psi(d_\infty(y(t),\widehat{0}))] \leq \\ &\leq 2p(t)\psi(M).\end{aligned}$$

As a result,

$$\lim_{n\to\infty}\int_0^1 \rho_n(s)ds = \int_0^1 \lim_{n\to\infty}\rho_n(s)ds = 0.$$

Thus

$$H_1(Ny_n,Ny) \to 0 \quad \text{as} \quad n\to\infty,$$

so $N:\mathcal{A}\to\mathcal{A}$ is continuous.

Step 3: *$N(\mathcal{A})$ is an equicontinuous set of $C([0,1],E^n)$* .

Let $l_1,l_2\in[0,1]$, $l_1<l_2$, and let $y\in\mathcal{A}$. Then

$$d_\infty(Ny(l_2),Ny(l_1)) = d_\infty\Bigg(\int_0^{l_2}(l_2-s)f(s,y(s))ds+$$
$$+\frac{l_2}{1-\eta}\int_0^{\eta}(\eta-s)f(s,y(s))ds - \frac{l_2}{1-\eta}\left[\int_0^1(1-s)f(s,y(s))ds\right],$$

$$
\begin{aligned}
&\int_0^{l_1}(l_1-s)f(s,y(s))ds+\frac{l_1}{1-\eta}\int_0^{\eta}(\eta-s)f(s,y(s))ds-\\
&\quad-\frac{l_1}{1-\eta}\Big[\int_0^1(1-s)f(s,y(s))ds\Big]\Big)=\\
&=d_\infty\Big(\int_0^{l_1}(l_1-s)f(s,y(s))ds+\int_0^{l_1}(l_2-l_1)f(s,y(s))ds+\\
&\quad+\int_{l_1}^{l_2}(l_2-s)f(s,y(s))ds+\\
&\quad+\frac{l_2-l_1}{1-\eta}\int_0^{\eta}(\eta-s)f(s,y(s))ds+\frac{l_1}{1-\eta}\int_0^{\eta}(\eta-s)f(s,y(s))ds-\\
&\quad-\frac{l_2-l_1}{1-\eta}\int_0^1(1-s)f(s,y(s))ds-\frac{l_1}{1-\eta}\int_0^1(1-s)f(s,y(s))ds,\\
&\int_0^{l_1}(l_1-s)f(s,y(s))ds+\frac{l_1}{1-\eta}\int_0^{\eta}(\eta-s)f(s,y(s))ds-\\
&\quad-\frac{l_1}{1-\eta}\Big[\int_0^1(1-s)f(s,y(s))ds\Big]\Big).
\end{aligned}
$$

As a result,

$$
\begin{aligned}
d_\infty(Ny(l_2),Ny(l_1))&=d_\infty\Big(\int_0^{l_1}(l_2-l_1)f(s,y(s))ds+\int_{l_1}^{l_2}(l_2-s)f(s,y(s))ds+\\
&\quad+\frac{l_2-l_1}{1-\eta}\int_0^{\eta}(\eta-s)f(s,y(s))ds-\\
&\quad-\frac{l_2-l_1}{1-\eta}\int_0^1(1-s)f(s,y(s))ds,\widehat{0}\Big)\le\\
&\le l_2\int_{l_1}^{l_2}d_\infty(f(s,y(s)),\widehat{0}))ds+\int_0^{l_1}(l_2-l_1)d_\infty(f(s,y(s)),\widehat{0}))ds+
\end{aligned}
$$

$$+2\frac{l_2-l_1}{1-\eta}\int_0^1 d_\infty(f(s,y(s)),\widehat{0})ds \leq$$

$$\leq l_2\int_{l_1}^{l_2} p(s)\psi(y(s),\widehat{0}))ds + \int_0^{l_1}(l_2-l_1)p(s)\psi(y(s),\widehat{0}))ds +$$

$$+2\frac{l_2-l_1}{1-\eta}\int_0^1 p(s)\psi(y(s),\widehat{0})ds \leq$$

$$\leq \int_{l_1}^{l_2} l_2 p(s)\psi(M)ds + \int_0^{l_1}(l_2-l_1)\psi(M)ds +$$

$$+2\frac{l_2-l_1}{1-\eta}\int_0^1 p(s)\psi(M)ds.$$

Now Steps 1 to 3, (A2) and the Arzela–Ascoli theorem guarantee that $N : \mathcal{A} \to \mathcal{A}$ is continuous and completely continuous. Theorem 3.2 implies that N has a fixed point y which is a solution of the problem (1)–(2). □

Next we study the four-point problem (3)–(4).

Definition 3.5. A function $y \in C^2((0,1), E^n)$ is said to be a solution of (3)–(4) if y satisfies the equation $y''(t) = f(t, y(t))$ on $[0,1]$ and the condition (4).

Theorem 3.6. *Let $f : [0,1] \times E^n \to E^n$ be continuous and assume that the following conditions hold:*

(A3) *There exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$d_\infty(f(t,y),\widehat{0}) \leq p(t)\psi(d_\infty(y,\widehat{0})) \quad \text{for} \quad t \in J, \quad y \in E^n;$$

(A4) *There exists $M_1 > 0$ with*

$$\frac{M_1}{\psi(M_1)\left((3+\frac{2}{1-\tau})\int_0^1 p(s)ds\right)} \geq 1$$

such that for each $t \in J$ the set

$$\left\{ \begin{array}{c} \int_0^t (t-s)f(s,y(s))ds + \int_0^\eta f(s,y(s))ds + \\ +\frac{1+t}{1-\tau}\left[\int_0^\tau (\tau-s)f(s,y(s))ds + \int_0^1 (1-s)f(s,y(s))ds\right] : y \in \mathcal{A}_1 \end{array} \right\}$$

is a totally bounded subset of E^n, where

$$\mathcal{A}_1 = \{y \in C(J,E^n) : d_\infty(y(t),\widehat{0}) \le M_1,\ t \in J\}.$$

Then the problem (3)–(4) *has at least one fuzzy solution on J.*

Proof. We transform the problem (3)–(4) into a fixed point problem. A simple computation shows that the solutions of the problem (3)–(4) are fixed points of the operator $N_1 : C(J,E^n) \to C(J,E^n)$ defined by

$$N_1(y)(t) := \int_0^t (t-s)f(s,y(s))ds + \int_0^\eta f(s,y(s))ds +$$

$$+\frac{1+t}{1-\tau}\Big[\int_0^\tau (\tau-s)f(s,y(s))ds + \int_0^1 (1-s)f(s,y(s))ds\Big].$$

Set

$$\mathcal{A}_1 = \{y \in C(J,E^n) : d_\infty(y(t),\widehat{0}) \le M_1,\ t \in J\}.$$

Clearly, $\mathcal{A}_1$ is an absolute retract. Now we prove that $N_1(\mathcal{A}_1) \subset \mathcal{A}_1$. Let $y \in \mathcal{A}_1$. Then

$$d_\infty(N_1y(t),\widehat{0}) = d_\infty\Big(\int_0^t (t-s)f(s,y(s))ds + \int_0^\eta f(s,y(s))ds +$$

$$+\frac{1+t}{1-\tau}\Big[\int_0^\tau (\tau-s)f(s,y(s))ds\Big] - \int_0^1 (1-s)f(s,y(s))ds, \widehat{0}\Big) \le$$

$$\le \int_0^t (t-s)d_\infty(f(s,y(s)),\widehat{0})ds + \int_0^\eta d_\infty(f(s,y(s)),\widehat{0})ds +$$

$$+\frac{1+t}{1-\tau}\int_0^\tau (\tau-s)d_\infty(f(s,y(s)),\widehat{0})ds + \int_0^1 (1-s)d_\infty(f(s,y(s)),\widehat{0})ds \le$$

$$\le \int_0^1 p(s)\psi(M_1)ds + \int_0^1 p(s)\psi(M_1)ds +$$

$$+\frac{2}{1-\tau}\int_0^1 p(s)\psi(M_1)ds + \int_0^1 p(s)\psi(M_1)ds =$$

$$= \psi(M_1)\Big(3 + \frac{2}{1-\tau}\Big)\int_0^1 p(s)ds \le M_1.$$

Thus $N_1(\mathcal{A}_1) \subset \mathcal{A}_1$.

Essentially the same reasoning as in Theorem 3.4 guarantees that $N_1 : \mathcal{A}_1 \to \mathcal{A}_1$ is continuous and completely continuous. Now Theorem 3.2 implies that N_1 has a fixed point y which is a solution to the problem (3)–(4). $\square$

References

1. R. P. Agarwal, D. O'Regan and S. Park, Fixed point theory for multimaps in extension type spaces. *J. Korean Math. Soc.* **39** (2002), No. 4, 579–591.
2. J. J. Buckley and Th. Feuring, Fuzzy differential equations. *Fuzzy Sets and Systems* **110** (2000), No. 1, 43–54.
3. J. Dugundji and A. Granas, Fixed point theory. I. *Monografie Matematyczne* [*Mathematical Monographs*], 61. *Państwowe Wydawnictwo Naukowe (PWN), Warsaw,* 1982.
4. C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation. *J. Math. Anal. Appl.* **168** (1992), No. 2, 540–551.
5. V. A. Il'in and E. I. Moiseev, A nonlocal boundary value problem of the first kind for the Sturm-Liouville operator in differential and difference interpretations. (Russian) *Differentsial'nye Uravneniya* **23** (1987), No. 7, 1198–1207.
6. V. A. Il'in and E. I. Moiseev, A nonlocal boundary value problem of the second kind for the Sturm–Liouville operator. (Russian) *Differentsial'nye Uravneniya* **23** (1987), No. 8, 1422–1431.
7. A. Kandel and W. J. Byatt, Fuzzy differential equations. *Proceedings of the International Conference on Cybernetics and Society,* Vol. I, II (*Tokyo/Kyoto,* 1978), 1213–1216, *IEEE, New York,* 1978.
8. O. Kaleva, Fuzzy differential equations. *Fuzzy Sets and Systems* **24** (1987), No. 3, 301–317.
9. O. Kaleva, The Cauchy problem for fuzzy differential equations. *Fuzzy Sets and Systems* **35** (1990), No. 3, 389–396.
10. Y. K. Kim, Measurability for fuzzy valued functions. *Fuzzy Sets and Systems* **129** (2002), No. 1, 105–109.
11. J. J. Nieto, The Cauchy problem for continuous fuzzy differential equations. *Fuzzy Sets and Systems* **102** (1999), No. 2, 259–262.
12. D. O'Regan, V. Lakshmikantham and J. J. Nieto, Initial and boundary value problems for fuzzy differential equations. *Nonlinear Anal.* **54** (2003), No. 3, 405–415.
13. M. Puri and D. Ralescu, Fuzzy random variables. *J. Math. Anal. Appl.* **114** (1986), No. 2, 409–422.
14. S. Seikkala, On the fuzzy initial value problem. *Fuzzy Sets and Systems* **24** (1987), No. 3, 319–330.

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