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RIEMANN–HILBERT PROBLEMS AND YANG–MILLS THEORY

Abstract. Two-dimensional Yang–Mills equations on Riemann surfaces and Bogomol'ny equation are studied using methods of the theory of Riemann–Hilbert problem. In particular, representations of solutions in terms of connections are given and solvability conditions of arising Riemann–Hilbert problems are established.

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1. Motivation of the Problem

Applications of methods of the theory of Riemann–Hilbert problems in modern mathematical and theoretical physics are well known. To mention only the latest spectacular example, A. Connes and D. Kreimer [14] successfully applied those methods to the investigation of the renormalization problem which is of fundamental importance in theoretical physics.

In this paper, we give another example of application of Riemann–Hilbert problems to the investigation of two-dimensional Yang–Mills equations [5]. To this end, we use the methods based on the results of Georgian mathematicians presented in the monographs of N.Mushkhelishvili [40], I. Vekua [53], N. Vekua [54], G. Manjavidze [38], E. Obolashvili [44], G. Khimshiashvili [30], as well as some results of the author [23].

Under the Riemann–Hilbert monodromy problem it will be understood the following problem: a compact Riemann surface X is given together with its discrete finite subset S. Moreover, a representation $\varrho : \pi_1(X \setminus S, z_0) \to \mathrm{GL}_n(\mathbb{C})$ is given. The problem consists in constructing such a system $df = \omega f$ of differential equations on X whose singular set coincides with S, while the group of monodromy induced by this system is $G = \operatorname{im} \varrho \subset \mathrm{GL}_n(\mathbb{C})$. One might require of the sought for system of differential equations to have regular singular points, be of the Fuchs type, or just some of the singular points to be regular, or the system to have apparent singular points.

Systems of equations of *Fuchs type* have always been object of special interest. The reason was probably that by the I. Lappo-Danilevsky theorem such a system can be explicitly constructed from the monodromy matrices $M_1, M_2, \ldots, M_m \in \mathrm{GL}_n(\mathbb{C})$. Riemann–Hilbert problem for Fuchsian systems is also called *Hilbert's* 21*st problem*.

The monodromy representation ϱ enables one also to construct a holomorphic bundle $E' \to X \setminus S$ on the noncompact Riemann surface $X \setminus S$ for which $\nabla' = d - \omega$ will be a holomorphic connection. There exists a construction (which we will present in Section 3) which permits to extend the bundle (E', ∇') to a holomorphic bundle (E, ∇) with a regular connection. Extension is not unique, but there exists a so-called canonical extension (E°, ∇°) whose holomorphic triviality for $X \cong \mathbb{CP}^1$ is a sufficient condition for the solvability of Hilbert's 21st problem [9]. Irreducibility of the representation ϱ is also a sufficient condition for the existence of a system of Fuchs type [9].

Holomorphic classification of holomorphic bundles on compact Riemann surfaces has a long history. It arose in several contexts and after the works of G. Birkhoff, A. Grothendieck, M. Atiyah, D. Mumford, M. Narasimhan and T. Seshadri comprises a finalized theory. A synthesis of the theorems by G. Birkhoff and A.Grothendieck is known in the literature as the Birkhoff–Grothendieck theorem and amounts to the following:

Each holomorphic vector bundle $E \to \mathbb{CP}^1$ on the Riemann sphere $\mathbb{CP}^1$ decomposes into the sum of line bundles: $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$, the integers $k_1 \geq \cdots \geq k_n$ being the Chern numbers of the line bundles.

Classification of holomorphic vector bundles on Riemann surfaces of genus $g \geq 1$ has been accomplished with the aid of holomorphic connections by M. Atiyah [4], who assigned to each bundle $E \to X$ an element $b(E) \in H^1(X; \Omega^1)$ of the cohomology group $H^1(X; \Omega^1)$ whose triviality is necessary and sufficient for the existence of a holomorphic connection on $E \to X$.

D. Mumford [39] determined an important subclass of holomorphic bundles $E \to X$, $g \geq 2$, the so-called *semistable bundles*, while Narasimhan and Seshadri showed that a bundle is semistable if and only if it is induced by an irreducible unitary representation $\varrho : \pi_1(X \setminus \{x_0\}; z_0) \to \mathrm{U}(n)$ of the fundamental group of the surface $X \setminus \{x_0\}$, where $x_0 \in X$ is some point. Let us reproduce here a formulation of this theorem due to S.Donaldson [16]:

An indecomposable holomorphic bundle $E \to X$ is stable if and only if there is a unitary connection ∇ on E having constant central curvature $*F_\nabla = -2\pi i \mu(E)\mathbf{1}$, where $\mu(E) = \mathrm{degree}(E)/\mathrm{rank}(E)$, $*$ is a Hodge operator, and $\mathbf{1}$ is the identity matrix.

This result relates to the Riemann–Hilbert monodromy problem as follows: for a representation $\varrho : \pi_1(X \setminus \{x_0\}, z_0) \to \mathrm{U}(n)$ there exists a system $df = \omega f$ of differential equations on X for which x_0 is a regular singular point and its monodromy coincides with ϱ. Thus $\nabla = d - \omega$ will be a connection with a regular singularity on the holomorphic bundle $E_\varrho \to X$, and since $*F_\nabla$ is constant, one has $D_\nabla * F_\nabla = 0$, which means that ∇ is a Yang–Mills connection [4]. A wider class of Yang–Mills connections can be obtained from the linear elliptic system $\frac{\partial}{\partial \bar{z}} f(x) = A(z) f(z)$ [6], where $\frac{\partial}{\partial \bar{z}}$ is the derivative in the Sobolev sense, $A(z)$ is a square matrix function of rank n with entries of the class L_p. This system is interesting in relation with the following *linear conjugation problem* which can be formulated as follows.

Suppose we are given a matrix function $g : \Gamma \to \mathrm{GL}_n(\mathbb{C})$ of the Hölder class. One must find a piecewise holomorphic vector function $\varphi(t)$ on $U_+ \cup U_-$ which extends continuously to Γ, satisfies the boundary condition $\varphi_+(t) = g(t)\varphi_-(t)$ for all $t \in \Gamma$, and is of finite order at infinity.

This problem is solved with the aid of the so-called *Wiener–Hopf factorization* (which is also often called the *Birkhoff factorization* [46]) of the Hölder class matrix function $g(t)$, which means that $g(t)$ can be represented in the form $g(t) = g_-(t) d_K(t) g_+(t)$, where $g_\pm(t)$ are holomorphic, respectively, on $U_\pm$ and satisfy a finiteness condition at ∞, $d_K(t) = \mathrm{diag}(t^{k_1}, \ldots, t^{k_n})$, with integers $k_1 \geq k_2 \geq \cdots \geq k_n$ [40].

To relate the linear conjugation problem and the Riemann–Hilbert monodromy problem, one must take for $g(t)$ a piecewise constant function which relates to the monodromy matrices $M_1, \ldots, M_m$ via the equality $g(t) = M_j \cdots M_1$ for t belonging to the arc $\langle s_j, s_{j+1} \rangle$, where $s_j \in S$,

$j = 1, \dots, m$. Traditionally such a problem is reduced to a problem of the Hölder class and is then solved using the Wiener-Hopf factorization. We will consider the monodromy problem for the system $\frac{\partial}{\partial \bar{z}} f(x) = A(z) f(z)$, and replace the Wiener-Hopf factorization by the so-called Φ-factorization [51].

A particular case of the aforementioned elliptic system is the *Beltrami equation*, used for investigation of deformations of the holomorphic structures of Riemann surfaces [31]. In our case deformation of holomorphic structure occurs via perturbation of the singular point of the system of equations, whose isomonodromy condition is realized by the *Schlesinger equation*.

We have noted above that according to Lappo-Danilevsky it is possible to express analytically the coefficients of a Fuchs type system by the monodromy matrices, provided these matrices satisfy certain conditions. Lappo-Danilevsky [34] showed that if the monodromy matrices $M_1, \dots, M_m$ are close to $\mathbf{1}$, then the coefficients A_j of the system of differential equations of the Fuchs type $\frac{df}{dz} = \left(\sum_{j=1}^{m} \frac{A_j}{z - s_j} \right) f$ are expressed by the singular points s_j and monodromy matrices M_j via the noncommutative power series

$$A_j = \frac{1}{2\pi i} \tilde{M}_j + \sum_{1 \le k, l \le n} \xi_{kl}(s) \tilde{M}_k \tilde{M}_l + \cdots ,$$

where ξ_{kl} are functions depending on the singular points which can be given as explicit functions of s, $s \in S$, and $\tilde{M}_j = M_j - \mathbf{1}$. Algebraic version of the Riemann–Hilbert monodromy problem is known in the differential Galois theory under the name of *inverse problem*.

In this context the Riemann–Hilbert monodromy problem in the class of Yang–Mills connections takes the following form: for a prescribed monodromy and a fixed finite set of points on a given Riemann surface, construct a Yang–Mills connections whose monodromy representation and singular points coincide with the given data.

2. Monodromy Problem for Generalized Analytic Vector

In this section we present some results on L_p-connections which owe much to numerous helpful discussions with B.Bojarski which the author had in last five years. Our approach and results are based on the theory of generalized analytic functions [53] and vectors [8]. A part of this section was already presented in [24].

Let $\mathrm{L}^p(\Gamma)$ be the space of Lebesgue measurable functions satisfying the condition that the norm $||f||_{\mathrm{L}^p(\Gamma)} = \left(\int_\Gamma |f(\tau)|^p |d\tau| \right)^{\frac{1}{p}} < \infty$, is finite. It is well known that $\mathrm{L}^p(\Gamma)$ is a Banach space with the above norm.

Consider the singular integral operator $(S_\Gamma f)(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} dt, \quad t \in \Gamma$. This operator is bounded on $\mathrm{L}^p(\Gamma)$ and $S_\Gamma^2 = \mathbf{1}$. Let us introduce the following projectors: $P_\Gamma = \frac{\mathbf{1} + S_\Gamma}{2}$, $Q_\Gamma = \frac{\mathbf{1} - S_\Gamma}{2}$.

Functions $f \in L^p_+(\Gamma)$ can be identified with functions $\hat{f}$ holomorphic in U_+ so that $\hat{f}$ is an analytic continuation of f to U_+. Here $L^p_+(\Gamma)$ denotes the space of those holomorphic functions on U_+ whose boundary values are functions from $L^p(\Gamma)$; similarly let $L^p_-(\Gamma)$ denote the space of those holomorphic functions on U_- whose extension to Γ gives an element of $L^p(\Gamma)$. Let also $L^\infty(\Gamma)$ be the Banach space of Lebesgue measurable and essentially bounded functions.

Definition 2.1 ([51])**.** Factorization of a matrix-function $G \in L^\infty(U)^{n\times n}$ in the space $L^p(\Gamma)$ is its representation in the form

$$G(t) = G_+(t)\Lambda(t)G_-(t), \quad t \in \Gamma, \tag{2.1}$$

where $\Lambda(t) = \operatorname{diag}(t^{k_1}, \dots, t^{k_n})$, $k_i \in \mathbb{Z}$, $i = 1, \dots, n$, $G_+ \in L_+(\Gamma)^{n\times n}$ and $G_+^{-1} \in L^q_+(\Gamma)^{n\times n}$, $G_- \in L^q_-(\Gamma)^{n\times n}$, and $G_-^{-1} \in L^p_-(\Gamma)^{n\times n}$, $\frac{1}{p} + \frac{1}{q} = 1$.

We say that G admits the *canonical* factorization in $L^p(\Gamma)$ if $k_1 = \dots = k_n = 0$. This definition implies that the operator $G_-^{-1}Q_\Gamma G_+^{-1}$ is defined on the dense subspace of the space $L^p(\Gamma)^n$ consisting of those rational vector-functions which are allowed to have poles on Γ, and maps this subspace onto $L^1(\Gamma)^n$. If this operator is bounded in the L^p norm, then it can be extended to the whole $L^p(\Gamma)^n$ and the obtained operator is still bounded, in which case the representation (2.1) from Definition 2.1 will be called the Φ-*factorization* of $G(t)$. It is known that a matrix-function $G \in L^\infty(\Gamma)^{n\times n}$ is Φ-factorizable in the space $L^p(\Gamma)$ if and only if the operator $P_\Gamma + GQ_\Gamma$ is Fredholm on the space $L^p(\Gamma)^n$ [17].

Let us consider the particular case concerned with the subspace $PC(\Gamma)^{n\times n}$ of piecewise continuous matrix-functions. For the elements of this subspace there exist one-sided limits $G(t+0)$ and $G(t-0)$ for each $t \in \Gamma$. For such matrix-functions a necessary and sufficient condition for the existence of Φ-factorization is given by the following theorem.

Theorem 2.1 ([17])**.** *A matrix-function $G \in PC(\Gamma)^{n\times n}$ is Φ-factorizable in the space $L_p(\Gamma)$ if and only if*

a) *the matrices $G(t+0)$ and $G(t-0)$ are invertible for each $t \in \Gamma$;*
b) *for each $j = 1, \dots, n$ and $t \in \Gamma$ one has $\frac{1}{2\pi}\arg\lambda_j(t) + \frac{1}{p} \notin \mathbb{Z}$.*

Here $\lambda_1(t), \dots, \lambda_n(t)$ are eigenvalues of the matrix-function $G(t-0)G(t+0)^{-1}$.

If a matrix-function G is Φ-factorizable, then $\xi_j(\tau) = \frac{1}{2\pi}\arg\lambda_j(\tau)$ is a single-valued function taking its values in the interval $\left(\frac{1}{p} - 1, \frac{1}{p}\right)$.

Suppose G has m singular points $s_1, \dots, s_m \in \Gamma$. Then

$$\kappa = \sum_{k=1}^{m}\left[\frac{1}{2\pi}\arg\det G(t)\right]_{t=a_k+0}^{a_{k+1}-0} + \sum_{k=1}^{m}\sum_{j=1}^{n}\xi_j(s_k). \tag{2.2}$$

It can be seen from (2.2) that κ depends on $L_p(\Gamma)$. If the $\lambda_j(\tau)$ are positive real numbers, then $\xi_j(\tau) = 0$ and consequently κ does not depend on the space $L_p(\Gamma)$.

Suppose now that $G \in PC(\Gamma)^{n\times n}$ is a piecewise constant matrix function with the singular points $s_1, \dots, s_m \in \Gamma$ occurring in this order on Γ. Suppose G is factorizable in the space $L_p(\Gamma)$. Let us denote $M_k = G(s_k - 0)G(s_k + 0)^{-1}$, $k = 1, \dots, m$. Thus G is constant on the arc (s_k, s_{k+1}), and clearly $M_1 M_2 \cdots M_k = 1$. Suppose that the monodromy matrices are similar to the matrices $\exp(-2\pi i E_k)$ and the eigenvalues of E_k belong to the interval $\left(\frac{1}{p} - 1, \frac{1}{p}\right)$, where the matrices E_k are determined uniquely up to similarity since the length of that interval is 1. The numbers $\xi_1(s_k), \dots, \xi_n(s_k)$ are equal to real parts of the eigenvalues of E_k. This implies that for the index κ one has the formula $\kappa = \sum_{k=1}^{m} \operatorname{tr} E_k$. Thus the matrices $E_1, \dots, E_k$ depend on the space $L_p(\Gamma)$. They also depend on the choice of the branches of logarithms for eigenvalues of the matrices M_j. Thus $G \in PC(\Gamma)^{n\times n}$ produces two m-tuples $(M_1, \dots, M_m)$ and $(E_1, \dots, E_m)$ of matrices.

Let

$$\frac{df}{dz} = \Omega(z)f(z) \tag{2.3}$$

be a system of differential equations with regular singularities, having $s_1, \dots, s_m$ as singular points, and ∞ as an apparent singular point. It is known that such a system has n linearly independent solutions in a neighborhood of any regular point.

Let us denote such a fundamental system of solutions by $F(\tilde{z})$. It is possible to characterize $F(\tilde{z})$ by its behavior near the singular points $s_1, \dots, s_m$, using the monodromy matrices $M_1, \dots, M_m$ which are determined by the matrices $E_1, \dots, E_m$, and by the behavior at ∞ which is characterized by partial indices $k_1, \dots, k_m$. Therefore it is said that the system (2.3) has the *standard form* with respect to the matrices $(M_1, \dots, M_m)$ and $(E_1, \dots, E_m)$ satisfying the condition $M_1 \cdots M_m = 1$ such that M_k are similar to $\exp(-2\pi i E_k)$, $k = 1, \dots, m$, and E_j are not resonant, with singular points $s_1, \dots, s_m$ and partial indices $k_1 \geq \cdots \geq k_n$, if

- i) $s_1, \dots, s_m$ are the only singular points of (2.3), with ∞ as an apparent singular point;
- ii) the monodromy group of (2.3) is conjugate to the subgroup of $GL_n(\mathbb{C})$ generated by the matrices $M_1, \dots, M_m$;
- iii) in a neighborhood U_j of the point s_j the solution has the form $F(\tilde{z}) = Z_j(z)(\tilde{z} - s_j)^{E_j} C$, where $Z_j(z)$ is an analytic and invertible matrix-function on $U_j \cup \{s_j\}$ and C is a nondegenerate matrix;
- iv) the solution of the system in a neighborhood U_∞ of ∞ has the form $F(z) = \operatorname{diag}(z^{k_1}, \dots, z^{k_n}) Z_\infty(z) C, \quad z \in U_\infty$, with $Z_\infty(z)$ holomorphic and invertible on U_∞.

Theorem 2.2 ([17]). *Suppose $G \in PC(\Gamma)^{n\times n}$ is a piecewise constant function with jump points $s_1, \dots, s_m$. Suppose G has a Φ-factorization in* *the space* $\mathrm{L}_p(\Gamma)$*,* $1 < p < \infty$*, and* $(M_1, \dots, M_m)$*,* $(E_1, \dots, E_m)$ *are matrices associated to* G *on* $\mathrm{L}_p(\Gamma)$*.*

Suppose that there exists a system of differential equations in the standard form (2.3) *with the singular points* $s_1, \dots, s_m$ *and partial indices* $\kappa_1, \dots, \kappa_m$*. Let* $F_1(z)$*,* $F_2(z)$ *be a fundamental system of its solutions in* U_+ *and* $U_- \setminus \{\infty\}$*.*

Then there exist nondegenerate $n \times n$*-matrices* C_1 *and* C_2 *such that* $G(t) = G_+(t)\Lambda(t)G_-(t)$ *is a* Φ*-factorization of* G *in* $\mathrm{L}_p(\Gamma)$*, where* $\Lambda(t) = \operatorname{diag}(t^{k_1}, \dots, t^{k_n})$*,* $G_+(z) = C_1^{-1}F_1^{-1}(z)$*,* $z \in U_+$*,* $G_-(z) = \Lambda^{-1}(z)F_2(z)C_2$*,* $z \in U_- \setminus \{\infty\}$*.*

Let Γ be a simple closed contour, $s_1, \dots, s_m \in \Gamma$ and $M_1, \dots, M_m \in GL_n(\mathbb{C})$. We say that the piecewise constant matrix function $G(t)$ is induced by the collections $s = \{s_1, \dots, s_m\}$, $M = \{M_1, \dots, M_m\}$ if it is constructed in the following manner: $G(t) = M_j \cdot \dots \cdot M_1$, if $t \in [s_j, s_{j+1})$, where M_j is the monodromy matrice corresponding to going along a small loop around singular point s_j.

Theorem 2.3. *Let*

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{s_1, \dots, s_m\}) \to GL_n(\mathbb{C}) \tag{2.4}$$

be a representation such that $(\rho(\gamma_1) = M_1, \dots, \rho(\gamma_m) = M_m)$ *and* $(E_1, \dots, E_m)$ *is admissible.*

Then the Riemann–Hilbert monodromy problem for the representation (2.4) *is solvable if* $G(t)$ *admits a canonical factorization in* $L^\alpha(\Gamma)$ *for some* $\alpha > 1$ *sufficiently close to 1.*

Proof. It is known that for the given monodromy matrices $M_1, \dots, M_m$ and singular points $s_1, \dots, s_m$ there exists a system of differential equations of the form

$$df = \omega f \tag{2.5}$$

such that $s_1, \dots, s_m$ are the poles of first order for (2.5) and ∞ is an apparent singular point, the matrices $M_1, \dots, M_m$ are monodromy matrices of (2.5), and the solution of (2.5) in the neighborhood of the singular point s_j has the form: $\Phi_j(\tilde{z}) = U_j(z)(\tilde{z} - s_j)^{E_j}C$, where the matrix function $U_j(z)$ is invertible and analytic in the neighborhood of s_j and C is a non-degenerate matrix; in the neighborhood of ∞ the solution has the form:

$$\Phi_\infty(\tilde{z}) = \operatorname{diag}(k_1, \dots, k_n)U_\infty(z)C, \tag{2.6}$$

where $U_\infty(z)$ is analytic and invertible at ∞ [17]. By theorem 2.1, the piecewise constant matrix function $G(t)$ admits a Φ-factorization, therefore $\xi_j(\tau) = \frac{1}{2\pi}\arg\lambda_j(\tau)$ is a single-valued function taking values in the interval $\left(\frac{1}{p} - 1, \frac{1}{p}\right)$. From the factorization condition $G(t) = G_+(t)\Lambda(t)G_-(t)$ and by Theorem 2.2 we have

$$G_+(z) = C_1^{-1}F_1^{-1}(z),\ z \in U_+,\ G_-(z) = \Lambda^{-1}(z)F_2(z)C_2,\ z \in U_- \setminus \{\infty\}.$$

By assumption $G(t)$ admits a canonical factorization, i.e., $k_1 = \cdots = k_n = 0$. From this it follows that ∞ is a regular point of the system (2.5). □

The global theory of generalized analytic functions, both in one-dimensional [53] and multi-dimensional cases [8], involves studying the space of horizontal sections of a holomorphic line bundle with connection on a complex manifold with singular divisor. In this context one needs to require that a connection is complex analytic. An interesting class of such connections is given by L_p-connections, and their moduli spaces have many applications. Such connections and their moduli spaces are the object of intensive study [52], [19].

We study the holomorphic vector bundles with L_p-connections from the viewpoint of the theory of generalized analytic vectors [8]. To this end, we consider a matrix elliptic system of the form:

$$\partial_{\bar{z}} \Phi(z) = A(z)\Phi(z). \tag{2.7}$$

The system (2.7) is a particular case of the *Carleman–Bers–Vekua system* [53]

$$\partial_{\bar{z}} f(z) = A(z)f(z) + B(z)\overline{f(z)}, \tag{2.8}$$

where $A(z)$, $B(z)$ are bounded matrix functions on a domain $U \subset \mathbb{C}$ and $f(z) = (f^1(z), \ldots, f^n(z))$ is unknown vector function. The solutions of the system (2.8) are called *generalized analytic vectors* by analogy with the one-dimensional case [53], [8].

Along with similarities between the one-dimensional and multi-dimensional cases, there also exist essential differences. One of them, as noticed by B.Bojarski [8], is that there can exist solutions of the system (2.7) for which there is no analog of the Liouville theorem on the constancy of bounded entire functions.

We present first some necessary fundamental results of the theory of generalized analytic functions [53], [6], [7], [8] in the form convenient for our purposes. A modern consistent exposition of this theory was given by A.Soldatov [48], [49], [50].

Let us define two differential operators on $\mathrm{W}_p(U)$

$$\partial_{\bar{z}} : \mathrm{W}_p(U) \to \mathrm{L}_p(U), \quad \partial_z : \mathrm{W}_p(U) \to \mathrm{L}_p(U),$$

by setting $\partial_{\bar{z}} f = \theta_1$, $\partial_z f = \theta_2$. The functions θ_1 and θ_2 are called the generalized partial derivatives of f with respect to $\bar{z}$ and z respectively. Sometimes we will use a shorthand notation $f_{\bar{z}} = \theta_1$ and $f_z = \theta_2$. It is clear that ∂_z and $\partial_{\bar{z}}$ are linear operators satisfying the Leibniz equality.

Define the following singular integral operator in the Banach space $\mathrm{L}_p(U)$:

$$T : \mathrm{L}_p(U) \to W_p(U), \quad T(\omega) = -\frac{1}{\pi} \iint\limits_U \frac{\omega(t)}{t-z} dU, \quad \omega \in L_p(U). \tag{2.9}$$

It is known [53] that in one-dimensional case a solution of (2.7) can be represented as

$$\Phi(z) = F(z)\exp(\omega(z)), \tag{2.10}$$

where F is a holomorphic function in U, and $\omega = -\frac{1}{\pi}\int\int_U \frac{A(z)}{\xi - z} dU$. In multi-dimensional case an analog of the factorization (2.10) is given by the following result.

Theorem 2.4 ([6])**.** *Each solution of the matrix equation* (2.7) *in* U *can be represented as*

$$\Phi(z) = F(z)V(z), \tag{2.11}$$

where $F(z)$ *is an invertible holomorphic matrix function in* U*, and* $V(z)$ *is a single-valued matrix function invertible outside* $\overline{U}$*.*

The above representation of solutions to (2.7) will be used for constructing a holomorphic vector bundle on the Riemann sphere and for computing the monodromy matrices of the elliptic system (2.7). We recall some properties of solutions to (2.7). The product of two solutions is again a solution. From Theorem 2.4 it follows (see also [20]) that the solutions constitute an algebra and the invertible solutions are a subfield of this algebra.

Proposition 2.1. *Let* $C(z)$ *be a holomorphic matrix function. Then* $[C(z), \partial_{\overline{z}}] = 0$.

Proof. Indeed,

$$[C(z), \partial_{\overline{z}}]\Phi(z) = C(z)\partial_{\overline{z}}\Phi(z) - \partial_{\overline{z}}C(z)\Phi(z) = C(z)\partial_{\overline{z}}\Phi(z) - C(z)\partial_{\overline{z}}\Phi(z) = 0.$$

Here we have used that $\partial_{\overline{z}}C(z) = 0$. □

Definition 2.2. Two systems $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ and $\partial_{\overline{z}}\Phi(z) = B(z)\Phi(z)$ are called gauge equivalent if there exists a non-degenerate holomorphic matrix function $C(z)$ such that $B(z) = C(z)A(z)C(z)^{-1}$.

Proposition 2.2. *Let the matrix function* $\Psi(z)$ *be a solution of the system* $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ *and let* $\Phi_1(z) = C(z)\Phi(z)$*, where* $C(z)$ *is a nonsingular holomorphic matrix function. Then* $\Phi(z)$ *and* $\Phi_1(z)$ *are solutions of the gauge equivalent systems. The converse is also true: if* $\Phi(z)$ *and* $\Phi_1(z)$ *satisfy systems of equations*

$$\partial_{\overline{z}}\Phi(z) = A(z)\Phi_(z), \quad \partial_{\overline{z}}\Phi_1(z) = B(z)\Phi_1(z)$$

and $A(z) = C^{-1}(z)B(z)C(z)$*, then* $\Phi_1 = D(z)\Phi(z)$ *for some holomorphic matrix function* $D(z)$*.*

Proof. By Proposition 2.1 we have $C(z)\partial_{\overline{z}}\Phi_1(z) = A(z)C(z)\Phi_1(z)$, and therefore $\Phi_1(z)$ satisfies the equation $\partial_{\overline{z}}\Phi_1(z) = C^{-1}(z)A(z)C(z)\Phi_1(z)$. To prove the converse, let us substitute in $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ in place of $A(z)$ the expression of the form $C^{-1}B(z)C(z)$ and consider $\partial_{\overline{z}}\Phi_1(z) = C^{-1}B(z)C(z)\Phi(z)$. Hence $C(z)\partial_{\overline{z}}\Phi(z) = B(z)C(z)\Phi(z)$. But for the left hand side of the latter equation we have $C(z)\partial_{\overline{z}}\Phi(z) = \partial_{\overline{z}}C(z)\Phi(z)$. Therefore

$$\partial_{\overline{z}}(C(z)\Phi(z)) = B(z)(C(z)\Phi(z)).$$

From this it follows that Φ and $C\Phi$ are solutions of equivalent systems, which means that $\Phi_1 = D\Phi$. □

The above arguments for solutions of (2.7) are of a local nature, so they are applicable for an arbitrary compact Riemann surface X, which enables us to construct a holomorphic vector bundle on X. Moreover, using the solutions of the system (2.7) one can construct a matrix 1-form $\Omega = D_{\overline{z}}FF^{-1}$ on X which is analogous to holomorphic 1-forms on Riemann surfaces.

Let X be a Riemann surface. Denote by $L_p^{\alpha,\beta}(X)$ the space of L_p-forms of the type $(\alpha,\beta), \alpha,\beta = 0,1$, with the norm $\|\omega\|_{L_p^{\alpha,\beta}(X)} = \sum_j \|\omega\|_{L_p^{\alpha,\beta}(U_j)}$, where $\{U_j\}$ is an open covering of X, and denote by $W_p(U) \subset L_p(U)$ the subspace of the functions which have generalized derivatives.

We define the operators

$$D_z = \frac{\partial}{\partial z} : W_p(U) \to L_p^{1,0}(U), \quad f \mapsto \omega_1 dz = \partial_z f dz,$$
$$D_{\bar z} = \frac{\partial}{\partial \bar z} : W_p(U) \to L_p^{0,1}(U), \quad f \mapsto \omega_2 d\,\bar z = \partial_{\overline{z}} f d\overline{z}.$$

It is clear that $D_{\bar z}^2 = 0$ and hence the operator $D_{\bar z}$ can be used to construct the *de Rham cohomology*.

Let us denote by $\mathbb{C}L_p^1(X)$ the *complexification* of $L_p^1(X)$, i.e., $\mathbb{C}L_p^1(X) = L_p^1(X) \otimes \mathbb{C}$. Then we have the natural decomposition

$$\mathbb{C}L_p^1(X) = L_p^{1,0}(X) \oplus L_p^{0,1}(X) \tag{2.12}$$

according to the eigenspaces of the Hodge operator $* : L_p^1(X) \to L_p^1(X)$, $* = -\imath$ on $L_p^{1,0}(X)$ and $* = \imath$ on $L_p^{0,1}(X)$. The decomposition (2.12) splits the operator $d : L_p^0(X) \to L_p^0(X)$ into the sum $d = D_z + D_{\bar z}$.

Next, let as above, $\mathcal{E} \to X$ be a C^∞-vector bundle on X, $L_p(X,\mathcal{E})$ be the sheaf of the L_p-sections of $\mathcal{E}$ and let $\Omega \in L_p^1(X,\mathcal{E}) \otimes GL_n(\mathbb{C})$ be a matrix valued 1-form on X. If the above arguments are applied to the complex $L_p^*(X,\mathcal{E})$ with covariant derivative $\bigtriangledown_\Omega$, we obtain again the decompositions of the space $\mathbb{C}L_p^1(X,\mathcal{E})$ and the operator $\bigtriangledown_\Omega$:

$$\mathbb{C}L_p^1(X,\mathcal{E}) = L_p^{1,0}(X,\mathcal{E}) \oplus L_p^{0,1}(X,\mathcal{E}), \quad \bigtriangledown_\Omega = \bigtriangledown'_\Omega + \bigtriangledown''_\Omega.$$

Locally, on the domain U, we have $\bigtriangledown_\Omega^U = d_U + \Omega$, where $\Omega \in L_p^1(X,U) \otimes GL_n(\mathbb{C})$ is a 1-form. Therefore $\bigtriangledown_\Omega^U = (D_z + \Omega_1) + (D_{\bar z} + \Omega_2)$, where Ω_1 and Ω_2 are, respectively, holomorphic and anti-holomorphic part of the matrix-valued 1-form on U. We say that a W_p-section f of the bundle $\mathcal{E}$ with L_p-connection is holomorphic if it satisfies the system of equations

$$\partial_{\bar z} f(z) = A(z) f(z), \tag{2.13}$$

where $A(z)$ is an $n\times n$ matrix-function with entries in $L_p^0(X)\otimes GL_n(\mathbb{C})$ and $f(z)$ is a vector function $f(z)=(f_1(z),f_2(z),\dots,f_n(z))$, or in equivalent form (2.13) reads: $D_{\overline{z}}f=\Omega f$, where $\Omega\in L_p^1(X)\otimes GL_n(\mathbb{C})$.

We now use the above arguments for constructing a holomorphic vector bundle over the Riemann sphere $\mathbb{CP}^1$ by means of the system (2.7). Let $\{U_j\}$, j=1,2, be an open covering of $\mathbb{CP}^1$. Then in any domain U_j, a solution $\Phi(z)$ can be represented as $\Phi(z)=V_j(z)F(z)$, where $V_j(z)$ is a holomorphic non-degenerate matrix function on $U_j^c-S_j$ where S_j is a finite set of points. Restrict $\Phi(z)$ on $(U_1^{\mathbb{C}}\cap U_2^{\mathbb{C}})-S=(U_1\cup U_2)^{\mathbb{C}}-S,\ S=S_1\cup S_2$ and consider the holomorphic matrix-function $\varphi_{12}=V_1(z)V_2(z)^{-1}$ on $(U_1\cup U_2)^{\mathbb{C}}-S$. It is a cocycle and therefore defines a holomorphic vector bundle $\mathcal{E}'$ on $\mathbb{CP}^1-S$. From the Proposition 2.2 it follows that $\mathcal{E}'\to\mathbb{CP}^1-S$ is independent of the choice of solutions in the same gauge equivalence class. The extension of this bundle to a holomorphic vector bundle $\mathcal{E}\to\mathbb{CP}^1$ can be done by a well-known construction (see Section 3) and the obtained bundle is holomorphically nontrivial.

It is now possible to verify that the operator $\frac{\partial}{\partial\bar{z}}+\Omega(z,\overline{z})$ is a L_p-connection of this bundle. It turns out that its index coincides with the index of Cauchy–Riemann operator on X. This follows since the index of Cauchy–Riemann operator is equal to the Euler characteristic of the sheaf of holomorphic sections of the holomorphic vector bundle $\mathcal{E}$.

Consider now a related problem. For a given loop $G:\Gamma\to GL_n(\mathbb{C})$, find a piecewise continuous generalized analytic vector $f(z)$ with the jump on the contour Γ such that on Γ it satisfies the conditions

$$\text{a) } f^+(t)=G(t)f^-(t),\ \ t\in\Gamma,\qquad \text{b) } |f(t)|\le c|z|^{-1},\ \ |z|\to\infty.$$

It is known that for G there exists a Birkhoff factorization, i.e., $G(t)=G_+(t)d_K(t)G_-(t)$. Setting this equality in a) we obtain the following boundary problem $G_+^{-1}(t)f^+(t)=d_K(t)G_-(t)f^-(t)$. Since $G_+^{-1}(t)f^+(t),\ f^+(t)$ and $G_-(t)f^-(t),\ f^-(t)$ are solutions of the gauge equivalent systems, the holomorphic type of the corresponding vector bundle on the Riemann sphere is defined by $K=(k_1,\dots,k_n)$.

Proposition 2.3. *The cohomology groups $H^i(\mathbb{CP}^1,\mathcal{O}(\mathcal{E}))$, $H^i(\mathbb{CP}^1,\mathcal{G}(\mathcal{E}))$ are isomorphic for $i=0,1$, where $\mathcal{O}(\mathcal{E})$ and $\mathcal{G}(\mathcal{E})$, respectively, are the sheaves of holomorphic and generalized analytic sections of $\mathcal{E}$.*

From this proposition it follows that the number of linearly independent solutions of the Riemann–Hilbert boundary problem is equal to $\sum_{k_j<0}k_j$. Its holomorphic type is determined by an integer vector. In terms of cohomology groups $H^i(\mathbb{CP}^1,\mathcal{O}(\mathcal{E}))$ and $H^i(\mathbb{CP}^1,\mathcal{G}(\mathcal{E}))$ one can describe the number of solutions and stability of the Riemann–Hilbert problem [8]. The topological constructions related with the sheaf $\mathcal{O}(\mathcal{E})$ can be extended to the sheaf $\mathcal{G}(\mathcal{E})$ [23].

Theorem 2.5. *There exists a one-to-one correspondence between the space of gauge equivalent Carleman–Bers–Vekua systems and the space of holomorphic structures on the bundle $E \to X$.*

For the investigation of the monodromy problem for a Pfaff system, an important role is played by a representation of solution of the system in exponential form, which in one-dimensional case was studied by W.Magnus in [36]. We use iterated path integrals and the theory of formal connections (as a paralel transport operator) developed by K.-T.Chen [12].

Let $\Omega_1, \ldots, \Omega_r$ be $m \times m$ matrix forms with entries from $L^1_p(X)$. The iterated integral of $\Omega_1, \ldots, \Omega_r$ is introduced as follows: consider the form product of matrix forms $\Omega = \Omega_1, \ldots, \Omega_r$ and define the iterated integral of Ω element-wise.

Proposition 2.4. *The parallel transport corresponding to the elliptic system* (2.7) *has an exponential representation.*

Since the elliptic system (2.7) defines a connection, the proof of the proposition follows from the general theory of formal connections. From the identity $\partial_{\overline{z}}\Phi\Phi^{-1} = \Omega$ it follows that the singular points of Ω are the zeros of the matrix function Φ, in particular, this refers to ∞. This means that it makes sense to speak of singular and apparent singular points of the system (2.7).

From the integrability of (2.7) it follows that for the iterated integral $\int \Omega\Omega\ldots\Omega$ we have $d\int \Omega\Omega\ldots\Omega = 0$ and therefore we have a representation of the fundamental group $\pi_1(X - S, z_0)$. We can say that $z_i \in \{z_1, \ldots, z_m\}$ is a regular singular point of (2.7) if any element of $F(z)$ has at most polynomial growth as $z \to z_i$. If the solution $\Phi(z)$ at any singular point z_i, $i = 1, \ldots, m$, has a regular singularity, then we call the system (2.7) a regular system.

In case $n = 1$ the singular integral (2.10) is well studied. In particular, it is known that $\omega(z)$ is holomorphic in $\mathbb{C}_m \setminus \overline{U_{z_0}}$ and equal to zero at infinity. Here $\mathbb{C}_m = \mathbb{CP}^1 \setminus \{z_1, \ldots, z_m\}$.

Let $\tilde{z} \in U_{z_0}$ be any point and let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be loops at $\tilde{z}$ such that γ_i goes around z_i without going around any $z_j \neq z_i$. Consider the holomorphic continuation of the function $F(z)$ around γ_i. Then we obtain an analytic element $\tilde{F}(z)$ of the holomorphic function $F(z)$ related to the latter by the equality $\tilde{F}_i(z) = m_i F(z)$, where $m_i \in \mathbb{C}^*$. It is independent of the choice of the homotopy type of the loop γ_i. Therefore, we obtain a representation of the fundamental group $\pi_1(\mathbb{CP}^1 \setminus \{z_1, \ldots, z_m\}, \tilde{z}) \to \mathbb{C}^*$, which is defined by the correspondence $\gamma_i \to m_i$. Let us sum up all what was said above.

Proposition 2.5. *Let the system* (2.7) *have regular singularities at the points $z_1, \ldots, z_m$. Then it defines a monodromy representation of the fundamental group*

$$\rho : \pi_1(\mathbb{C} \setminus \{z_1, \ldots, z_m\}, \tilde{z}) \to GL_n(\mathbb{C}).$$

In this situation the monodromy matrices are given by Chen's iterated integrals

$$\rho(\gamma_j) = 1 + \int\limits_{\gamma_j} \Omega + \int\limits_{\gamma_j} \Omega\Omega + \int\limits_{\gamma_j} \Omega\Omega\Omega + \cdots + \cdots \tag{2.14}$$

The convergence properties of the series (2.14) can be described as follows. Let a 1-form Ω be smooth except the points $s_1, \dots, s_m \in X$. Let, as above, $S = \{s_1, s_2, \dots, s_m\}$ and $X_m = X - S$. Thus, for every $\gamma \in PX_m$, there exists a constant $C > 0$ such that

$$\Big| \int\limits_{\gamma_j} \overbrace{\Omega \dots \Omega}^{r} \Big| = O\Big(\frac{C^r}{r!}\Big)$$

and the series (2.14) converges absolutely [27].

3. G-Systems of Differential Equations

The concept of G-system of differential equations emerged in relation with investigation of connections with regular singularities on principal bundles over Riemann surfaces. It is well-known that in the classical case there exists a direct connection between the Riemann–Hilbert boundary problem and the Riemann–Hilbert monodromy problem. An analog of this connection exists in the context of Lie groups and G-bundles (see [23], [30]) and we describe it below.

Let G be a connected complex Lie group, M a complex manifold and P a holomorphic principal G-bundle on M. Then there is an exact sequence of vector bundles on M

$$0 \to \operatorname{ad} P \to Q(P) \to TM, \tag{3.1}$$

where TM is the tangent bundle of M, $\operatorname{ad} P$ is the vector bundle associated to P and $Q(P)$ is the bundle of G-invariant tangent vector fields on P. Here and in the sequel P also denotes the total space of the bundle.

Definition 3.1 ([5])**.** A holomorphic connection on a principal bundle $P \to M$ is called integrable if the splitting of (3.1) is G-invariant.

The following proposition was established by M. Atiyah.

Proposition 3.1 ([5])**.** *A holomorphic principal bundle $P \to M$ with the structure group G possesses an integrable connection if and only if it is induced by a representation of the fundamental group $\varrho : \pi_1(M) \to G$.*

Let $G = \mathrm{GL}_n(\mathbb{C})$ and let $E \to M$ be a vector bundle. If E is induced by a representation $\varrho : \pi_1(M) \to G$, then there is a system of differential equations with holomorphic coefficients $df = \omega f$ whose monodromy coincides with the given representation, moreover, ω will be a connection of this bundle, and its holomorphy implies its complete integrability. Proposition

3.1 and the Birkhoff–Grothendieck theorem imply that a holomorphic vector bundle $E \to \mathbb{CP}^1$ possesses a holomorphic connection if and only if the type of the splitting has the form $\kappa = (0, \dots, 0)$, i.e., iff the bundle is trivial.

Let G be a compact connected Lie group of rank r and let $G_{\mathbb{C}}$ be the complexification of G, so that $G_{\mathbb{C}}$ is a reductive group. Let us denote by $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ the Lie algebras of the groups G and $G_{\mathbb{C}}$ respectively. Note that any Lie algebra can be complexified: $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. If G is a Lie group and $\mathfrak{g}$ its Lie algebra, then under the complexification of G is understood a complex Lie group $G_{\mathbb{C}}$ whose Lie algebra is $\mathfrak{g}_{\mathbb{C}}$. Such a complexification need not exist in general. If G is isomorphic to a subgroup of a unitary group $\mathrm{U}(n)$ for sufficiently large n, then $G_{\mathbb{C}}$ can be considered as a subgroup of the complexification of the unitary group $\mathrm{U}(n)_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$. Thus for compact groups there always exists a complexification, unique up to isomorphism.

Denote by $L_{an}G$ the group of real analytic loops. If G is embedded in the unitary group U_n, so that a loop γ in G is a matrix-valued function and can be expanded in Fourier series $\gamma(z) = \sum_{j=-\infty}^{\infty} \gamma_j z^j$, then the real-analytic loops are those for which this series converges in some annulus $r \le |z| \le r^{-1}$ with $r < 1$, i.e., such that $||\gamma_j r^{-|j|}||$ is bounded for all j for some $r < 1$. The natural topology on $L_{an}G$ is got by regarding it as the direct limit of the Banach Lie groups $L_{an,r}G$ consisting of the functions holomorphic in $r \le |z| \le r^{-1}$; the group $L_{an,r}G$ has the topology of uniform convergence. $L_{an}G$ is a Lie group with the Lie algebra $L_{an}\mathfrak{g}$.

Denote by $L_{rat}G$ the subgroup of rational loops, i.e., loops which, when regarded as matrix-valued functions, have entries which are rational functions of z with no poles on $|z| = 1$. Denote by $L^{\pm}G_{\mathbb{C}}$ the subgroups of $LG_{\mathbb{C}}$ which consist of the loops from $LG_{\mathbb{C}}$, which are boundary values of holomorphic $G_{\mathcal{C}}$-valued functions defined on $U^{\pm}$, respectively. Here $U^+ = \{z : |z| \le 1\}$ and $U^- = \{z : |z| \ge 1\}$ as above. Analogously, denote by $PCL^{\pm}G_{\mathbb{C}}$ the group of piecewise continuous loops $S^1 \to G$. Suppose there exist the one-side limits $g(t+0)$ and $g(t-0)$ for each $t \in S^1$.

We say that a loop $g \in \mathrm{L}^{\infty}(\Gamma, G)$ is Φ-factorizable in the space $\mathrm{L}^p(\Gamma, G)$ if and only if the operator $P_\Gamma + GQ_\Gamma$ is Fredholm on the space $\mathrm{L}^p(\Gamma, G)$ [28],[29]. Then we have the following sufficient condition of solvability of the Riemann–Hilbert monodromy problem for G-systems.

Theorem 3.1. *If a loop $g(t) \in \Omega G$ has the factorization $g(t) = g^+(t)g^-(t)$, then the Riemann–Hilbert problem is solvable.*

Thus to establish solvability it is sufficient to check that all partial G-indices vanish which can be done using formulae from [1].

As it was remarked in the previous chapter, for any vector bundle there exists a connection which has regular singularities in the given points. This result can be generalized for holomorphic principal G-bundles. For this purpose a system of the form $Df = \alpha$ is considered in this section, where α is a $\mathfrak{g}$-valued 1-form defined on the manifold M, and $f : M \to G$ is a G-valued unknown function.

Let $r_g : G \to G$ be the right shift on the group G, let $\mathrm{C}(X, G)$ be the group of all smooth functions $f : X \to G$ and let $\Lambda^p(X, G)$, $p = 0, 1, 2$, be the space of all $\mathfrak{g}$-valued p-forms on X. We define now the operator

$$D : \Lambda^0(X, \mathfrak{g}) \to \Lambda^1(X, \mathfrak{g}) \tag{3.2}$$

by the formula $D_x(f)(u) = dr^{-1}_{f(x)}(df)_x(u)$.

Definition 3.2. An expression of the form

$$Df = \alpha, \tag{3.3}$$

where α is a $\mathfrak{g}_{\mathbf{C}}$-valued 1-form on X and $f : X \to G_{\mathbf{C}}$ is an unknown smooth function, is called a G-system of differential equations.

For G-systems, it is possible to formulate the Riemann–Hilbert problem as follows: whether for a given homomorphism $\rho : \pi_1(M) \to G$ there exists a G-system whose monodromy coincides with ρ. It is known that solution of this problem depends on the group G.

If $G = \mathrm{U}(n)$, then $Df = df \cdot f^{-1}$ and α is a matrix of 1-forms on X, so that one obtains a usual system of the form $df = \omega f$. If $n = 1$, then $G_{\mathbb{C}} = \mathbb{C}^*$ and $Df = d \log f$, the logarithmic derivative of the function f.

Let $* : \Lambda^1(X; \mathfrak{g}) \to \Lambda^1(X; \mathfrak{g})$ be the Hodge operator. Then the complexification of the de Rham complex $\Lambda^p_{\mathbb{C}}(X; \mathfrak{g})$, $p = 0, 1, 2$, decomposes into the direct sum $\Lambda^1_{\mathbf{C}}(X; \mathfrak{g}) = \Lambda^{1,0}(X; \mathfrak{g}) \oplus \Lambda^{0,1}(X; \mathfrak{g})$ by the requirement that $* = -i$ on $\Lambda^{1,0}(X; \mathfrak{g})$ and $* = i$ on $\Lambda^{0,1}(X; \mathfrak{g})$. The operator D decomposes into the direct sum $D = D' \oplus D''$, where

$$D' : \Lambda^0(X; \mathfrak{g}) \to \Lambda^{1,0}(X; \mathfrak{g}), \quad D'' : \Lambda^0(X; \mathfrak{g}) \to \Lambda^{0,1}(X; \mathfrak{g})$$

are determined by the formulae

$$D'_x(f)(u) = d'_{r^{-1}_{f(x)}}(d'f)_x(u), \quad D''_x(f)(u) = d'' r^{-1}_{f(x)}(d''f)_x(u).$$

A $G_{\mathbf{C}}$-valued function $f : X \to G_{\mathbb{C}}$ is called holomorphic (resp. antiholomorphic) if $D''f = 0$ (resp. $D'f = 0$).

The operator D has the following properties:

1) it is a crossed homomorphism, i.e., $D(f \cdot g) = (Df)_x + (\operatorname{ad} f(x)) \circ (Dg)_x$ for any $f, g \in \mathrm{C}(X, G)$. Note that the operator D'' is also a crossed homomorphism.
2) the kernel $\ker D$ consists of constant functions.

Definition 3.3. We will say that the system (3.3) is completely integrable if for any $x_0 \in X$ and $g_0 \in G$ there exists in a neighborhood of x_0 a solution f of this system with $f(x_0) = g_0$. A point x_0 is called isolated singular point of a map $f : U \to G_{\mathbb{C}}$ if there is a punctured neighborhood U_{x_0} such that the map f is analytic along any path $\gamma \subset U_{x_0}$ circling around x_0.

The properties 1), 2) of the operator D imply that if f_0 is some solution of the system (3.3), then $f = f_0 h$ is also a solution for any $h \in \ker D$, i.e., the solution is uniquely determined up to multiplication by a constant.

Definition 3.4. We will say that a $G_{\mathbb{C}}$-valued function $f \in \Omega(U_\varepsilon(x_0))$ is of polynomial growth if for each sector

$$S = \{z \mid \theta_0 \le \arg z \le \theta_1,\ 0 \le |z| < \varepsilon\},$$

where z denotes a local coordinate system on X, there exist, for sufficiently small ε, an integer $k > 0$ and a constant c such that the inequality $d(f(z), \mathbf{1}) < c|z|^{-k}$ is valid, where $d(\_, \mathbf{1})$ denotes the distance from the unit of the group $G_{\mathbb{C}}$.

Under integration of $G_{\mathbb{C}}$-valued functions is understood the multiplicative integral for Lie groups and algebras. Let $\gamma \subset U$ be a smooth arc with the parameterizing map $z : [a, b] \to U$. Multiplicative integral along the arc γ is by definition $\int\limits_\gamma (1 + f(z))dz := \int\limits_a^b (1 + f(z))z'(t)dt$, where 1 denotes the unit element of $G_{\mathbb{C}}$. If γ is a closed arc, then $M_f(\gamma) = \oint\limits_\gamma (1 + f(z))dz$ is an invertible element of $\mathfrak{g}$ called the holonomy of the map f with respect to γ.

Consider the system (3.3) on $\mathbb{CP}^1$. Let f_0 be a solution of the G-system (3.3) in the neighborhood $U \subset \mathbb{CP}^1$ of the point z_0 having polynomial growth at the points from the set $S = \{z_1, \ldots, z_m\}$. After continuation of f_0 along a path $\gamma_i \in \pi_1(\mathbb{CP}^1 \setminus S, z_0)$ starting and ending in z_0 and circling around a singular point z_i, the solution f_0 transforms into another solution f_1. As noted before, $\gamma_i^* f_0 = g_i f_1$ for some $g_i \in G$. Thus f_0 determines a representation

$$\varrho : \pi_1(\mathbb{CP}^1 \setminus S) \to G_{\mathbb{C}}. \tag{3.4}$$

The image $\operatorname{im} \varrho \subset G_{\mathbb{C}}$ is called the *monodromy group* of the G-system (3.3) and the representation (3.4) induces a principal $G_{\mathbb{C}}$-bundle $P'_\varrho \to \mathbb{CP}^1 \setminus S$, the form α being a holomorphic connection for this bundle.

Let us extend the bundle $P'_\varrho \to \mathbb{CP}^1 \setminus S$ to a holomorphic principal bundle $P_\varrho \to \mathbb{CP}^1$. Let $\gamma_1, \ldots, \gamma_m \in \pi_1(\mathbb{CP}^1 \setminus S, z_0)$ be generators satisfying the relation $\gamma_1 \cdots \gamma_m = e$. Let us denote $B_i = \varrho(\gamma_i)$ and let A_i be elements of $G_{\mathbb{C}}$ with $B_i = \exp A_i$, $i = 1, \ldots, m$. To extend the bundle $P'_\varrho \to \mathbb{CP}^1 \setminus S$ into some point $z_i \in S$, let us cover $\mathbb{CP}^1 \setminus S$ in the same way as in Section 2, with transition functions on $V_j \cap U_{i_1}$, for $z_j \in V_j$, chosen to be $g_{01} := \exp(A_j \ln(z - z_j))$. Then on the intersections $V_j \cap U_{ik} \cap U_{i_1}$ one will have the equality $g_{0k} = g_{01} \cdot B_i = g_{01} \cdot g_{1k}$. In such a way one obtains a holomorphic $G_{\mathbb{C}}$-bundle $P_\varrho \to \mathbb{CP}^1$ with connection α, i.e., $P_\varrho \to \mathbb{CP}^1$ is induced by a system of the form (3.3) and the Atiyah class $a(P_\varrho)$ is nontrivial. This means that $P_\varrho \to \mathbb{CP}^1$ does not admit holomorphic connections and hence the system (3.3) must have singular points. Here and in the sequel under singular points will be meant critical singular points, i.e., ramification points of the solution.

The Birkhoff stratum Ω_κ consists of the loops from $\mathrm{L}\, G_{\mathbb{C}}$ with fixed partial indices $K = (k_1, \ldots, k_r)$. Topology of Ω_K is investigated in [29]. Existence of a one-to-one correspondence between the Birkhoff strata Ω_K and holomorphic equivalence classes of principal bundles on $\mathbb{CP}^1$ is a straightforward generalization of the analogous theorem for holomorphic vector bundles. More precisely, the following theorem holds.

Theorem 3.2 ([46])**.** *Each loop $f \in \Omega G$ determines a pair (P, τ), where P is a holomorphic principal $G_{\mathbb{C}}$-bundle on $\mathbb{CP}^1$ and τ is a smooth section of the bundle $P|_{\bar{X}_\infty}$ holomorphic in X_∞, and if (P', τ') and (P, τ) are holomorphically equivalent bundles, then f' and f lie in the same Birkhoff stratum.*

The theorem implies that to each principal bundle with a fixed trivialization there corresponds a tuple of integers $(k_1, \dots, k_r)$ which completely determine the holomorphic type of the principal bundle and hence if a holomorphic principal G-bundle is induced by a system of the form (3.3) without singular points, then this bundle is trivial.

Suppose G is a connected compact Lie group and $G_{\mathbb{C}}$ is its complexification; $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ are the Lie algebras of the groups G and $G_{\mathbb{C}}$, respectively; Z is the centrum of the group $G_{\mathbb{C}}$, and Z_0 is the connected component of the unit; X is a compact connected Riemann surface of genus $g \geq 2$. If $\tilde{X} \to X$ is a universal covering and $\varrho : \pi_1(X) \to G_{\mathbb{C}}$ is a representation, then the corresponding principal bundle will be denoted by P_ϱ.

Let $x_0 \in X$ be a fixed point and $p : \tilde{X} \to X \setminus \{x_0\}$ be a universal cover. Then the triple $(\tilde{X}, p, X \setminus \{x_0\})$ is a principal bundle whose structure group Γ is a free group on $2g$ generators, and if γ is a loop circling around x_0, then $\gamma = \prod_{i=1}^{g}[a_i, b_i]$, where a_i, b_i are generators of $\Gamma \cong \pi_1(X \setminus \{x_0\})$ and $[\_, \_]$ denotes the commutator.

Let $P'_\varrho \to X \setminus \{x_0\}$ be the principal bundle corresponding to the representation $\varrho : \pi_1(X \setminus \{x_0\}) \to G_{\mathbb{C}}$. Since by Theorem 3.2 each loop $f : S^1_X \to G$ determines a holomorphic principal $G_{\mathbb{C}}$-bundle, using f one can extend the bundle $P'_\varrho \to X \setminus \{X_0\}$ to X in the following way: let U_{x_0} be a neighborhood of x_0 homeomorphic to a unit disc and consider the trivial bundles $Ux_0 \times G_{\mathbb{C}} \to U_{x_0}$ and $P'_\varrho \to X \setminus \{x_0\}$. Let us glue these bundles over the intersection $(X \setminus \{x_0\}) \cap U_{x_0} = U_{x_0} \setminus \{x_0\}$ using the loop f. We thus obtain an extended bundle $P_\varrho \to X$.

Consider the homomorphism of fundamental groups $f_* : \pi_1(S^1_X) \to \pi_1(G_{\mathbb{C}})$ induced by f and suppose that γ is a generator of $\pi_1(S^1_X)$ mapped to $+1$ under the isomorphism $\pi_1(S^1_X) \cong \mathbb{Z}$. If $f' : S^1_X \to G_{\mathbf{C}}$ is homotopic to f, then $f'_* = f_*$, and f and f' correspond to topologically equivalent $G_{\mathbb{C}}$-bundles on X. Conversely, for any element $c \in \pi_1(G_{\mathbb{C}})$ there exists $f_* : \pi_1(S^1_X) \to \pi_1(G_{\mathbb{C}})$ with $f_*(\gamma) = c$.

Let $P \to X$ be a principal bundle and f the corresponding loop.

Definition 3.5. The element $\chi(P) := f_*(\gamma) \in \pi_1(G_{\mathbb{C}})$ of the fundamental group is called the characteristic class of the bundle P.

It is easy to see that the map $\chi : H^1(X; \mathrm{C}^\infty(G_\mathbb{C})) \to \pi_1(G_\mathbb{C})$ determined by the formula $\chi(P) = c$ for each $P \in H^1(X; \mathrm{C}^\infty(G_\mathbb{C}))$ is surjective. Here $\mathrm{C}^\infty(G_\mathbb{C})$ denotes the sheaf of germs of continuous maps $X \to G_\mathbb{C}$.

Let $\varrho : \pi_1(X \setminus \{x_0\}) \to G_\mathbb{C}$ be a representation such that $\varrho(S^1_X) = c \in Z_0$. If $\tilde{Z}_0$ is the Lie algebra of the group Z_0, then $\exp : \tilde{Z}_0 \to Z_0$ is a universal covering. Let us choose an element $\alpha \in \tilde{Z}_0$ such that $\exp \alpha = c$. Extend the bundle $P'_\varrho \to X \setminus \{x_0\}$ to X using the loop $f : S^1_X \to G$ with $f(z) = \exp(\alpha \ln(z - x_0))$ on S^1_X. Denote the obtained principal bundle by $P_{\varrho,\alpha} \to X$.

Definition 3.6. The space $H \subset G$ is called irreducible if

$$\{Y \in \mathfrak{g} \mid \forall h \in H \ \ \mathrm{ad}\, h(Y) = Y\} = \ \mathrm{center}\mathfrak{g}.$$

The representation $\varrho : \Gamma \to G_\mathbb{C}$ is called unitary if $\varrho(\Gamma) \subset G$, and $\varrho : \Gamma \to G$ is called irreducible if $\varrho(\Gamma)$ is irreducible.

The following theorem holds.

Theorem 3.3 ([47])**.** *Let ϱ and ϱ' be unitary representations of the group $\Gamma \cong \pi_1(X \setminus \{x_0\})$ in G. The bundles $P_{\varrho,\beta}$ and $P_{\varrho',\beta'}$ are holomorphically equivalent if and only if ϱ and ϱ' are equivalent in K and $\beta = \beta'$.*

Let M be any connected Riemann surface (compact or not) an let $\varrho : \pi_1(M) \to G_\mathbb{C}$ be any homomorphism. The following theorem from [45] is important for our considerations.

Theorem 3.4 ([45])**.** 1) *If $\pi_1(M)$ is a free group and $G_\mathbb{C}$ is connected, then ϱ is the monodromy homomorphism for the system* (3.3).

2) *If $\pi_1(M)$ is a free abelian group and G is a connected compact Lie group with torsion free cohomology, and if $\mathrm{im}\, \varrho \subset G$, then ϱ is the monodromy homomorphism for some system of the type* (3.3).

Theorem 3.4 is a solution of the Riemann–Hilbert problem for holomorphic systems of the type (3.3). In particular, 1) implies that if $M = X \setminus \{x_0\}$, then for any representation $\varrho : \pi_1(X \setminus \{x_0\}) \to G_\mathbb{C}$ there exists a G-system with the monodromy homomorphism ϱ. We also need some concepts and constructions used in [45].

Lemma 3.1 ([23])**.** *If there is a lifting of ϱ to $\tilde{\varrho} : \pi_1(M) \to \tilde{G}_\mathbb{C}$, then ϱ is the monodromy homomorphism of the G-system* (3.3).

Definition 3.7. A holomorphic principal $G_\mathbb{C}$-bundle $P \to X$ is called stable (resp. semistable) if for any reduction $\sigma : X \to P/B$ the degree of the vector bundle $T_{G/B}$ is positive (resp. nonnegative), where B is a maximal parabolic subgroup of G and $T_{G/B}$ is the tangent bundle along the fibres of the bundle $P/B \to X$.

The following theorem gives a criterion of stability of holomorphic principal bundles on X.

Theorem 3.5. *A holomorphic $G_{\mathbb{C}}$-bundle $P \to X$ is stable if and only if it is of the form $P_{\varrho,\alpha}$ for some irreducible unitary representation $\varrho : \pi_1(X \setminus \{x_0\}) \to G$ such that $\varrho(\gamma) = c \in Z_0$, $\alpha \in \tilde{Z}_0$ and $\exp \alpha = c$.*

The theorem implies that if G is a semisimple group, then a G-bundle is stable if and only if it is induced by some irreducible unitary representation of the fundamental group $\pi_1(X)$.

Consider on a Riemann surface X a G-system of differential equations

$$Df = \omega \tag{3.5}$$

which has a regular singularity at the point x_0 and the monodromy homomorphism of the system (3.5) is such that $\varrho(\gamma) = c \in Z_0$. Let P_ϱ be the principal G-bundle over the noncompact Riemann surface $X \setminus \{x_0\}$. Let us extend this bundle to the whole X in the following way: let $\alpha \in \tilde{Z}_0$ be an element with $\exp \alpha = c$, and let $\tilde{\varrho}(\gamma) = \beta$, where $\tilde{\varrho} : \pi_1(X \setminus \{x_0\}) \to \tilde{G}_{\mathbf{C}}$ is a lifting of ϱ to the covering of $G_{\mathbb{C}}$. As the transition function, let us take the $G_{\mathbb{C}}$-valued function $g_{12}(z) = \exp(-z\beta)$. After gluing trivial $G_{\mathbb{C}}$-bundles over U and $X \setminus \{x_0\}$ using the function $g_{12}(z)$, one obtains a $G_{\mathbb{C}}$-bundle $P_{\varrho,\alpha} \to X$ which is an extension of $P_\varrho \to X \setminus \{x_0\}$.

Theorem 3.6. *A stable holomorphic principal $G_{\mathbb{C}}$-bundle has a connection θ with regular singularity at the given point x_0.*

Proof. Indeed, let $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper half-plane and $H \to X$ be a covering with the single ramification point $x_0 \in X$ with ramification index m. Then the Fuchsian group Γ realizing X as a quotient $X = H/\Gamma$ is generated by elements $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma$ with the relations

$$\left(\prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}\right) \gamma = 1, \ \gamma^m = 1. \tag{3.6}$$

It is clear that $\Gamma \cong \pi_1(X \setminus \{x_0\})$, and by Theorem 3.5 the bundle $P \to X$ has the form $P_{\varrho,\alpha}$. By Lemma 3.1, for the representation $\varrho : \pi_1(X \setminus \{x_0\}) \to G_{\mathbb{C}}$ there exists a G-system $Df = \theta$ with a singularity at the point x_0 whose monodromy representation coincides with ϱ. The form θ is Γ-invariant and thus is a connection for the bundle P. The relations (3.6) imply that x_0 is a regular singular point of the equation $Df = \theta$. □

Proposition 3.2. *Let the monodromy representation of a G-system $Df = \omega$ with one regular singular point x_0 be unitary, irreducible and $\varrho(\gamma) = c \in Z_0$, where γ is a loop circling around x_0. Then the characteristic class of the principal bundle $P_{\varrho,\alpha}$ corresponding to this G-system equals $\beta - \alpha$.*

Proof. We apply the following fact which is well known in algebraic geometry. If G is a reductive group with connected center $Z(G)$, then $G_1 = [G, G]$ is a semisimple group and the homomorphism $Z(G) \times G_1 \to G$ has a finite kernel. Let $\tilde{G}_1$ be the universal cover of G_1 and $\tilde{Z}_0$ be the universal cover of $Z(G)$. Then $\tilde{G} = \tilde{Z}_0 \times \tilde{G}_1 \to G$ is the universal cover of G.

The embedding $Z_0 \hookrightarrow G$ canonically induces an embedding $\tilde{Z}_0 \subset \tilde{Z}_0 \times \tilde{G}_1$, and $\tilde{Z}_0$ can be identified with its image. Since $S_X^1 = \prod_{i=1}^{g}[a_i, b_i]$ and $\pi_1(X \setminus \{x_0\})$ is a free group, there exists a lifting of the homomorphism $\varrho : \pi_1(X \setminus \{x_0\}) \to G$, i.e., the diagram

$$\begin{array}{ccc} & & \tilde{G} \\ & {}^{\tilde{\varrho}}\nearrow & \\ \pi_1(X \setminus \{x_0\}) & & \downarrow^{\tilde{\varrho}} \\ & {}_{\varrho}\searrow & \\ & & G \end{array}$$

commutes. Since $\varrho(S_X^1)$ lies in the center of the group G and $\pi\tilde{\varrho} = \varrho$, it follows that $\tilde{\varrho}(\gamma) = \beta$ lies in the kernel of p, i.e., in $\pi_1(G)$. The element β does not depend on the lifting of the homomorphism ϱ and by the definition of the characteristic class one obtains $\chi(P_{\varrho,\alpha}) = \beta - \alpha$. □

The action of the gauge group $\mathcal{G}(P)$ is defined as $A \mapsto ad(g)A + dgg^{-1}$. The tangent space to the space of equivalence classes of G-representations of the group $\pi_1(X)$ is $T_\rho Hom(\pi_1(X), G)/G = H^1(\pi_1(X), ad\rho)$.

Definition 3.8. The logarithmic connection of the holomorphic principal bundle $P \to X$ is the first order differential operator

$$\nabla : \Omega^0(adP) \to \Omega^0(adP) \otimes \Omega_X^1(\log S).$$

We say that a connection ∇ on a principal bundle P is Fuchsian if (adP, ∇) is a holomorphic vector bundle with connection. Let $G_{\mathbb{C}} = GL_n(\mathbb{C})$ and let the system $df = \omega f$ be Fuchsian. Transform the monodromy matrices $M_j, j = 1, 2, \ldots, m$, to upper-triangular form by some matrices C_j. Assume that $\Psi_j, j = 1, 2, \ldots, m$, are diagonal integer matrices whose entries φ_j^i satisfy the inequalities $\varphi_j^1 \geq \varphi_j^2 \geq \cdots \geq \varphi_j^n$.

Consider the local section $U_j(z)$ of the principal $GL_n(\mathbb{C})$-bundle $\mathbf{P}_\rho \to X$ over $V_j \backslash s_j$ such that the corresponding $\Phi_j(z)$ has the form

$$\Phi_j\,(\tilde{z}) = U_j(z)(z - s_j)^{\Psi_j}(\tilde{z} - s_j)^{E_j}. \tag{3.7}$$

The following proposition describes extensions of $\mathbf{P}_\rho$.

Proposition 3.3. *Every extension of* $\mathbf{P}'_\rho \to X_m$ *to the points* s_j *which is induced by a connection* ∇ *with at most logarithmic singularities at* s_j *is determined by matrices* C_j *and* Ψ_j *such that*

1) $C_j^{-1} G_j C_j$ *is upper triangular,*
2) $\Psi_j = \mathrm{diag}(\varphi_j^1, \varphi_j^2, \ldots, \varphi_j^n), \varphi_j^i \in \mathbb{Z}, \varphi_j^1 \geq \varphi_j^2 \geq \cdots \geq \varphi_j^n$.

Extend $\mathbf{P}'_\rho \to X_m$ in a similar way to all singular points. Denote by C the collection $(C_1, C_2, \ldots, C_m)$ and by Ψ the collection $(\Psi^1, \Psi^2, \ldots, \Psi^m)$, where $\Psi^j = (\varphi_j^1, \varphi_j^2, \ldots, \varphi_j^n)$. Denote by $\mathbf{P}_\rho^{C,\Psi} \to X$ the corresponding extension of the bundle $\mathbf{P}'_\rho \to X_m$, and denote by $E_\rho^{C,\Psi} \to X$the vector bundle associated with the principal bundle $P_\rho^{C,\Psi}$. The collection C, Ψ is said to be admissible, if C_j, Ψ^j satisfy 1), 2) for every j.

Proposition 3.4. *There is a one-to-one correspondence between the set of all Fuchsian systems of ordinary differential equations on the Riemann surface with prescribed monodromy and the set* $\{H^0(X, \mathcal{O}(\mathbf{P}_\rho^{C,\Psi})\}$ *of holomorphic sections of all admissible extensions of the principal bundle* $\mathbf{P}'_\rho \to X_m$.

Theorem 3.7. *Let* ρ *be the monodromy representation of a stable holomorphic principal bundle with connection* (P, ∇) *and with Chern number* 0*. Then there exists a* $\mathcal{G}$*-valued* 1*-form* ω *with poles of first order at the points from* S *such that the monodromy of the* G*-valued system* $Df = \omega$ *coincides with* ρ.

The fundamental groups $\pi_1(\mathbb{CP}^1 \setminus \{s_1, \dots s_m\})$ and $\pi_1(X \setminus \{s_1, \dots, s_m)$ are free with $m - 1$ and $2g + m - 1$ generators correspondingly. From this it follows the following condition of solvability of the Riemann–Hilbert problem.

Theorem 3.8. *If* $\operatorname{Im}\rho$ *is connected, then the Riemann–Hilbert problem is solvable for any* m *points* $s_1, \dots, s_m$.

Proof. Let $\gamma_1, \dots, \gamma_m$ be the generators of $\pi_1(M)$. Put $\rho_1 = \rho(\gamma_1), \dots, \rho_m = \rho(\gamma_m)$. If $\operatorname{Im}\rho$ is a connected subgroup, then there exists a continuous path $\rho_\alpha(t)$ such that $\rho_\alpha(0) = \mathbb{I}$ and $\rho_\alpha(1) = \rho_\alpha$. From this it follows that there exists a homomorphism $\chi_t : \pi_1(M) \to G$ such that $\chi_t(\gamma_\alpha) = \rho_\alpha(t)$. We use the following general result from homological algebra: if G is connected, then the homomorphism $h : \pi_1(M) \to G$ is the monodromy homomorphism of a G-system if and only if it is possible to connect h to $\mathbb{I}$ by a continuous path in $Z^1(\pi_1(M), G)$. □

Proposition 3.5. *Suppose* $\rho(\gamma_j) \in \mathbf{T}$ *for some* j*. Then* $\rho : \pi_1(X \setminus S) \to G$ *is monodromy of Fuchsian system.*

Proof. Let $Df = \alpha$ be a G-system of equations with singular points $s_1, \dots, s_m$. Let $M_1, \dots, M_m$ be the monodromy matrices. We reduce the problem to a G-valued linear conjugation problem. To this end, we construct the curves $\Gamma_j, (z_0, s_j), \Gamma = \cup_{j=1}^m \Gamma_j$, where z_0 is an arbitrary point, and consider the boundary problem

$$\Phi^+ = G(t) f^-(t), \quad G(t) = M_j \quad \text{if} \quad t \in \Gamma.$$

Notice that $G(t)$ has a jump at the given points. This makes it necessary to study the G-valued linear conjugation problem. Consider the contour Γ_0 such that t-plane is decomposed into two domains U^+ and U^-, where $s_1, \dots, s_m \in U^+$, $\infty \in U^-$. Cover the domain U^+ by simply connected domains U_j, $j = 1, \dots, m$, such that $s_j \in U_j$, U_j contains no other points s_j. In every domain U_j we represent some solution of $Df = \alpha$ by

$$f^+ = \exp(E_j \log(t - s_j))\Phi_j(t), \quad t \in U_j,$$
$$M_j = \exp(2\pi i E_j), \quad j = 1, \dots, m,$$

where $\Phi_j \in L^+G_{\mathbb{C}}$. The coboundary of the cochain $\{\Phi_j\}$ is

$$\Phi_{jk} = \Phi_j\Phi_k^{-1} = \exp(E_j \log(t - s_j)) \exp(-E_k \log(t - s_k)), \quad t \in U_j \cap U_k.$$

It means that we can define a principal bundle with transition functions Φ_k over U^+. The domain U^+ being a Stein manifold, this bundle is trivial and there exist elements $h_j : U_j \to G_{\mathbb{C}}$ such that $\Phi_{jk} = h_j^{-1}h_k$ and hence $V^+(t) = h_j(t)\Phi_k(t) = h_k(t)\Phi_k(t)$ is holomorphic in U^+ and

$$f^+(t) = \exp(E_j \log(t - s_j))h_j^{-1}(t)V^+(t) = \exp(E_j \log(t - s_j)W_j(t), \ \ t \in U_j.$$

The G-valued function $f^+(t)$ satisfies the G-system $Df^+ = \alpha$. On the t-plane the solution has the form

$$f(t) = f^+(t)V^(+)(t), \quad t \in U^+,$$
$$f(t) = (\frac{1}{t})^{M_{m+1}}V^-(t), \quad t \in U^-,$$
$$\exp(2\pi i M_{m+1}) = (M_1 \cdot \cdots \cdot M_m)^{-1}.$$

If the point t moves along Γ_0 encircling infinity, then we see that it passes around all points $s_1, \ldots, s_m$ in the negative direction. We have $V^{\pm}(t) : U^{\pm} \to G_{\mathbb{C}}$ and obtain

$$V^+(t) = (f^+)^{-1}(t)(\exp(E_{m+1} \log(-t))V^-(t), \quad t \in \Gamma_0.$$

In the domain U^-, the function $V^-(t)$ is represented in the form $f(t) = \exp(D \log(t))W(t)P(t)$, where $D \in \mathbb{T}$ and $P(t)$ is a polynomial loop. Then $\widetilde{f}(t) = f(t)P^{-1}(t)$ has only first order poles at the points $s_1, \ldots, s_m$. If $M_{m+1} \in \mathbb{T}$, then $\widetilde{f}(t) = \exp(M_{m+1} - D) \log(-t))W(t)$, and hence infinity is a first order pole. □

In general, α may have a pole at infinity whose order is greater than 1. For example, see [3].

4. Yang–Mills Equations on Riemann Surfaces

The objects of investigation in Yang–Mills–Higgs theory are a connection ω on a principal bundle $P \to X$ with the structure group G, and a scalar field ϕ. In n-dimensional Yang–Mills–Higgs theory the connection has the form $\omega = \sum \omega_i dx^i$, and the scalar field is just a scalar function $\phi = \phi(x)$. The components of the connection $\omega_i(x)$ are G-valued functions, where G is the gauge group. The Higgs field is a complex function $\phi = \phi_1 + i\phi_2$, where ϕ_1 and ϕ_2 are real functions. More generally, ϕ is a global section of a Hermitian line bundle L over $\mathbb{R}^n$. The curvature of the Yang–Mills connection ω has the form $F(\omega) = d\omega + \omega \wedge \omega = \frac{1}{2}F_{ij}dx^i \wedge dx^j$ with the components $F_{ij}(x) = \partial_j\omega_j(x) - \partial_j\omega_i(x) + [\omega_i(x), \omega_j(x)]$.

Denote by $\mathcal{A}(P)$ the space of smooth connections of P and by $\Gamma(L)$ the space of Higgs fields. In Euclidean Yang–Mills–Higgs theory (in this case the metric is flat, i.e., the components of the metric q satisfy the condition $q_{ij} = \delta_{ij}$) the Yang–Mills–Higgs action is of the form

$$\mathcal{IMH} : \mathcal{A}(P) \times \Gamma(L) \to \mathcal{R}$$

and it is defined by the formula

$$\mathcal{IMH}(\omega, \phi) = \frac{1}{2} \int\limits_{\mathcal{R}^n} ((F(\omega), F(\omega)) + (D_\omega \phi, D_\omega \phi) + \frac{\lambda}{4} \star (|\phi|^2 - 1)^2), \quad (4.1)$$

where λ is a real parameter, $\star$ is the Hodge star operator, $(,)$ denotes scalar product in ad-representation of Lie group, $\omega \in \mathcal{A}(P)$ and $\phi \in \Gamma(L)$.

If $\mathcal{IMH}(\omega, \phi) < \infty$, then (4.1) is called a finite action. From the finiteness it follows that ω and ϕ satisfy the conditions: $|\phi| \to 1$, $|D_\omega \phi| = |d\phi - i\omega\phi| \to 0$, and $F(\omega) \to 0$ as $x \to \infty$. The Euler-Lagrange equations for the action (4.1) have the form

$$D_\omega \star F(\omega) = \star J, \quad (4.2)$$

$$D_\omega \star D_\omega \phi = \frac{\lambda}{2} \phi(|\phi|^2 - 1). \quad (4.3)$$

In case $\lambda = 0$, $\phi \equiv 0$, $n = 4$, the action $\mathcal{IMH}$ is called the Yang–Mills action and is denoted by $\mathcal{IM}$. Thus the Yang–Mills action has the form:

$$\mathcal{IM}(\omega) = \frac{1}{2} \int\limits_{\mathbb{R}^n} (F(\omega, F(\omega)). \quad (4.4)$$

From (4.3) it follows that for the Yang–Mills action the Euler-Lagrange equations have the form

$$D_\omega \star F(\omega) = 0 \quad (4.5)$$

called the Yang–Mills equation. The Yang–Mills equation together with the Bianchi identity $D_\omega F(\omega) = 0$ means that the connection is closed and coclosed. From this it follows that Yang–Mills theory is a nonlinear generalization of the Hodge theory. Nonlinearity is caused by the non-commutativity of G. The Yang–Mills equation from the partial differential equations theory viewpoint was considered in [13].

If $n > 4$, for the finite action there exist no nontrivial solutions of the equation (4.3). In the case where $n = 4$, avtodual solutions are called instantons. In the case where $n = 3$, nontrivial solution are called monopoles. For $n = 2$, nontrivial solutions always exist, they are called vortices, and the two-dimensional version of the equation (4.3) is called the vortex equation.

In abelian Yang–Mills–Higgs theory in case $n = 2$ the structural group G is $U(1)$ and the action has the form:

$$\mathcal{IMH}(\omega, \phi) = \frac{1}{2} \int\limits_{\mathbb{R}^n} (D_\omega \phi \wedge \star D_\omega \overline{\phi}) + F(\omega) \wedge \star F(\omega)) + \frac{\lambda}{4} \star (\phi\overline{\phi} - 1)^2. \quad (4.6)$$

The Euler-Lagrange equations for this action have the form

$$d \star F(\omega) = \frac{i}{2} \star (\phi D_\omega \overline{\phi} - \overline{\phi} D_\omega \phi), \quad (4.7)$$

$$D_\omega \star D_\omega \phi = \frac{\lambda}{2} \star (\phi\overline{\phi} - 1)\phi. \quad (4.8)$$

The number $N = \frac{1}{2\pi}\int_{\mathbb{R}^n} F(\omega)$ is integer and it is called the vortex number. Its four-dimensional analog $N = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} tr(F(\omega) \wedge F(\omega))$ is called the topological charge.

If $N \geq 0$, then the relation $\mathcal{IMH}(\omega, \phi) = \pi N$ takes place if and only if the following system of differential equations is satisfied by (ω, ϕ):

$$(\partial_1\phi_1 + \omega_1\phi_2) \mp (\partial_2\phi_2 - \omega_2\phi_1) = 0, \quad (4.9)$$

$$(\partial_2\phi_1 + \omega_2\phi_2) \pm (\partial_1\phi_2 - \omega_1\phi_1) = 0, \quad (4.10)$$

$$F_{12}(\omega) \pm \frac{1}{2}(|\varphi_1|^2 + |\phi_2|^2 - 1) = 0. \quad (4.11)$$

It is known [32] that, given an integer $N > 0$ and a set $\{z_1, \dots, z_m\} \subset \mathbb{C}$, for a finite action (4.6) there exists the unique class of gauge equivalent solutions of the system (4.9-4.11) which satisfy the conditions:

1. ϕ is smooth;
2. the zeros of ϕ are $\{z_1, \dots, z_n\}$ and when $z \to z_j$ one has $\phi(z, \overline{z}) \sim c_j(z - z_j)^{n_j}, c_j \neq 0$, where n_j are multiplicities of z_j.
3. $N = \frac{1}{2\pi i}\int_{\mathbb{R}^2} F(\omega) = \sum n_j$.

The action (4.6) in case $\lambda = 1$ has an absolute minimum if and only if a pair (ω, ϕ) satisfies the following system of equations:

$$D_\omega\phi = \mp \star D_\omega\phi, \quad (4.12)$$

$$\star F(\omega) = \pm\frac{1}{2}(|\phi|^2 - 1). \quad (4.13)$$

The equations (4.12-4.13) are called Bogomol'ny equations. It turns out that solutions to this system can be constructed using the Riemann–Hilbert boundary problem with the coefficients in a loop group considered by G.Khimshiashvili in [28] (cf. also [46]).

Theorem 4.1. *On a Riemann surface X, the space of smooth solutions ϕ to the equation*

$$D_\omega\phi - i \star D_\omega\phi = 0 \quad (4.14)$$

coincides with the space of solutions to a certain Riemann–Hilbert boundary problem on X.

Before explaining the proof of this theorem, we present some general remarks. Let $\pi : E \to M$ be a holomorphic vector bundle over the complex manifold M with canonical connection ω. Fix a local basis $\{e_i\}$ in the fibres of bundle and consider the following objects defined with respect to this basis: $\langle , \rangle$ which is a metric in E, ω_{ij} which is the matrix of connection, $F(\omega)_{ij}$ which is the matrix of curvature $F(\omega)$. Then we have:

1. $\omega \in \Gamma(\Lambda^{1,0}M)$ and if $H_{ij} = (\langle e_i, e_j\rangle)$ then $\omega_{ij} = H_{ij}^{-1}\partial H_{ij}$;
2. $F(\omega)_{ij} = \partial\omega_{ij},\ F(\omega)_{ij} \in \Gamma(\Lambda^{1,1}M)$.

Let $(E, \pi, M, \langle , \rangle)$ be an Hermitian vector bundle and ω an Hermitian connection in E such that the curvature $F(\omega)$ lies in $\Gamma(\Lambda^{1,1}M) \otimes E$. Then there exists a holomorphic structure on (E, π, M) such that ω is the canonical connection with respect to this structure. Suppose ω and ω_1 are gauge equivalent connections on E, then there exist holomorphic structures J_{ω_1} and J_{ω_2} such that ω_1 and ω_2 are their canonical connections. A simple calculation shows that J_{ω_1} and J_{ω_2} will be equivalent holomorphic structures.

Lemma 4.1 ([43]). *Let $p : L \to X$ be an Hermitian line bundle over a compact Riemann surface X, ω be the canonical Hermitian connection of L and J_ω be the corresponding complex structure on L. The smooth sections $\phi \in H^0(X, C^\infty(L))$ are holomorphic with respect to the complex structure J_ω if and only if the pair (ω, ϕ) satisfies the Bogomol'ny equation* (4.14).

Two important conclusions follow from this lemma:

1. there exists a one-to-one correspondence between the Hermitian connections and holomorphic structures on the line bundles;

2. the Bogomol'ny equation (4.14) can be considered as a criterion of holomorphicity of sections with respect to the holomorphic structure induced from a fixed Hermitian connection (thus, in a sense, Bogomol'ny equations are non-Euclidean analogs of Cauchy–Riemann equations).

Now we are prepared to prove the theorem. It is known [46] that for a holomorphic line bundle L there exists a loop $\gamma \in L^-\mathbb{C}^* \setminus L\mathbb{C}^*/L_X^+\mathbb{C}^*$ which defines the equivalence class of L in the moduli space of holomorphic line bundles on X, which means that the loop γ defines the holomorphic structure J on the line bundle L. From the lemma it follows that there exists a connection ω on L such that the complex structure J_ω induced from ω is equivalent to J. According to [28], with the loop γ one can associate the Riemann–Hilbert problem $\mathcal{P}_\gamma$ for sections of L. From 2) above it follows that $\phi \in H^0(X, \mathcal{O}(L))$ if and only if the pair (ω, ϕ) satisfies the equation (4.14). On the other hand, from the construction of L by the loop γ it follows that the space of solutions to the Riemann–Hilbert boundary problem $\mathcal{P}_\gamma$ coincides with the space $H^0(X, \mathcal{O}(L))$. The two preceding conclusions obviously prove the statement of the theorem.

Let us now describe some related algebraic constructions. If a connection ω is flat, then ω is a solution to (4.5) and the given G-bundle is induced from a representation of the fundamental group $\rho : \pi_1(X) \to G$. Of course the equation (4.5) may also have nontrivial solutions. These nontrivial solutions are obtained from the central extension of the fundamental group $\pi_1(x)$. Let $\Gamma_\mathbb{R}$ be the central extension of the fundamental group obtained by extending the center of Γ by the additive group of real numbers $\mathbb{R}$. In other words, we have the exact sequence:

$$1 \to \mathbb{R} \to \Gamma_\mathbb{R} \to \pi_1(X) \to 1. \tag{4.15}$$

One also has another central extension described by the exact sequence

$$1 \to \mathbb{Z} \to \Gamma \to \pi_1(X) \to 1. \tag{4.16}$$

The exact sequences (4.15) and (4.16) yield the third exact sequence needed in the proof of Theorem 4.2 below:

$$1 \to \mathbb{Z} \to \Gamma_{\mathbb{R}} \to U(1) \times \pi_1(X) \to 1. \tag{4.17}$$

Let $\widetilde{X} \to X$ and $P \to X$ be principal bundles with structural groups $\pi_1(X)$ and $U(1)$ correspondingly. Suppose $P \to X$ is topologically nontrivial with the Chern number equal to 1. Then $\widetilde{X} \times_X P \to X$ is a principal bundle with the structural group $\pi_1(X) \times U(1)$ and after normalization of the metric on X the connection ω has a constant curvature equal to $-2\pi i\mu$, where μ is the volume form on X.

The representation $\rho : \Gamma_{\mathbb{R}} \to G$ induces a principal $G_{\mathbb{C}}$-bundle on X with connection ω_ρ. From the functoriality of the Yang–Mills equations it follows that ω_ρ will be a solution of (4.5). By Atiyah–Bott theorem [4], there exists a bijective homomorphism between the equivalence classes of homomorphisms $\rho : \Gamma_{\mathbb{R}} \to G$ and the gauge equivalence classes of Yang–Mills connections, which is merely a consequence of the results due to M. Narasimhan with T. Seshadri [42] and a theorem of S. Donaldson [16]. We are now in position to establish an analytic property of connections having only one singular point provided by Theorem 3.6.

Theorem 4.2. *Let P be a stable holomorphic principal $G_{\mathbb{C}}$-bundle. Then any connection θ on P having a single singular point is a Yang–Mills connection.*

Proof. Let $Q \to X$ be a $U(1)$-bundle with the Chern number 1. Suppose μ is a normalized volume form on X. Then there exists a harmonic connection ω on Q with a constant curvature equal to $-2\pi i\mu\mathbf{1}$, where $\mathbf{1}$ is the identity automorphism of Q. Let $P \simeq Q \times_X \widetilde{X} \to X$ be the associated $U(1) \times \pi_1(X)$-bundle. From the Atyah–Bott theorem it follows that for the representation $\rho : \Gamma_{\mathbb{R}} \to G$ there exists a connection ω_ρ on P such that ω_ρ satisfies the Yang–Mills equation and the correspondence $\rho \to \omega$ is one-to-one. Let $\widetilde{\rho} : \Gamma \to G_{\mathbb{C}}$ be the representation induced from the G-system $Df = \alpha$ with one regular singular point and let $i : \Gamma \to \Gamma_{\mathbb{R}}$ be an embedding. Then the representation $\widetilde{\rho} = \rho \circ i$ corresponds to the Yang–Mills connection $\omega_{\rho\circ i}$. On the other hand $\widetilde{\rho}$ is the monodromy representation of the above G-system with one regular singular point. Therefore connection θ satisfies the Yang–Mills equation, as was claimed. □

The both theorems of this section give examples of nontrivial applications of Riemann–Hilbert problems to Yang–Mills equations. The author hopes that the interplay between these two topics may lead to further results in the Yang–Mills theory.

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