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**EXTENSION OF THE CLASS OF
EFFECTIVELY SOLVABLE TWO-DIMENSIONAL
PROBLEMS WITH PARTIALLY UNKNOWN
BOUNDARIES IN THE THEORY OF FILTRATION**

Abstract. In the present paper, first we briefly describe effective methods for solving such two-dimensional problems of the theory of filtration to which on the plane of complex velocity there correspond circular pentagons of special type and removable singular points. Of five vertices of the circular pentagon at least one is a cut end formed by two neighboring sides, with the angle 2π . Then we present effective methods of solving two problems of the theory of filtration dealing with the motion of an incompressible liquid through the plane earth dams of trapezoidal shape with water non-permeable bases. To each problem there corresponds one removable singular point. The first problem deals with the plane earth dam whose lower slope is vertical and the upper one is inclined to the horizon. The second problem deals with the plane earth dam whose lower slope is inclined to the horizon and the upper one is vertical. To these problems on the plane of complex velocity there correspond circular pentagons one of whose vertices is a cut end with the angle 2π . To find unknown parameters, we write a system of equations which is decomposed into three systems. We substitute the solutions of the first system into the second and third systems, and we substitute the solution of the second system into the third system.

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რეზიუმე. ნაშრომში ჩვენ ჯერ მოკლედ აღვწერთ ფილტრაციის თეორიის ისეთი ორგანზომილებიანი ამოცანების ამოხსნის ეფექტურ მეთოდებს, რომელთაც კომპლექსური სიჩქარის სიბრტყეზე შეესაბამება სპეციალური სახის წრიული ხუთკუთხედი და აცილებადი სინგულარული წერტილები. წრიული ხუთკუთხედის ერთი წვერო მაინც წარმოადგენს რომელიმე ორი მეზობელი გვერდით შექმნილი ჭრილის ბოლოს 2π კუთხით. შემდეგ ჩვენ წარმოვადგენთ ფილტრაციის თეორიის ორი ამოცანის ამოხსნის ეფექტურ მეთოდებს, რომლებიც ეხება არაკუმშვადი სითხის მოძრაობას წყალგაუმტარი ფუძეების მქონე ტრაპეციის ფორმის მიწაყრილში. თითოეულ ამოცანას თითო აუცილებადი სინგულარული წერტილი შეესაბამება. პირველი ამოცანა ეხება მიწაყრილს, რომლის ქვედა ფერდი ვერტიკალურია, ხოლო ზედა ფერდი დახრილია ჰორიზონტისადმი. მეორე ამოცანა ეხება მიწაყრილს, რომლის ქვედა ფერდი დახრილია ჰორიზონტისადმი, ხოლო ზედა-ვერტიკალურია. ამ ამოცანებს კომპლექსური სიჩქარის სიბრტყეზე შეესაბამება წრიული ხუთკუთხედები, რომელთა ერთი წვერო წარმოადგენს ჭრილის ბოლოს 2π კუთხით. უცნობი პარამეტრების საპოვნელად ჩვენ ვაგებთ განტოლებათა სისტემას, რომელიც სამ სისტემად იშლება. პირველი სისტემის ამონახსნებს ჩვენ ვსვამთ მეორე და მესამე სისტემებში, ხოლო მეორე სისტემის ამონახსნს მესამეში.

INTRODUCTION

In the present paper we give an algorithm for the effective solution of such two-dimensional problems of the theory of filtration with partially unknown boundaries to which on the plane of complex velocity there correspond circular pentagons in which at least one of five vertices is a cut end with the angle 2π , while the remaining four angles are arbitrary ones. So we can, in particular, consider arbitrary circular quadrangles and triangles. Moreover, in these problems we may additionally come across removable singular points. We give the solution of the problem of incompressible liquid motion through earth dams with water-nonpermeable bases whose lower and upper slopes are segments of straight lines which make with the horizon the following angles: (I). $\pi/2, \pi\alpha$; (II). $\pi(1-\beta), \pi/2$; (III). $\pi/2, \pi/2$. Problem III was solved by the well-known authors (see, e.g., [1], [2], [7], [8]). It should be noted that the most complete solution of that problem is given by P.Ya. Polubarinova-Kochina in [1, 2], therefore the solution of problem III will be omitted here. Each of the problems I and II has one removable singular point and to these problems on the plane of complex velocity there correspond circular pentagons, while problem III has two removable singular points and to that problem there corresponds a circular triangle.

To solve problems I and II, we construct three analytic functions $z(\zeta)$, $\omega(\zeta)$ and $w(\zeta)$ which map conformally the half-plane of the complex $\zeta = t + i\tau$ plane onto the domains of: liquid motion, complex potential and complex velocity, respectively. In constructing the above-mentioned functions we solve Fuchs class equations and nonlinear Schwartz equation. A system of equations is derived with respect to the unknown parameters; it decomposes into three systems. The first system is reduced to solution of three higher transcendental equations with respect to three unknown essential parameters, the second system is reduced to a system of equations with respect to parameters (constants) of integration of the Schwartz equation, and the third system of equations is connected with removable singular points, with the equation for determination of liquid discharge via filtration, and also with some other parameters. First of all, we solve the first, then the second and finally the third system. The solutions of these systems are used subsequently. Finally, we define unknown parts of the boundaries of the domain of liquid motion.

1. EFFECTIVE METHODS OF SOLVING TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF FILTRATION WITH PARTIALLY UNKNOWN BOUNDARIES.

Before we proceed to solving the above-mentioned specific problems, let us consider in short some new effective methods of solving the plane problems of the theory of filtration with partially unknown boundaries. Using the methods presented in this section, we construct solutions of Problems I and II [26]–[34].

The plane of steady motion of incompressible liquid in a porous medium subjected to the Darcy law coincides with the plane of a complex variable $z = x + iy$. The porous medium is assumed to be isotropic, homogeneous and undeformable. The boundary $l(z)$ of the domain $s(z)$ of liquid motion consists of an unknown depression curve to be defined, and of known segments, half-lines and straight lines.

In the domain $s(z)$ with the boundary $l(z)$ we seek for a reduced complex potential (divided by the coefficient of filtration) $\omega(z) = \varphi(x, y) + i\psi(x, y)$, where $\varphi(x, y)$ is the velocity potential, $\psi(x, y)$ is the flow function satisfying both the Cauchy-Riemann conditions and the following boundary conditions [1]–[8]:

$$a_{k1}\varphi(x, y) + a_{k2}\psi(x, y) + a_{k3}x + a_{k4}y = f_k, \quad k = 1, 2, \quad (x, y) \in l(z), \quad (1.1)$$

where a_{kj} , f_k , $k = 1, 2$, $j = \overline{1, 4}$ are known piecewise constant real functions, f_k , $k = 1, 2$, depend on the parameter Q , where Q is the liquid discharge per filtration.

Using the boundary conditions (1.1), we can define a part of the boundary $l(\omega)$ of $s(\omega)$ and the boundary $l(w)$ of the domain of complex velocity $w(z) = \omega'(z) = d\omega(z)/dz$, except some coordinates of vertices of circular polygons $s(\omega)$ [1]–[6]. By means of the functions $\omega(z)$ and $w(z)$ the domain $s(z)$ with the boundary $l(z)$ is conformally mapped respectively onto the domains $s(\omega)$ and $s(w)$ with the boundaries $l(\omega)$ and $l(w)$, where the domain $s(w)$ is a circular polygon with the boundary $l(w)$ consisting of a finite number of circular arcs and, in particular, of segments of straight lines, half-lines and straight lines [1]–[8].

Angular points of the boundaries $l(z)$, $l(\omega)$ and $l(w)$ which may be encountered at least on one of them upon the circuit in the positive direction will be denoted by A_k , $k = \overline{1, n}$.

To solve the problem of filtration, we map conformally the half-plane $\text{Im}(\zeta) > 0$ (or $\text{Im}(\zeta) < 0$) of the plane $\zeta = t + i\tau$, $i = \sqrt{-1}$ onto the domains $s(z)$, $s(\omega)$ and $s(w)$. The corresponding mapping functions are denoted by $z(\zeta)$, $\omega(\zeta)$ and $w(\zeta) = \omega'(\zeta)/z'(\zeta)$, $d\omega(\zeta)/d\zeta = \omega'(\zeta)$, $dz(\zeta)/d\zeta = z'(\zeta)$. To the angular points A_k , $k = \overline{1, n}$, along the axis t there correspond the points $t = e_k$, $k = \overline{1, n}$; note that $-\infty < e_1 < e_2 < \dots < e_n < +\infty$, where the point $t = e_{n+1}$ is mapped into a nonangular point A_∞ of the boundary $l(z)$ which lies between the points A_n and A_1 .

The boundary values of the functions $z(\zeta)$, $\omega(\zeta)$ and $w(\zeta)$, as $\zeta \rightarrow t$, $\zeta \in \text{Im}(\zeta) > 0$ are denoted as follows: $z(t) = x(t) + iy(t)$, $\omega(t) = \varphi(t) + i\psi(t)$, $w(t) = u(t) - iv(t)$. By $\overline{z(t)}$, $\overline{\omega(t)}$ and $\overline{w(t)}$ we denote the functions which are complex-conjugate respectively to the functions $z(t)$, $\omega(t)$ and $w(t)$.

Introduce the vectors $\overline{\Phi(t)} = [\overline{\omega(t)}, \overline{z(t)}]$, $\overline{\Phi'(t)} = [\overline{\omega'(t)}, \overline{z'(t)}]$, $\overline{f(t)} = [\overline{f_1(t)}, \overline{f_2(t)}]$. Then by means of these vectors the boundary conditions (1.1) can be written as [1]–[8], [11],

[12], [26]–[34]

$$\Phi(t) = g(t)\overline{\Phi(t)} + i2G^{-1}f(t), \quad -\infty < t < +\infty, \quad (1.2)$$

where $g(t) = G^{-1}(t)\overline{G(t)}$ is a piecewise constant nonsingular matrix of the second order with the points $t = e_k$, $k = \overline{1, n}$, of discontinuity, $G^{-1}(t)$ and $\overline{G(t)}$ are, respectively, the inverse and complex-conjugate matrices to the matrix $G(t)$, and $f(t)$ is a piecewise constant vector. The matrix $G(t)$ and the vector $f(t)$ are defined by (1.1).

Differentiating (1.2) along the boundary t , we get

$$\Phi'(t) = g(t)\overline{\Phi'(t)}, \quad -\infty < t < +\infty. \quad (1.3)$$

We can easily verify that the equality $\overline{g(t)} = g^{-1}(t) = \overline{G^{-1}(t)}G(t)$ holds.

For the points $t = e_j$, $j = \overline{1, n}$, let us consider the characteristic equations [12]

$$\det[g_{j+1}^{-1}(e_j + 0)g_j(e_j - 0) - \lambda E] = 0 \quad (1.4)$$

with respect to the parameter λ , where E is the unit matrix, $g_j(t)$, $e_j < t < e_{j+1}$, $g_{j+1}^{-1}(e_j + 0)$, and $g_j(e_j - 0)$ are limiting values of the matrices $g_{j+1}^{-1}(t)$ and $g_j(t)$ at the point $t = e_j$ respectively from the right and from the left.

To solve the problem (1.3), we have first to find $w(\zeta)$ and then, using it and the boundary conditions (1.3), we construct $\omega'(\zeta)$ and $z'(\zeta)$. Finally, integrating (1.3) with regard for (1.2), we find $\omega(\zeta)$ and $z(\zeta)$. With the help of $\omega(\zeta)$ and $z(\zeta)$ we can find unknown parts of the boundaries $l(z)$ and $l(\omega)$, Q being the liquid discharge via filtration.

Using the roots λ_{kj} of the equation (1.4), we define uniquely the numbers $\alpha_{kj} = (2\pi i)^{-1} \ln \lambda_{kj}$, $k = 1, 2$, $j = \overline{1, n}$ [1], [2].

Suppose that among the points A_k , $k = \overline{1, n}$, of the boundaries $l(z)$ and $l(\omega)$ there exist removable angular points to which on the boundary $l(w)$ of $s(w)$ there correspond regular nonangular points; such angular points of the boundaries $l(z)$ and $l(\omega)$ are commonly called removable singular points [1]–[6].

For the sake of simplicity, we assume that the number of removable singular points equals two. Suppose that removable singular points coincide with the points $t = e_j$, $t = e_{j+k}$. To these points on the contours $l(z)$ and $l(\omega)$ there correspond the angles $\pi/2$, and on the boundary $l(w) - \pi$ there corresponds the angle $l(w) - \pi$. To remove these singular points from the boundary conditions (1.3), we introduce a new unknown vector $\Phi_1(t)$ by the formula [1], [2], [26]–[34]

$$\Phi'(t) = \chi_{01}(t)\Phi_1(t), \quad -\infty < t < +\infty, \quad (1.5)$$

where

$$\chi_{01}(t) = \sqrt{(t - e_{j-1})(t - e_j)^{-1}(t - e_{j+k+1})(t - e_{j+k})^{-1}} > 0, \quad (1.6)$$

$$t > e_{j+k+1}.$$

After passing from the vector $\Phi'(t)$ to $\Phi_1(t)$, we multiply the matrices $g_{j-1}(t)$ and $g_{j+k}(t)$ in the intervals (e_{j-1}, e_j) and (e_{j+k}, e_{j+k+1}) by (-1) .

The boundary condition with respect to $\Phi_1(t)$ takes the form

$$\Phi_1(t) = g^*(t)\overline{\Phi_1}(t), \quad -\infty < t < +\infty, \quad (1.7)$$

where

$$g^*(t) = [\chi_{01}(t)]^{-1}g(t)[\overline{\chi_{01}}(t)]. \quad (1.7_1)$$

We renumerate singular points on the contour $l(w)$ and denote them by B_j , $j = \overline{1, m}$, while the corresponding points along the axis t are denoted by a_j , $j = \overline{1, m}$. Uniquely defined characteristic numbers corresponding to the points $t = a_j$ we denote again by α_{kj} , $k = 1, 2$, $j = \overline{1, m}$. They satisfy the Fuchs condition.

Let us set up the Fuchs class [1]–[6], [26]–[34] equation

$$u''(\zeta) + p(\zeta)u'(\zeta) + q(\zeta)u(\zeta) = 0, \quad (1.8)$$

where

$$p(\zeta) = \sum_{j=1}^m (1 - \alpha_{1j} - \alpha_{2j})(\zeta - a_j)^{-1}, \quad (1.9)$$

$$q(\zeta) = \sum_{j=1}^m [\alpha_{1j}\alpha_{2j}(\zeta - a_j)^{-2} + c_j(\zeta - a_j)^{-1}]. \quad (1.10)$$

c_j are the unknown accessory parameters satisfying as yet the condition

$$\sum_{j=1}^m c_j = 0. \quad (1.10_1)$$

We write the equation (1.8) in the form of a system [26]–[34]

$$\chi'(t) = \chi(t)\mathcal{P}(t), \quad (1.11)$$

where

$$\mathcal{P}(t) = \begin{pmatrix} 0, & -q(t) \\ 1, & -p(t) \end{pmatrix}, \quad \chi(t) = \begin{pmatrix} u_1(t), & u_1'(t) \\ u_2(t), & u_2'(t) \end{pmatrix}. \quad (1.12)$$

Using linearly independent solutions $u_1(t)$ and $u_2(t)$ of the equation (1.8), we construct

$$w(t) = [Au_1(t) + Bu_2(t)][Cu_1(t) + Du_2(t)]^{-1}, \quad (1.12_1)$$

the general solution of the Schwartz equation [14]–[16]

$$\{w, t\} \equiv w'''(t)/w'(t) - 1, 5[w''(t)/w'(t)]^2 = R(t), \quad (1.13)$$

where

$$\begin{aligned} R(t) &= 2q(t) - p'(t) - 0, 5[p(t)]^2 = \\ &= \sum_{j=1}^m \{0, 5[1 - (\alpha_{1j} - \alpha_{2j})^2](t - a_j)^{-2} + c_j^*(t - a_j)^{-1}\}, \end{aligned} \quad (1.14)$$

$$\alpha_{1j} - \alpha_{2j} = \nu_j, \quad j = \overline{1, m}, \quad c_j^* = 2c_j - \beta_j \sum_{k=1, k \neq j}^m \beta_k (a_j - a_k)^{-1}, \quad (1.15)$$

with

$$\beta_k = 1 - \alpha_{1k} - \alpha_{2k}, \quad k = \overline{1, m}.$$

A, B, C and D are integration constants of (1.13) which satisfy the condition

$$AD - BC \neq 0. \quad (1.16)$$

We can see from (1.14) that the equation (1.13) depends on $\alpha_{1j} - \alpha_{2j} = \nu_j$, $j = \overline{1, m}$. By (1.12₁) the half-plane $\text{Im}(\zeta) > 0$ (or $\text{Im}(\zeta) < 0$) is conformally mapped onto the domain $s(w)$ with the boundary $l(w)$.

Now we express the function $R(\zeta)$ in the vicinity of $\zeta = \infty$ as a power series in $1/\zeta$ and obtain

$$R(\zeta) = \sum_{k=1}^{\infty} M_k \zeta^{-k}. \quad (1.17)$$

Since the point $\zeta = \infty$ is the image of a nonangular point of the boundary $l(w)$, the conditions ([14]–[16])

$$\begin{aligned} M_1 = \sum_{k=1}^m c_k^* = 0, \quad M_2 = \sum_{k=1}^m [a_k c_k^* + 0, 5(1 - \nu_k^2)] = 0, \\ M_3 = \sum_{k=1}^m [a_k^2 c_k^* + a_k(1 - \nu_k^2)] = 0 \end{aligned} \quad (1.18)$$

should be fulfilled. From the condition $M_1 = 0$ it follows the condition (1.10₁), and vice versa, from the condition (1.10₁) it follows the condition $M_1 = 0$. The conditions (1.18) will be obtained below in somewhat different way. These conditions will allow us to define three parameters c_j , $j = \overline{1, 3}$. Moreover, of the parameters $t = a_k$, $k = \overline{1, m}$ we choose arbitrarily and fix only three. Therefore $R(\zeta)$, defined by the formula (1.14), depends on $2(m-3)$ unknown parameters a_j , c_j , $j = \overline{1, m-3}$ ([14]–[16]).

The equation (1.8) in the vicinity of the point $t = a_j$ can be rewritten as

$$(t - a_j)^2 u''(t) + (t - a_j) p_j(t) u'(t) + q_j(t) u(t) = 0, \quad (1.19)$$

where

$$\begin{aligned} p_j(\zeta) &= p_{0j} + \sum_{j=1}^{\infty} p_{nj} (t - a_j)^n, \quad p_{nj} = (-1)^{n-1} \sum_{k=1, k \neq j}^m \beta_k (\alpha_j - \alpha_k)^{-n}, \\ p_{0j} &= \beta_j, \quad \beta_k = 1 - \alpha_{1k} - \alpha_{2k}, \\ q_j(t) &= \alpha_{1j} \alpha_{2j} + c_j (t - a_j) + \sum_{n=2}^{\infty} q_{nj} (t - a_j)^n, \\ q_{nj} &= (-1)^{n-2} \sum_{k=2, k \neq j}^m [\alpha_{1k} \alpha_{2k} (n-1) + c_k (a_j - a_k)] (a_j - a_k)^{-n}, \\ &n = 2, 3, \dots, \\ q_{0j} &= \alpha_{1j} \alpha_{2j}, \quad q_{1j} = c_j, \quad j = \overline{1, m}, \quad n = 0, 1. \end{aligned} \quad (1.20)$$

The local solutions (1.19) for the points $t = a_j, j = \overline{1, m}$ are sought in the form

$$u_j(t) = (t - a_j)^{\alpha_j} \tilde{u}_j(t), \quad \tilde{u}_j(t) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}(t - a_j)^n,$$

where $\gamma_{nj}, n = \overline{1, \infty}, j = \overline{1, m}$, are defined by the recursion formulas

$$f_{0j}(\alpha_j) = \alpha_j(\alpha_j - 1) + p_{0j}\alpha_j + q_{0j} = 0, \tag{1.21}$$

$$\gamma_{1j}f_{0j}(\alpha_j + 1) + f_{1j}(\alpha_j) = 0, \tag{1.22}$$

$$\gamma_{2j}f_{0j}(\alpha_{j+2}) + \gamma_{1j}f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j) = 0, \tag{1.23}$$

.....

$$\begin{aligned} &\gamma_{nj}f_{0j}(\alpha_j + n) + \gamma_{(n-1)j}f_{1j}(\alpha_j + n - 1) + \gamma_{(n-2)j}f_{2j}(\alpha_j + m - 2) + \\ &+ \dots + \gamma_{1j}f_{(n-1)j}(\alpha_j + 1) + f_{nj}(\alpha_j) = 0, \end{aligned} \tag{1.24}$$

.....

where

$$f_n(\alpha_j) = \alpha_j p_{nj} + q_{nj}. \tag{1.25}$$

If the difference $\alpha_{1j} - \alpha_{2j}, j = \overline{1, m}$, is not an integer, then using the formulas (1.22)–(1.24), we can construct the linearly independent solutions (1.8),

$$u_{kj}(t) = (t - a_j)^{\alpha_{kj}} \tilde{u}_{kj}(t), \quad \tilde{u}_{kj}(t) = 1 + \sum_{n=1}^{\infty} \gamma_{nj}^k (t - a_j)^n, \tag{1.26}$$

$$k = 1, 2, \quad j = \overline{1, m}.$$

If, however, $\alpha_{1j} - \alpha_{2j} = n, n = 0, 1, 2$, then $u_{1j}(t)$ can be constructed by the formulas (1.22)–(1.24), while $u_{2j}(t)$ by the Frobenius method [13], [17]. Note that if $\alpha_{1j} - \alpha_{2j} = 0$, then $u_{2j}(t)$ is of a simple form

$$u_{2j}(t) = u_{1j}(t) \ln(t - a_j) + (t - a_j)^{\alpha_{1j}} \sum_{n=1}^{\infty} \gamma_{nj}^2 (t - a_j)^n, \tag{1.27}$$

where

$$\gamma_{nj}^2 = \left\{ \frac{d\gamma_{1j}(\alpha_j)}{d\alpha_j} \right\}_{\alpha_j = \alpha_{2j}}.$$

If $\alpha_{1j} - \alpha_{2j} = n, n = 1, 2$, then for constructing $u_{2j}(t)$ we have to differentiate the equality

$$u_{2j}(t) = (t - a_j)^{\alpha_j} \left[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty} \gamma_{nj}(\alpha_j)(t - a_j)^n \right] \tag{1.28}$$

with respect to α_j and let $\alpha_j \rightarrow \alpha_{2j}$. Thus we obtain

$$u_{2j}(t) = (t - a_j)^{\alpha_{2j}} \left[\sum_{n=1}^{\infty} \lim_{\alpha_j \rightarrow \alpha_{2j}} \gamma_{nj}(\alpha_j)(t - a_j)^n \right] \ln(t - a_j) +$$

$$+(t - a_j)^{\alpha_{2j}} \left\{ 1 + \sum_{n=1}^{\infty} \left[\frac{d\gamma_n(\alpha_j)}{d\alpha_j} \right]_{\alpha_j=\alpha_{2j}} (t - a_j)^n \right\}. \quad (1.281)$$

P. Ya. Polubarinova-Kochina has proved that a solution for the cut end $u_{2j}(t)$, where $\alpha_{1j} - \alpha_{2j} = 2$, does not contain a logarithmic term. Moreover, for such points she has obtained the equation which connects the parameters $a_j, c_j, j = \overline{1, m}$. To such points $t = a_j$ on the contour $l(w)$ there correspond cut ends (the cuts may be circular or rectangular) with angle 2π . To construct $u_{2j}(t)$ uniquely, we have proposed [26]–[34] the following method. The equality (1.23) for the point $t = a_j$ fails to be fulfilled since

$$f_{0j}(\alpha_j + 2) = 0, \quad \alpha_j \rightarrow \alpha_{2j}. \quad (1.29)$$

For the equality (1.23) to take place as $\alpha_j \rightarrow \alpha_{2j}$, it is necessary and sufficient to require that

$$\gamma_{1j} f_1(\alpha_j + 1) + f_2(\alpha_j) = 0, \quad \alpha_j \rightarrow \alpha_{2j}. \quad (1.30)$$

After transformation, the condition (1.30) takes the form [26]–[34]

$$q_{2j} + q_{1j}^2 + q_{1j} p_{1j} = 0. \quad (1.31)$$

To construct $u_{2j}(t)$ uniquely, it suffices to construct uniquely $\gamma_{2j}^2(\alpha_{2j})$; the remaining $\gamma_{nj}^2(\alpha_{2j}), n = 1, 3, 4, \dots$, can be calculated by means of (1.22) and (1.24). Indeed, suppose $\alpha_j \neq \alpha_{2j}$. Then (1.23) yields

$$\gamma_{2j}(\alpha_j) = -[\gamma_{1j}(\alpha_j) f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j)] / f_0(\alpha_j + 2). \quad (1.32)$$

After developing indeterminacy in (1.30) as $\alpha_j \rightarrow \alpha_{2j}$ we obtain uniquely that

$$\gamma_{2j}^2 = -0, 5[p_{1j}(p_{1j} + 2q_{1j}) + p_{2j}] \quad (1.33)$$

Now we can define local solutions in the vicinity of the point $t = \infty$. We represent the functions $p(t)$ and $q(t)$ near $t = \infty$ as

$$p(t) = t^{-1} \sum_{n=0}^{\infty} p_{n\infty} t^{-n}, \quad q(t) = t^{-2} \sum_{n=0}^{\infty} q_{n\infty} t^{-n}, \quad (1.34)$$

where

$$p_{n\infty} = \sum_{k=1}^m \beta_k a_k^n, \quad p_{0\infty} = 6, \quad (1.35)$$

$$q_{n\infty} = \sum_{k=1}^m [\alpha_{1k} \alpha_{2k} (n+1) + c_k a_k] a_k^n, \quad (1.36)$$

$$q_{0\infty} = \sum_{k=1}^m [\alpha_{1k} \alpha_{2k} + c_k a_k], \quad (1.37)$$

$$q_{1\infty} = \sum_{k=1}^m [\alpha_{1k} \alpha_{2k} 2 + c_k a_k] a_k. \quad (1.38)$$

Local solutions in the vicinity of the point $t = \infty$ are sought in the form

$$u_\infty(t) = t^{-\alpha_\infty} + \sum_{n=1}^{\infty} \gamma_{n\infty} t^{-(\alpha_\infty+n)}, \tag{1.39}$$

where $\gamma_{n\infty}$, $n = \overline{1, \infty}$, are defined by the formulas

$$f_{0\infty}(\alpha_\infty) = \alpha_\infty(\alpha_\infty + 1) - p_{0\infty}\alpha_\infty + q_{0\infty} = 0, \tag{1.40}$$

$$\gamma_{1\infty} f_{0\infty}(\alpha_\infty + 1) - p_{1\infty}\alpha_\infty + q_{1\infty} = 0, \tag{1.41}$$

$$\gamma_{2\infty} f_{0\infty}(\alpha_\infty + 2) + \gamma_{1\infty}(\alpha_\infty + 1) - p_{2\infty}\alpha_\infty + q_{2\infty} = 0, \tag{1.42}$$

.....

$$\begin{aligned} &\gamma_{n\infty} f_{0\infty}(\alpha_\infty + n) + \gamma_{(n-1)\infty} f_{1\infty}(\alpha_\infty + \alpha - 1) + \\ &\quad + \gamma_{(n-2)\infty} f_{2\infty}(\alpha_\infty + n - 2) + \dots + \\ &\quad + \gamma_{1\infty} f_{(n-1)\infty}(\alpha_\infty + 1) - p_{n\infty}\alpha_\infty + q_{n\infty} = 0, \end{aligned} \tag{1.43}$$

.....

where

$$f_{k\infty} = q_{k\infty} - (\alpha_\infty + k)p_{k\infty}. \tag{1.44}$$

Taking into account that $t = \infty$ is the image of a nonangular point, the equation (1.40) must have the roots $\alpha_{1\infty} = 3$, $\alpha_{2\infty} = 2$ and thus

$$q_{0\infty} = \sum_{k=1}^m [\alpha_{1k}\alpha_{2k} + a_k c_k] = 6. \tag{1.45}$$

Because of the fact that $\alpha_{1\infty} - \alpha_{2\infty} = 1$, the equality (1.41) fails to be fulfilled. Therefore the formulas (1.41)–(1.43) allow one to define only one solution $u_{1\infty}(t)$. To define $u_{2\infty}(t)$, we act as follows [26]–[34]: for the equality (1.41) to take place as $\alpha_\infty \rightarrow \alpha_{2\infty}$, it is necessary and sufficient that the condition

$$q_{1\infty} - p_{1\infty}\alpha_{2\infty} = 0 \tag{1.46}$$

be fulfilled.

To define $\gamma_{1\infty}^2$, we act in the following manner: from (1.41) for $\alpha_\infty \neq \alpha_{2\infty}$ we define $\gamma_{1\infty}$ and get

$$\gamma_{1\infty} = [p_{1\infty}\alpha_\infty - q_{1\infty}] / f_{0\infty}(\alpha_\infty + 1). \tag{1.47}$$

Since the numerator and denominator in (1.47) vanish as $\alpha_\infty \rightarrow \alpha_{2\infty}$, developing indeterminacy we obtain uniquely [32]–[34] that

$$\gamma_{1\infty}^2 = p_{1\infty}. \tag{1.48}$$

Next, having found $\gamma_{1\infty}^2$, by the formulas (1.42) and (1.43) we define $\gamma_{n\infty}^2$, $n = \overline{2, \infty}$, and consequently, the solution $u_{2\infty}(t)$.

Finally, we have

$$u_{k\infty}(t) = t^{-\alpha_{k\infty}} + \sum_{n=1}^{\infty} \gamma_{n\infty}^k t^{-\alpha_{k\infty}-n}, \quad k = 1, 2. \tag{1.49}$$

It can be proved that the system (1.18), $M_k = 0$, $k = \overline{1, 3}$ coincides respectively with the systems (1.10₁), (1.45) and (1.46).

Local solutions $u_{k_j}(t)$, $k = 1, 2$, $j = \overline{1, m}$, contain many-valued functions of which we select one-valued branches as follows:

$$\begin{aligned} \exp[\beta_{k_j}(t - a_j)] &> 0, \quad t > a_j, \\ \{\exp[\alpha_{k_j} \ln(t - a_j)]\}^+ &= \exp[i\pi\alpha_{k_j}][\exp[\alpha_{k_j} \ln(a_j - t)]], \quad a_j > t, \\ \{\exp[\alpha_{k_\infty} \ln(t - a_j)]\}^- &= \exp[-i\pi\alpha_{k_j}][\exp[\alpha_{k_j} \ln(a_j - t)]], \quad a_j > t. \end{aligned}$$

For the equation (1.8) in the vicinity of each singular point $t = a_j$, $j = \overline{1, m+1}$, and in the vicinity of the points $t = a_j^* = (a_j + a_{j+1})/2$, $j = \overline{1, n-1}$, we construct respectively $u_{k_i}(t)$, $k = 1, 2$, $j = \overline{1, m+1}$.

A solution of (1.7) will be sought by means of the matrix $T\chi(t)$, where $\chi(t)$ is the solution of (1.11). Consequently, $T\chi(t)$ is likewise the solution of (1.11), where

$$T = \begin{pmatrix} p, & q \\ r, & s \end{pmatrix}, \quad \det T \neq 0, \quad (1.50)$$

p, q, r, s are the integration constants of the equation (1.13).

The local fundamental matrices $\Theta_j(t)$, $\sigma_j(t)$, $\Theta_j^*(t)$, $\Theta_j^\pm(t)$ are defined as follows:

$$\Theta_j(t) = \begin{pmatrix} u_{1j}(t), & u'_{1j}(t) \\ u_{2j}(t), & u'_{2j}(t) \end{pmatrix}, \quad a_j < t < a_{j+1}, \quad j = \overline{1, j-1}, \quad t = a_j, \quad j = \overline{1, n}, \quad (1.51)$$

$$\Theta_j^*(t) = \begin{pmatrix} u_{1j}^*(t), & u'_{1j}^*(t) \\ u_{2j}^*(t), & u'_{2j}^*(t) \end{pmatrix}, \quad a_j < t < a_{j+1}, \quad (1.52)$$

$$\sigma_j(t) = \begin{pmatrix} \sigma_{1j}(t), & \sigma'_{1j}(t) \\ \sigma_{2j}(t), & \sigma'_{2j}(t) \end{pmatrix}, \quad t = (a_j + a_{j+1})/2 = a_j^*, \quad j = \overline{1, m-1}, \quad (1.53)$$

$$\Theta_j^\pm(t) = \theta_j^\pm \Theta_j^*(t), \quad a_{j-1} < t < a_j, \quad (1.54)$$

$$\Theta_\infty(t) = \begin{pmatrix} u_{1\infty}(t), & u'_{1\infty}(t) \\ u_{2\infty}(t), & u'_{2\infty}(t) \end{pmatrix}, \quad (1.54_1)$$

where the matrices θ_j^\pm for $\alpha_{1j} - \alpha_{2j} \neq n$, $n = 0, 1, 2$, are defined as

$$\theta_j^\pm = \begin{pmatrix} \exp(\pm i\pi\alpha_{1j}), & 0 \\ 0, & \exp(\pm i\pi\alpha_{2j}) \end{pmatrix}, \quad (1.55)$$

while for $\alpha_{1j} - \alpha_{2j} = n$, $n = 0, 1, 2$, they are defined by the equalities

$$\theta_j^\pm = \exp[\pm i\pi\alpha_{2j}] \begin{pmatrix} 1, & 0 \\ \mp\pi i, & 1 \end{pmatrix}, \quad n = 0, 2,$$

$$\theta_j^\pm = \exp[\pm i\pi\alpha_{2j}] \begin{pmatrix} -1, & 0 \\ \mp\pi i, & 1 \end{pmatrix}, \quad n = 1.$$

It should be noted that the series $u_{k_j}(t)$, $k = 1, 2$, $j = \overline{1, m+1}$, converge slowly making the process of calculations difficult. To remove this drawback, we replace the series $u_{k_j}(t)$, $k = 1, 2$, $j = \overline{1, m+1}$, by rapidly and uniformly

convergent fundamental series. Towards this end, it is sufficient to write the series $u_{k_j}(t)$, $k = 1, 2$, $j = \overline{1, m+1}$, in the form [30-34]:

$$u_{k_j}(t) = (t - a_j)^{\alpha_{k_j}} \tilde{u}_{k_j}(t - a_j), \quad \tilde{u}_{k_j}(t - a_j) = 1 + \sum_{n=1}^{\infty} \gamma_{n_j}^k(t - a_j), \quad (1.56)$$

$$k = 1, 2; \quad j = \overline{1, m}, \quad u_{k\infty}(t) = t^{-\alpha_{k\infty}} \left(1 + \sum_{n=1}^{\infty} \gamma_{n\infty}^k(t) \right), \quad (1.57)$$

where $\gamma_{n_j}^k$, $\gamma_{n\infty}^k$ are defined through $f_{n_j}(\alpha_j)$ and $f_{k\infty}(\alpha_j)$ as follows:

$$f_{n_j}[(t - a_j), \beta_k] = \alpha_{k_j} p_{n_j}(t - a_j) + q_{n_j}(t - a_j), \quad (1.58)$$

$$f_{n_j}(t - a_j) = (-1)^{n-1} \sum_{k=1, k \neq j}^m \beta_j \left(\frac{t - a_j}{a_j - a_k} \right)^n, \quad n = 1, 2, \dots,$$

$$q_{1_j}(t - a_j) = c_j(t - a_j),$$

$$q_{n_j}(t - a_j) = (-1)^{n-2} \sum_{k=1, k \neq j}^m [\alpha_{1_k} \alpha_{2_k} (n-1) + c_k(a_j - a_k)] \left(\frac{t - a_j}{a_j - a_k} \right)^n, \quad (1.59)$$

$$\left| \frac{t - a_j}{a_j - a_k} \right| < 1, \quad k \neq j,$$

$$p_{n\infty}(t) = \sum_{k=1}^m \beta_k (a_k/t)^n, \quad q_{n\infty} = \sum_{k=1}^n [\alpha_{1_j} \alpha_{2_j} (n+1) + c_k a_k] (a_k/t)^n, \quad (1.60)$$

$$n = 0, 1, 2, \dots$$

The local matrix $\Theta_j(t)$ is complex conjugate with respect to the matrix $\Theta_j^+(t)$. The real matrices $\Theta_{j-1}(t)$, $\Theta_j^*(t)$ are the local solutions of the system of equations (1.11) in the vicinity of the points $t = a_{j-1}$, $t > a_{j-1}$, $t = a_j$, $t < a_j$. Suppose that the elements of these matrices converge on a part of the interval $a_{j-1} < t < a_j$, where the matrices $\Theta_j^*(t)$ and $\Theta_{j-1}(t)$ are connected by the matrix identity [26]-[34]

$$\Theta_j^*(t) = T_{j-1} \Theta_{j-1}(t), \quad (1.61)$$

which allows one to define the matrix T_{j-1} uniquely. Assume that the domains of convergence of the matrices $\Theta_1^*(t)$ and $\Theta_{j-1}(t)$ are nonintersecting. In this case we construct at the point $t = a_j^* = (a_{j-1} + a_j)/2$ the fundamental local matrix $\sigma_j(t)$ which converges on the interval $a_{j-1} < t < a_j$. It is evident that one can always pass from the matrix $\Theta_j^*(t)$ to the matrix $\Theta_{j-1}(t)$ successively:

$$\Theta_j^*(t) = T_{a_j^*} \sigma_j(t), \quad (1.62)$$

$$\Theta_j^*(t) = T_{j-1}^* \Theta_{j-1}(t). \quad (1.63)$$

$T_{a_j^*}$ and T_{j-1}^* are defined uniquely from (1.62) and (1.63). It follows from the above-said that $\theta_m(t)$ can be extended analytically along the whole axis t .

To define the functions $\omega'(t)$ and $z'(t)$ on the interval $(-\infty, +\infty)$, we consider the matrices [26]–[34]

$$\chi^*(t) = T\Theta_m^\pm(t), \quad t > a_m, \quad \Theta_m^+(t) = \Theta_m^-(t), \quad \Theta_m < t < +\infty, \quad (1.64)$$

where the matrix T is defined by the formula (1.50). From (1.64) it follows that $T = \overline{T}$.

The matrices $\chi^\pm(t)$ are the solutions of (1.11), where the signs $+$ and $-$ denote, respectively, the limiting values of the matrix $\chi(\zeta)$ from $\text{Im}(\zeta) > 0$ as $\zeta \rightarrow t$ and from $\text{Im}(\zeta) < 0$ as $\zeta \rightarrow t$.

Below we will define the matrix $\chi^+(t)$ and take into account that $\overline{\chi^+}(t) = \chi^-(t)$. The use will be made of the following notation: $\chi^+(t) = \chi(t)$, $\theta_j^+ = \theta_j$, $j = \overline{1, m}$.

$$\begin{aligned} \chi(t) &= T\theta_m\Theta_m^*(t), \quad a_{m-1} < t < a_m, \\ \chi(t) &= T\theta_m T_{m-1}\Theta_{m-1}(t), \quad \Theta_m^*(t) = T_{m-1}\Theta_{m-1}(t), \\ &\quad a_{m-1} < t < a_m, \\ \chi(t) &= T\theta_m T_{m-1}\theta_{m-1}\Theta_{m-1}^*(t), \quad a_{m-2} < t < a_{m-1}, \\ &\dots\dots\dots (1.65) \\ \chi(t) &= T\theta_m T_{m-1}\theta_{m-1}T_{m-2}\theta_{m-2}\dots T_1\Theta_1(t), \quad a_1 < t < a_2, \\ \chi(t) &= T\theta_m T_{m-1}\theta_{m-1}\dots T_1\theta_1\Theta_j^*(t), \quad -\infty < t < a_1, \\ \chi(t) &= T\theta_m T_{m-1}\theta_{m-1}\dots T_1\theta_1 T_{-\infty}\Theta_\infty(t), \quad -\infty < t < a_1, \\ \chi(t) &= TT_{+\infty}\Theta_\infty(t), \quad a_m < t < +\infty. \end{aligned}$$

Note that

$$\begin{aligned} \Theta_m^*(t) &= T_{m-1}\theta_{m-1}(t), \quad a_{m-1} < t < a_m, \\ \Theta_{m-1}^*(t) &= T_{m-2}\Theta_{m-2}(t), \quad a_{m-2} < t < a_{m-1}, \\ &\dots\dots\dots (1.66) \\ \Theta_2^*(t) &= T_1\Theta_1(t), \quad a_1 < t < a_2, \\ \Theta_1^*(t) &= T_{-\infty}\Theta_\infty(t), \quad -\infty < t < a_1, \\ \Theta_m(t) &= T_m\Theta_\infty(t), \quad a_m < t < +\infty. \end{aligned}$$

The matrices T_j , $j = \overline{1, m-1}$, $T_{-\infty}$, T_∞ from the system (1.66) are defined by means of the matrices $\Theta_j(t)$, $\Theta_j^*(t)$, $j = \overline{1, m}$, $\Theta_\infty(t)$, which depend on a_j , c_j , $j = \overline{1, m}$.

Substituting the matrices $\chi^+(t)$, $\overline{\chi(t)}$ defined on the intervals (a_{j-1}, a_j) , $j = m, m-1, \dots, 2, 1$, successively into the boundary condition (1.7) and then multiplying successively from the right each of the equalities by $[\Theta_j^*(t)]^{-1}$, $j = m, m-1, \dots, 1$, we obtain the system of matrix equations

$$T\theta_m = g_{m-1}T\overline{\theta}_m, \quad t = a_m; \quad (1.67)$$

$$T\theta_m T_{m-1}\theta_{m-1} = g_{m-2}T\overline{\theta}_m T_{m-1}\overline{\theta}_{m-1}, \quad t = a_{m-1}; \quad (1.68)$$

$$T\theta_m T_{m-1}\theta_{m-1}T_{m-2}\theta_{m-2} =$$

Further, for all the points $t = a_{m-1}, \dots, t = a_1$, we can write out similar matrices, which proves the above statement.

Now from the system (1.66) we define the elements of the matrices T_j , $j = \overline{1, m}$, depending on the parameters a_j, c_j , $j = \overline{1, m}$, and substitute them in (1.67)–(1.70); thus we obtain a system of matrix equations with respect to p, q, r, s, a_j, c_j , $j = \overline{1, m}$; from the matrix equations, as is said above, for each of the points $t = a_k$, $k = \overline{1, m}$, we obtain two scalar equations with respect to the parameters $a_j, c_j, p/s, q/s, r/s$, $j = \overline{1, m}$. Thus we obtain a system consisting of $2m$ equations.

According to Riemann's theorem, we can choose arbitrarily and fix three parameters from $t = a_j$, $j = \overline{1, m}$.

From the system (1.10₁), (1.45) and (1.46) we can define the next three parameters, for instance, c_1, c_2 and c_3 . Consequently, the number of unknown essential parameters a_j, c_j turns out to be equal to $2(m-3)$ [14]–[16]. In the general case, to the above-given parameters a_j, c_j we have to add three complex parameters of integration of the Schwartz equation. Hence the number of unknown parameters will be equal to $2(m-3) + 6 = 2m$. In our case, for $g_m(t) = E$ we have three integration parameters: $p/s, q/s$ and r/s . Thus the number of unknown parameters is equal to $2m - 6 + 3 = 2m - 3$, and the number of equations is $2m$. The difference is $2m - (2m - 3) = 3$.

The contour $l(w)$ of the domain $s(w)$ may contain vertices with the angles 2π formed by circular or linear cuts. For each of the vertices of $l(w)$ we obtain one equation of the type (1.30), because in the theory of filtration the coordinates of such vertices of the contour $l(w)$ are unknown beforehand. Suppose that the number of such vertices is two, then the number of equations will be equal to $2(m-1)$. In the theory of filtration, as it will be seen below, we may come across circular pentagons with only one cut. Then the number of essentially unknown parameters reduces to three, and the number of equations is equal to five. The difference between the number of equations and that of the essentially unknown parameters is equal to two. As is known, in the case of linear polygons the number of equations is by two units more than that of the essentially unknown parameters.

It is very difficult, but quite possible, to solve a system of three higher transcendent equations with respect to three essential parameters. If we denote by $u_1(t)$ and $u_2(t)$ the components of the vector $\Phi_1(t)$, then using the formula

$$w(t) = u_1(t)/u_2(t), \quad -\infty < t < +\infty, \quad (1.82)$$

we obtain the general solution of (1.3). The components $\omega'(t)$ and $z'(t)$ of the vector $\Phi'(t)$ are defined by the equalities

$$d\omega(t) = u_1(t)\chi_{01}(t)dt, \quad -\infty < t < +\infty, \quad (1.83)$$

and

$$dz(t) = u_2(t)\chi_{01}(t)dt, \quad -\infty < t < +\infty, \quad (1.84)$$

where $\omega'(t) = u_1(t)\chi_{01}(t)$ and $z'(t) = u_2(t)\chi_{01}(t)$ satisfy both the boundary conditions (1.3) and the conditions at the singular points $t = e_j$, $j = \overline{1, n+1}$.

The integration of the equalities (1.83) and (1.84) on the intervals $(-\infty, t)$, (e_j, t) , $j = \overline{1, n}$, results in

$$\omega(t) = \int_{-\infty}^t u_1(t)\chi_{01}(t)dt + \omega(-\infty), \quad (1.85)$$

$$z(t) = \int_{-\infty}^t u_2(t)\chi_{01}(t)dt + z(-\infty), \quad (1.86)$$

$$\omega(t) = \int_{e_j}^t u_2(t)\chi_{01}(t)dt + \omega(e_j + 0), \quad (1.87)$$

$$z(t) = \int_{e_j}^t u_2(t)\chi_{01}(t)dt + z(e_j + 0). \quad (1.88)$$

Considering (1.87) and (1.88) for $t = e_{j+1}$, we obtain a system of equations with respect to the removable singular points $t = e_j$, $t = e_{j+k}$ (parameters e_j , e_{j+k}) and parameters S , Q , where Q is the liquid discharge via filtration.

Equations (1.87) and (1.88) allow us to determine the parametric equation of the depression curve.

Remark. Sometimes it is advisable to map an arbitrary angular point of the contour $l(w)$ into the point $t = \infty$. Assume that the point B_{m+1} of $l(w)$ is an angular point with the angle $\pi\nu_{m+1}$. Let the point $t = a_{m+1} = \infty$ be the image of the angular point B_{m+1} with the characteristic numbers $\alpha_{1\infty}$ and $\alpha_{2\infty}$ which must satisfy the conditions

$$\alpha_{1\infty} - \alpha_{2\infty} = \nu_{m+1}, \quad (1.89)$$

$$p_{0\infty} \equiv \sum_{k=1}^m (1 - \alpha_{1k} - \alpha_{2k}) = 1 + \alpha_{1\infty} + \alpha_{2\infty}, \quad (1.90)$$

$$q_{0\infty} \equiv \sum_{j=1}^m [\alpha_{1j}\alpha_{2j} + a_j c_j] = \alpha_{1\infty} \cdot \alpha_{2\infty}. \quad (1.91)$$

Note that the condition $M_1 = 0$ in (1.8) remains true, while the condition $M_2 = 0$ is replaced by the condition (1.91). The condition $M_3 = 0$ fails to be fulfilled, i.e., $M_3 \neq 0$. The condition (1.90) is the Fuchs condition [13], [16].

2. LIQUID MOTION THROUGH THE PLANE EARTH DAM WHOSE LOWER AND UPPER SLOPES MAKE WITH THE HORIZON THE ANGLES, RESPECTIVELY, $\pi/2$ AND $\pi\alpha$

Schemes of the domains $s(z)$, $s(\omega)$ and $s(w)$ are given below in Fig. 1. The boundary conditions along the boundary $l(z)$ have the form: along the water boundaries A_1A_2 : $\varphi(x, y) = -H_2$, $x = L$; A_5A_6 : $\varphi(x, y) = -H_1$, $y = \operatorname{tg}(\pi\alpha)x$; along the leaking interval A_2A_3 : $\varphi(x, y) + y = 0$, $x = L$; along the unknown depression curve $A_3A_4A_5$: $\varphi(x, y) + y = 0$, $\psi(x, y) = Q$; along the dam base A_6A_1 : $\psi(x, y) = 0$, $y = 0$, where H_1 and H_2 is water depth, respectively, in the upper and lower pools, L is the length of water nonpermeable base of the plane earth dam, and Q is the liquid discharge via filtration.

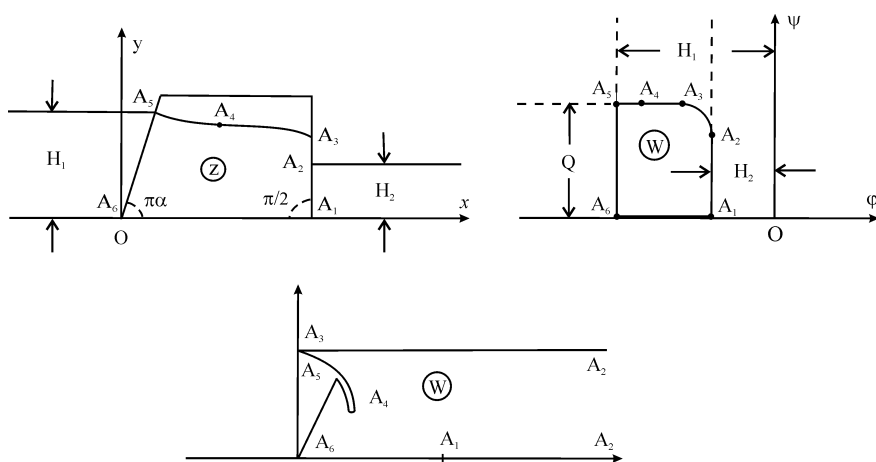


Fig. 1

The matrix $g(t)$ and the vector $f(t)$ are defined as follows:

$$\begin{aligned}
 g_{-\infty}(t) &= E, & f_{-\infty}(t) &= [0, 0], & -\infty < t < +\infty, \\
 g_1(t) &= E, & f_1(t) &= 2[-H_2; L], & e_1 < t < e_2, \\
 g_2(t) &= (-1) \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}, & f_2(t) &= 2L[i, 1], & e_2 < t < e_3, \\
 g_3(t) &= g_4(t) = \begin{pmatrix} 1 & 0 \\ -2i & 1 \end{pmatrix}, & f_3(t) &= f_4(t) = 2Q[i, 1], & e_3 < t < e_5, \\
 g_5(t) &= \begin{pmatrix} -1 & 0 \\ 0 & \exp(i2\pi\alpha) \end{pmatrix}, & f_5(t) &= -H_1[1; 0], & e_5 < t < e_6, \\
 g_6(t) &= E, & f_6(t) &= [0, 0], & e_6 < t < +\infty,
 \end{aligned} \tag{2.1}$$

where E is the unit matrix.

For the functions $\omega'(t)$ and $z'(t)$ at the singular points $t = e_j$, $j = \overline{1, 7}$, we set up the equations (1.4) and define the roots λ_{kj} , $k = 1, 2$, $j = \overline{1, 7}$, and then define uniquely the characteristic numbers $\alpha_{kj}^* = (2\pi i)^{-1} \ln \lambda_{kj}$,

$A_j[\alpha_{1j^*}, \alpha_{2j^*}]$, $j = \overline{1, 7}$. We have $A_1[1/2; -1/2]$, $A_2[0; 0]$, $A_3[0; 0]$, $A_4[2; 0]$, $A_5[-1/2; -\alpha]$, $A_6[-1/2; \alpha - 1]$, $A_7[3; 2]$.

The point $A_1[1/2; -1/2]$ is the removable singular point [1]–[8]. To remove it from the boundary conditions (1.3), we introduce the vector

$$\Phi'(t) = \chi_{02}(t)\Phi_2(t), \quad -\infty < t < +\infty, \quad (2.2)$$

where $\chi_{02}(t) = \sqrt{(t - e_2)(t - e_1)^{-1}} > 0$, $t > e_2$.

After transformations, on the contour $l(w)$ we renumerate the angular points and the corresponding points along the axis t and introduce the following notation: $B_j[\alpha_{1j}; \alpha_{2j}]$, $t = a_j$. $e_{j+1} = a_j$, $j = \overline{1, 5}$, $a_6 = a_7 = \infty$. We fix the points $t = a_j$ as follows: $a_1 = -b$, $a_2 = -a$, $a_3 = 0$, $a_4 = a$, $a_5 = b$.

For the points $B_j[\alpha_{1j}; \alpha_{2j}]$, $j = \overline{1, 6}$, we define the characteristic exponents α_{kj} , $k = 1, 2$, $j = \overline{1, 6}$, and the corresponding matrices $\theta_j^+ = \theta_j$, $\bar{\theta}^+ = \theta_j^-$, $j = 1, 6$:

$$B_1[-1/2; -1/2], \quad B_2[0; 0], \quad B_3[2; 0], \quad B_4[-1/2; 1 - \alpha], \quad B_5[-1/2; \alpha - 1], \\ B_6[3; 2] = B_\infty[3; 2],$$

$$\theta_1 = (-i) \begin{pmatrix} 1, & 0 \\ -i\pi, & 1 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 1, & 0 \\ \pi i, & 1 \end{pmatrix}, \quad \theta_3 = E, \\ \theta_4 = (-1) \begin{pmatrix} -1, & 0 \\ 0, & \exp(-i\pi\alpha) \end{pmatrix}, \quad \theta_5 = (-1) \begin{pmatrix} i, & 0 \\ 0, & \exp(i\pi\alpha) \end{pmatrix}.$$

Note that $\alpha_{1j} - \alpha_{2j} = \nu_j$, $j = \overline{1, 6}$, where $\pi\nu_j$ is the angle at the vertex b_j .

By means of the numbers α_{kj} , $k = 1, 2$, $j = \overline{1, 5}$, we write the Fuchs class equation

$$u''(t) + p(t)u'(t) + q(t)u(t) = 0, \quad (2.3)$$

where

$$p(t) = \sum_{j=1}^5 [1 - \alpha_{1j} - \alpha_{2j}](t - a_j)^{-1}, \\ q(t) = \sum_{j=1}^5 [\alpha_{1j}\alpha_{2j}(t - a_j)^{-2} + c_j(t - a_j)]. \quad (2.3_1)$$

Following Section 1, for the equation (2.3) we construct $u_{kj}(t)$, $\sigma_{kj}(t)$, $\Theta_j(t)$, $\Theta_j^*(t)$, $\sigma_j(t)$, $j = \overline{1, 6}$, $k = 1, 2$, and $\chi^+(t)$, $\chi^-(t)$, $\bar{\chi}^+(t) = \chi^-(t)$. Below we will use the notation $\chi^+(t) = \chi(t)$.

The system of equations (1.10₁), (1.45) and (1.46) for the problem under consideration has the form

$$\begin{aligned} \sum_{j=1}^5 c_j = 0, \quad \sum_{k=1}^5 [\alpha_{1k}\alpha_{2k} + a_k c_k] - 6 = 0, \\ \sum_{k=1}^5 [2\alpha_{1k}\alpha_{2k} + c_k a_k] a_k - p_{1\infty}\alpha_{2\infty} = 0, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} p_{1\infty} &= \sum_{j=1}^5 (1 - \alpha_{1j} - \alpha_{2j}) a_k, \quad p_{13} = \sum_{k=1, k \neq 3}^5 (1 - \alpha_{1k} - \alpha_{2k}) (a_3 - a_k)^{-1}, \\ q_{n3} &= (-1)^{n-2} \sum_{k=1, k \neq 3}^5 [\alpha_{1k}\alpha_{2k}(n-1) + c_k(a_3 - a_k)] (a_3 - a_k)^{-n}, \quad n = \overline{2, \infty}, \\ q_{0j} &= \alpha_{1j}\alpha_{2j}, \quad q_{1j} = c_j, \quad n = 0, 1. \end{aligned}$$

Using the formulas (1.65) and (1.66) for $m = 5$, we construct the matrix $\chi(t)$, and according to the formulas (1.67) for the points $t = a_k$, $k = \overline{1, 5}$, we construct the matrix equations

$$\begin{aligned} t = a_5 : T\theta_5 = g_4 T\bar{\theta}_5; \quad t = a_4 : T\theta_5 T_4 \theta_4 = g_3 T\bar{\theta}_5 T_4 \bar{\theta}_4; \\ t = a_2 : T\theta_5 T_4 \theta_4 T_{32} \theta_2 = g_2 T\bar{\theta}_5 T_4 \bar{\theta}_4 T_{32} \bar{\theta}_2, \\ t = a_1 : T\theta_5 T_4 \theta_4 T_{32} \theta_2 T_1 \theta_1 = T\bar{\theta}_5 T_4 \bar{\theta}_4 T_{32} \bar{\theta}_2 T_1 \bar{\theta}_1, \end{aligned} \quad (2.5)$$

where $T_{32} = T_3 T_2$, $\theta_j^*(t) = T_{j-1} \Theta_{j-1}(t)$, $j = \overline{2, 5}$, $\Theta_1^*(t) = T_{-\infty} \Theta_{inf ty}(t)$, $\Theta_5(t) = T_{+\infty}$, $\Theta_\infty(t)$.

From the matrix equations (2.5) we obtain respectively the scalar equations

$$t = a_5 : q = 0, \quad r = 0; \quad (2.6)$$

$$t = a_4 : q_4 = 0, \quad (2.7)$$

$$pp_4 - sr_4 \cos(\pi\alpha) = 0, \quad (2.8)$$

$$t = a_3 : q_{23} + q_{13}^2 + q_{13}p_{13} = 0, \quad (2.9)$$

$$t = a_2 : r_4 q_{32} \sin(\pi\alpha) - s_4 s_{32} = 0, \quad (2.10)$$

$$r_4 (p_{32} \sin(\pi\alpha) + \pi q_{32} \cos(\pi\alpha)) - s_4 r_{32} = 0, \quad (2.11)$$

$$t = a_1 : q_1 p_{32} + s_1 q_{32} = 0, \quad (2.12)$$

$$p_1 p_{32} + (r_1 - \pi^2 q_1) q_{32} = 0. \quad (2.13)$$

For the point $t = a_3$ we get only one equation (2.9). The compatibility conditions for the systems (2.10), (2.11) and (2.12), (2.13) have the form

$$\operatorname{tg}(\pi\alpha) \det T_{32} + \pi s_{32} q_{32} = 0, \quad (2.14)$$

$$\det T_1 + \pi q_1^2 = 0. \quad (2.15)$$

From the matrix equations $\theta_j^*(t) = T_{j-1}\Theta_{j-1}(t)$, $j = \overline{2,5}$, we define the elements of the matrices T_j , $j = \overline{1,4}$, and substitute them in the system (2.7)–(2.15).

The systems (2.4) and (2.9) allow us to find the unknown parameters c_1 , c_2 , c_4 , and c_5 . We substitute them in the system of equations (2.7)–(2.15) and in the matrices $\theta_j(t)$, $j = \overline{1,6}$. The number of unknown essential parameters reduces to three: a , b and c_3 .

The number of equations for the determination of the parameters a , b and c_3 is equal to five: they are (2.7), (2.10), (2.11), (2.12) and (2.13). The difference between the number of equations and that of the unknown parameters a , b and c_3 is equal to two. To define a , b , and c_3 from the system (2.7), (2.10)–(2.15), we take the system (2.7), (2.14) and (2.15) and solve it, we substitute the obtained parameters a , b and c_3 in (2.4), (2.8)–(2.13), (1.66) (for $m = 5$) and find c_j , $j = \overline{1,5}$, $p/s = r_4 \cos(\pi\alpha)/p_4$. Moreover, we define the matrices T_j , $j = \overline{1,4}$, $T_{-\infty}$, $T_{+\infty}$ where the parameter α is fixed. Having found the functions $\omega(t)$ and $z(t)$, we can define the remaining parameters, for instance, $t = e_1$ and Q .

The matrix $\chi(t)$ along the axis t is defined as follows:

$$\begin{aligned}
\chi(t) &= \begin{pmatrix} p, & 0 \\ 0, & s \end{pmatrix} \Theta_5(t), \quad t > a_5, \\
\chi(t) &= \begin{pmatrix} pp_{+\infty}, & pq_{+\infty} \\ sr_{+\infty}, & ss_{+\infty} \end{pmatrix} \Theta_{\infty}(t), \quad a_5 < t < +\infty, \\
\chi(t) &= (-1) \begin{pmatrix} ip, & 0 \\ 0, & \exp(i\pi\alpha) \end{pmatrix} \Theta_5^*(t), \quad a_4 < t < a_5, \\
\chi(t) &= (-1) \begin{pmatrix} pp_4, & 0 \\ sr_4 \exp(i\pi\alpha), & ss_4 \exp(i\pi\alpha) \end{pmatrix} \Theta_5^*(t) \Theta_4, \quad a_4 < t < a_5, \\
\chi(t) &= \begin{pmatrix} (-1)pp_4, & 0 \\ isru \exp(i\pi\alpha), & ss_4 \end{pmatrix} \Theta_4^*(t), \quad t < a_4, \\
\chi(t) &= \begin{pmatrix} -pp_4p_3, & -pp_4q_3 \\ isrup_3 \exp(i\pi\alpha) + ss_4r_3, & isr_4q_3 \exp(i\pi\alpha) + ss_4s_3 \end{pmatrix} \Theta_3(t), \\
&\quad a_2 < t < a_4, \\
\chi(t) &= pp_4 \begin{pmatrix} -p_{32}, & -q_{32} \\ ip_{32} + \pi q_{32}, & iq_{32} \end{pmatrix} \Theta_2(t), \quad a_2 < t < a_4, \\
\chi(t) &= pp_4 \begin{pmatrix} -p_{32} - i\pi q_{32}, & -q_{32} \\ ip_{32}, & iq_{32} \end{pmatrix} \Theta_2^*(t), \quad a_1 < t < a_2, \\
\chi(t) &= (-1)\pi pp_4 q_{32} \begin{pmatrix} \pi q_1 + ip_1, & iq_1 \\ -i\pi q_1, & 0 \end{pmatrix} \Theta_1(t), \quad a_1 < t < a_2, \\
\chi(t) &= (-1)\pi pp_4 q_{32} \begin{pmatrix} p_1, & q_1 \\ -\pi q_1, & 0 \end{pmatrix} \theta_1^*(t), \quad -\infty < t < a_1.
\end{aligned} \tag{2.16}$$

The linearly independent solutions $u_1(t)$ and $u_2(tr)$ of the equation (2.3) which are defined by the elements of the first column of the matrix $\chi(t)$ can

be represented along the axis t as follows:

$$\begin{aligned}
u_1(t) &= pu_{15}(t), \quad u_2(t) = su_{25}(t), \quad a_5 < t < +\infty, \\
u_1(t) &= (-i)pu_{15}^*(t), \quad u_2(t) = (-1)s \exp(i\pi\alpha)u_{25}^*(t), \quad a_4 < t < a_5, \\
u_1(t) &= (-i)pp_4u_{14}(t), \quad u_2(t) = (-1)s \exp(i\pi\alpha)[r_4u_{14}(t) + su_{24}(t)], \\
&\quad a_4 < t < a_5, \\
u_1(t) &= -pp_4u_{14}^*(t), \quad u_2(t) = isr_4 \exp(i\pi\alpha)u_{14}^*(t) + ss_4u_{24}^*(t), \\
&\quad a_3 < t < a_4, \\
u_1(t) &= -pp_4[p_3u_{13}(t) + q_3u_{23}(t)], \quad a_3 < t < a_4, \\
u_2(t) &= [isru \exp(i\pi\alpha)p_3 + ss_4r_3]u_{13}(t) + \\
&\quad + [isr_4q_3 \exp(i\pi\alpha) + ss_3s_4]u_{23}(t), \quad a_3 < t < a_4, \\
u_1(t) &= -pp_4[p_{32}u_{12}(t) + q_{32}u_{22}(t)], \quad a_2 < t < a_3, \\
u_2(t) &= pp_4[(ip_{32} + \pi q_{32})u_{12}(t) + iq_{32}u_{22}(t)], \quad a_2 < t < a_3 \\
u_1(t) &= -pp_4[(p_{32} + i\pi q_{32})u_{12}^*(t) + q_{32}u_{22}^*(t)], \quad a_1 < t < a_2, \\
u_2(t) &= ipp_4[(p_{32}u_{12}^*(t) + q_{32}u_{22}^*(t))], \quad a_1 < t < a_2, \\
u_1(t) &= (-1)\pi pp_4q_{32}[(\pi q_1 + ip_1)u_{11}(t) + iq_1u_{21}(t)], \quad a_1 < t < a_2, \\
u_2(t) &= i\pi^2 pp_4q_{32}q_1u_{11}(t), \quad a_1 < t < a_2, \\
u_1(t) &= (-1)\pi pp_4q_{32}[p_1u_{11}^*(t) + q_1u_{21}^*(t)], \quad -\infty < t < a_1, \\
u_2(t) &= \pi^2 pp_4q_{32}q_1u_{11}^*(t), \quad -\infty < t < a_1, \\
u_1(t) &= (-1)\pi pp_4q_{32}[p_1v_{-1}(t) + q_1v_{-2}(t)], \quad -\infty < t < a_1, \\
u_2(t) &= \pi^2 pp_4q_{32}q_1v_{-1}(t), \quad -\infty < t < a_1,
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
v_{-1}(t) &= p_{-\infty}u_{1\infty}(t) + q_{-\infty}u_{2\infty}(t), \\
v_{-2}(t) &= r_{-\infty}u_{1\infty}(t) + s_{-\infty}u_{2\infty}(t).
\end{aligned}$$

The components $\omega'(t)$ and $z'(t)$ of the vector $\phi'(t)$ are defined by the equalities

$$d\omega(t) = \chi_{02}(t)u_1(t)dt, \quad -\infty < t < +\infty, \tag{2.18}$$

$$dz(t) = \chi_{02}(t)u_2(t)dt, \quad -\infty < t < +\infty. \tag{2.19}$$

Integrating (2.18) and (2.19) on the intervals (e_j, t) , $j = \overline{1, 6}$, we obtain

$$\omega(t) = p \int_{a_5}^t \chi_{02}(t)u_{15}(t)dt - H_1, \tag{2.20}$$

$$z(t) = s \int_{a_5}^t \chi_{02}(t)u_{25}(t)dt, \tag{2.21}$$

$$\omega(t) = (-i) \int_{a_4^*}^t \chi_{02}(t) u_{15}^*(t) dt - H_1 + i\psi(a_4^*), \quad (2.22)$$

$$z(t) = (-1)s \exp(i\pi\alpha) \int_{a_4}^t \chi_{02}(t) u_{25}(t) dt + z(a_4^*), \quad (2.23)$$

$$\omega(t) = -ipp_4 \int_{a_4}^t \chi_{02}(t) u_{14}(t) dt - H_1 + iQ, \quad (2.24)$$

$$z(t) = -s \exp(i\pi\alpha) [r_4 \int_{a_4}^t \chi_{02}(t) u_{14}(t) dt + s_4 \int_{a_4}^t \chi_{02}(t) u_{24}(t) dt] + H_1 [\text{ctg}(\pi\alpha) + i], \quad (2.25)$$

$$\omega(t) = -pp_4 \int_{a_3^*}^t \chi_{02}(t) u_{14}^*(t) dt + \varphi(a_3^*) + iQ, \quad (2.26)$$

$$z(t) = isr_4 \exp(i\pi\alpha) \int_{a_3^*}^t \chi_{02}(t) u_{14}^*(t) dt + ss_4 \int_{a_3^*}^t \chi_{02}(t) u_{24}^*(t) dt + z(a_3^*), \quad (2.27)$$

$$\omega(t) = -pp_4 \int_{a_2^*}^t \chi_{02}(t) [p_3 u_{13}(t) + q_3 u_{23}(t)] dt + \varphi(a_2^*) + iQ, \quad (2.28)$$

$$z(t) = [isr_4 p_3 \exp(i\pi\alpha) + ss_4 r_3] \int_{a_2^*}^t \chi_{02}(t) u_{13}(t) dt + [isr_4 q_3 \exp(i\pi\alpha) + ss_3 s_4] \int_{a_2^*}^t \chi_{02}(t) u_{23}(t) dt + z(a_2^*), \quad (2.29)$$

$$\omega(t) = -pp_4 \int_{a_2}^t \chi_{02}(t) [p_{32} u_{12}(t) + q_{32} u_{22}(t)] dt - y(a_2) + iQ, \quad (2.30)$$

$$z(t) = pp_4 (ip_{32} + \pi q_{32}) \int_{a_2}^t \chi_{02}(t) u_{12}(t) dt +$$

$$+ipp_4q_{32} \int_{a_2}^t \chi_{02}(t)u_{22}(t)dt + L + iy(a_2), \quad (2.31)$$

$$\begin{aligned} \omega(t) = -pp_4[(p_{32} + i\pi q_{32}) \int_{a_1^*}^t \chi_{02}(t)u_{12}^*(t)dt + q_{32} \int_{a_1^*}^t \chi_{02}(t)u_{22}^*(t)dt] + \\ + \varphi(a_1^*) + i\psi(a_1^*), \end{aligned} \quad (2.32)$$

$$\begin{aligned} z(t) = ipp_4 \left[p_{32} \int_{a_1^*}^t \chi_{02}(t)u_{12}^*(t)dt + q_{32} \int_{a_1^*}^t \chi_{02}(t)u_{22}^*(t)dt \right] + \\ + L + iy(a_1^*), \end{aligned} \quad (2.33)$$

$$\begin{aligned} \omega(t) = (-1)\pi pp_4q_{32} \left[(\pi q_1 + ip_2) \int_{a_1}^t \chi_{02}(t)u_{11}(t)dt + ia_1 \int_{a_1}^t \chi_{02}(t)u_{21}(t)dt \right] - \\ - H_2 + iQ', \end{aligned} \quad (2.34)$$

$$z(t) = i\pi^2 pp_4q_{32}q_1 \int_{a_1}^t \chi_{02}(t)u_{11}(t)dt + L + iH_2, \quad (2.35)$$

$$\begin{aligned} \omega(t) = -i\pi pp_4q_{32} \left[p_1 \int_{e_1^*}^t \tilde{\chi}_{02}(t)u_{11}^*(t)dt + q_1 \int_{e_1}^t \tilde{\chi}_{02}(t)u_{21}^*(t)dt \right] - \\ - H_2 + i\psi(e_1^*), \end{aligned} \quad (2.36)$$

$$z(t) = i\pi^2 pp_4q_{32}q_1 \int_{e_1^*}^t \tilde{\chi}_{02}(t)u_{11}^*(t)dt + L + iy(e_1^*), \quad (2.37)$$

$$\omega(t) = (-i)\pi pp_4q_{32} \int_{e_1^*}^t \tilde{\chi}_{02}(t)[p_1v_{-1}(t) + q_1v_{-2}(t)]dt - H_2, \quad (2.38)$$

$$z(t) = i\pi^2 pp_4q_{32}p_1 \int_{e_1}^t \tilde{\chi}_{02}(t)y_{-1}(t)dt + L, \quad (2.39)$$

where

$$\tilde{\chi}_{02}(t) = \sqrt{(e_2 - t)(t - e_1)^{-1}}, \quad e_1 < t < e_2, \quad (2.40)$$

Q' is the liquid discharge via filtration though the interval

$$(e_1, e_2), e_j^* = (e_{j-1} + e_j)/2, \quad a_j^* = (a_{j-1} + a_j)/2, \quad j = \overline{2, 5}. \quad (2.41)$$

Consider (2.39) and (2.38) for $t = e_1^*$,

$$x = L, \quad y(e_1^*) = \pi^2 pp_4 q_{32} p_1 \int_{e_1}^{e_1^*} \tilde{\chi}_{02}(t) V_{-1}(t) dt, \quad (2.42)$$

$$\varphi(e_1^*) = -H_2, \quad \psi(e_1^*) = -\pi pp_4 q_{32} \int_{e_1}^{e_1^*} \tilde{\chi}_{02}(t) [p_1 v_{-1}(t) + q_1 v_{-2}(t)] dt. \quad (2.43)$$

If we substitute the values of $y(e_1^*)$ and $\psi(e_1^*)$ in (2.37) and (2.36), and consider them for $t = a_1$, we obtain

$$x(a_1) = L, \quad y(a_1) = \pi^2 pp_4 q_{32} q_1 \int_{e_1^*}^{a_1} \tilde{\chi}_{02}(t) u_{11}^*(t) dt + y(e_1^*), \quad (2.44)$$

$$\varphi(a_1) = -H_2, \quad (2.45)$$

$$Q' = -\pi pp_4 q_{32} \left[p_1 \int_{e_1^*}^{a_1} \tilde{\chi}_{02}(t) u_{11}^*(t) dt + q_1 \int_{e_1^*}^{a_1} \tilde{\chi}_{02}(t) u_{21}^*(t) dt \right] + \psi(e_1^*). \quad (2.46)$$

Having defined the parameters, we can define the liquid discharge Q' through the interval $[e_1, e_2]$ by the formula (2.46).

Define now the values of the functions (2.34) and (2.35) for $t = a_1^*$. We have

$$x(a_1^*) = L, \quad y(a_1^*) = \pi^2 pp_4 q_{32} q_1 \int_{a_1}^{a_1^*} \chi_{02}(t) u_{11}(t) dt + H_2, \quad (2.47)$$

$$\varphi(a_1^*) = (-1)\pi^2 pp_4 q_{32} p_2 \int_{a_1}^{a_1^*} \chi_{02}(t) u_{11}(t) dt - H_2, \quad (2.48)$$

$$\begin{aligned} \psi(a_1^*) &= (-1)\pi pp_4 q_{32} \times \\ &\times \left[p_2 \int_{a_1}^{a_1^*} \chi_{02}(t) u_{11}(t) dt + a_1 \int_{a_1}^{a_1^*} \chi_{02}(t) u_{21}(t) u_{21}(t) dt \right] + Q'. \end{aligned} \quad (2.49)$$

By the formulas (2.33) and (2.32) we can define $z(a_2)$ and $\omega(a_2)$. We get

$$x(a_2) = L, \quad (2.50)$$

$$y(a_2) = pp_4 \left[p_{32} \int_{a_1^*}^{a_2} \chi_{02}(t) u_{12}^*(t) dt + q_{32} \int_{a_1^*}^{a_2} \chi_{02}(t) u_{22}^*(t) dt \right] + y(a_1^*), \quad (2.51)$$

$$\varphi(a_2) = -pp_4 \left[p_{32} \int_{a_1^*}^{a_2} \chi_{02}(t) u_{12}^*(t) dt + q_{32} \int_{a_1^*}^{a_2} \chi_{02}(t) u_{22}^*(t) dt \right] - y(a_1^*), \quad (2.52)$$

$$Q = \psi(a_2) - \pi p p_4 q_{32} \int_{a_1^*}^{a_2} \chi_{02}(t) u_{12}^*(t) dt + \psi(a_1^*). \quad (2.53)$$

Since $\varphi(a_2) = -y(a_2)$, the equations (2.51) and (2.52) coincide. Considering (2.31) and (2.30) for $t = a_2^*$, we find that

$$x(a_2^*) = \pi p p_4 q_{32} \int_{a_2}^{a_2^*} \chi_{02}(t) u_{12}(t) dt + L, \quad (2.54)$$

$$y(a_2^*) = p p_4 \left[p_{32} \int_{a_2}^{a_2^*} \chi_{02}(t) u_{12}(t) dt + q_{32} \int_{a_2}^{a_2^*} \chi_{02}(t) u_{22}(t) dt \right] + y(a_2), \quad (2.55)$$

$$\begin{aligned} \varphi(a_2^*) = & -p p_4 \int_{a_2}^{a_2^*} \chi_{02}(t) \left[p_{32} u_{12}(t) dt + q_{32} u_{22}(t) dt \right] dt - \\ & -y(a_2), \quad \psi(a_2^*) = Q. \end{aligned} \quad (2.56)$$

If we consider (2.29) and (2.28) for $t = a_3$, then we obtain

$$\begin{aligned} x(a_3) = & [-s r_4 p_3 \sin(\pi\alpha) + s s_4 r_3] \int_{a_2^*}^{a_3} \chi_{02}(t) u_{13}(t) dt + \\ & + [-s r_4 q_3 \sin(\pi\alpha) + s s_3 s_4] \int_{a_2^*}^{a_3} \chi_{02}(t) u_{23}(t) dt + x(a_2^*), \end{aligned} \quad (2.57)$$

$$\begin{aligned} y(a_3) = & s r_4 p_3 \cos(\pi\alpha) \int_{a_2^*}^{a_3} \chi_{02}(t) u_{13}(t) dt + \\ & + s r_4 q_3 \cos(\pi\alpha) \int_{a_2^*}^{a_3} \chi_{02}(t) u_{23}(t) dt + y(a_2^*), \end{aligned} \quad (2.58)$$

$$\begin{aligned} \varphi(a_3) = & -p p_4 \int_{a_2^*}^{a_3} \chi_{02}(t) [p_3 u_{13}(t) + q_3 u_{23}(t)] dt + \\ & + \varphi(a_2^*), \quad \psi(a_3) = Q. \end{aligned} \quad (2.59)$$

From (2.29) and (2.28) we find $z(a_3^*)$ and $\omega(a_3^*)$. We substitute the obtained values $z(a_3^*)$ and $\omega(a_3^*)$ in (2.27) and (2.26). We do not write out these values, because they can be obtained from (2.57)–(2.59) if instead of $t = a_3$ we take $t = a_3^*$.

Considering (2.27) and (2.26) for $t = a_4$, we obtain

$$x(a_4) = -sr_4 \sin(\pi\alpha) \int_{a_3^*}^{a_4} \chi_{02}(t) u_{14}^*(t) dt + ss_4 \int_{a_3^*}^{a_4} \chi_{02}(t) u_{24}^*(t) dt + x(a_3^*), \quad (2.60)$$

$$y(a_4) = sr_4 \cos(\pi\alpha) \int_{a_3^*}^{a_4} \chi_{02}(t) u_{14}^*(t) dt + y(a_3^*), \quad (2.61)$$

$$\varphi(a_4) = -pp_4 \int_{a_3^*}^{a_4} \chi_{02}(t) u_{14}^*(t) dt + \varphi(a_3^*), \quad \psi(a_4) = Q. \quad (2.62)$$

By means of the formulas (2.25) and (2.24) we find $z(a_4^*)$ and $\omega(a_4^*)$. Thus we have

$$x(a_4^*) = -s \cos(\pi\alpha) \left[r_4 \int_{a_4}^{a_4^*} \chi_{02}(t) u_{14}(t) dt + s_4 \int_{a_4}^{a_4^*} \chi_{02}(t) q_{24}(t) dt \right] + H_1 \operatorname{ctg}(\pi\alpha), \quad (2.63)$$

$$y(a_4^*) = H_1 - \sin(\pi\alpha) \left[r_4 \int_{a_4}^{a_4^*} \chi_{02}(t) u_{14}(t) dt + s_4 \int_{a_4}^{a_4^*} \chi_{02}(t) u_{24}(t) dt \right], \quad (2.64)$$

$$\varphi(a_4^*) = -H_1, \quad \psi(a_4^*) = -pp_4 \int_{a_4}^{a_4^*} \chi_{02}(t) u_{14} dt + Q. \quad (2.65)$$

Finally, considering (2.23) and (2.22) for $t = a_5$, we get

$$(-1)s \cos(\pi\alpha) \int_{a_4^*}^{a_5} \chi_{02}(t) u_{25}^*(t) dt + x(a_4^*) = 0, \quad (2.66)$$

$$(-1)s \sin(\pi\alpha) \int_{a_4^*}^{a_5} \chi_{02}(t) u_{25}^*(t) dt + y(a_4^*) = 0, \quad (2.67)$$

$$\varphi(a_5) = -H_1, \quad \psi(a_5) = -p \int_{a_4^*}^{a_5} \chi_{02}(t) u_{15}^*(t) dt + \psi(a_4^*) = 0. \quad (2.68)$$

Taking into account (2.63)–(2.65), let us consider the system of equations (2.66), (2.67) and (2.68). From the system (2.66) and (2.67) we find the parameters e_1 and s and then substitute them in (2.20)–(2.65) and (2.68). Then from the formula (2.65), with regard for (2.68), we define Q , and the formula (2.53), with regard for (2.68) we define Q' . Formula (2.51), with regard for (2.47) allows us to define $y(a_2)$. When all the unknown

parameters are determined, we can find the parametric equation of the depression curve by means of the formulas (2.26)–(2.31).

3. LIQUID MOTION THROUGH THE PLANE EARTH DAM WHOSE LOWER AND UPPER SLOPES MAKE WITH THE HORIZON THE ANGLES $\pi(1 - \beta)$ AND $\pi/2$, RESPECTIVELY

Schemes of the domains $s(z)$, $s(\omega)$ and $s(w)$ are given in Fig. 2, and the boundary conditions along $l(z)$ have the form: along the water boundaries A_1A_2 : $\varphi(x, y) = -H_2$, $y = -\text{tg}(\pi\beta)(x - L)$; A_4A_5 : $\varphi(x, y) = -H_1$, $x = 0$; along the leaking interval A_2A_3 : $\varphi(x, y) + y = 0$, $y = -\text{tg}(\pi\beta)(x - L)$; along the unknown depression curve A_3A_4 : $\varphi(x, y) + y = 0$, $\psi(x, y) = Q$; along the dam base A_5A_1 : $\psi(x, y) = 0$ $y = 0$, where H_1 and H_2 are water depth, respectively, in the upper and lower pools, L is length of water nonpermeable dam base, and Q is liquid discharge via filtration.

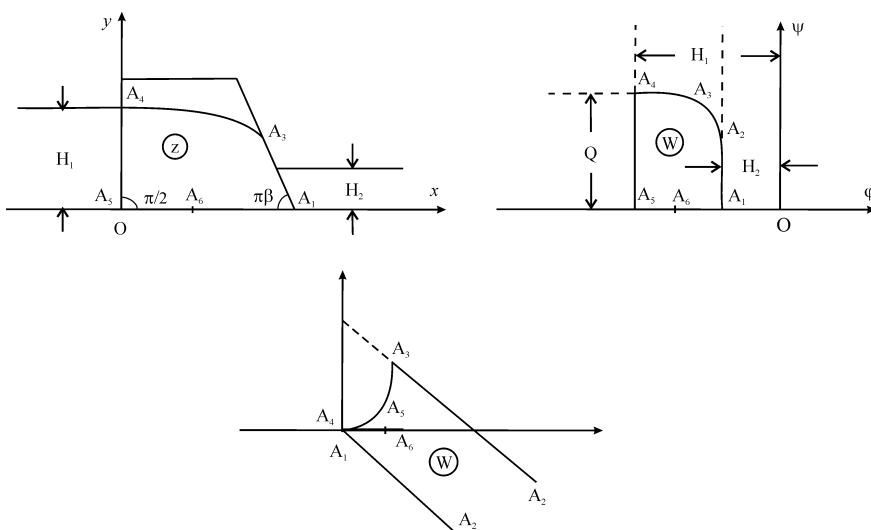


Fig. 2

The piecewise constant matrix $g(t)$ and the vector $f(t)$ are defined as follows:

$$\begin{aligned}
 g_{-\infty}(t) &= [0, 0], \quad f_{-\infty}(t) = [0, 0], \quad -\infty < t < e_1, \\
 g_1(t) &= \begin{pmatrix} -1, & 0 \\ 0, & \exp(-i2\pi\beta) \end{pmatrix}, \\
 f_1(t) &= 2i[-H_2, L \sin(\pi\beta) \exp(-i\pi\beta)], \quad e_1 < t < e_2, \\
 g_2(t) &= \begin{pmatrix} -1, & 2 \sin(\pi\beta) \exp(-i\pi\beta) \\ 0, & \exp(-i2\pi\beta) \end{pmatrix}, \\
 f_2(t) &= 2L \sin(\pi\beta) \exp(i\pi\beta)[-1, i], \quad e_2 < t < e_3,
 \end{aligned} \tag{3.1}$$

$$\begin{aligned}
g_3(t) = g_4(t) &= \begin{pmatrix} 1, & 0 \\ -2i, & 1 \end{pmatrix}, \quad f_3(t) = f_4(t) = 2Q[1, i], \quad e_3 < t < e_4, \\
g_5(t) &= \begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}, \quad f_5(t) = -2H_1[1, 0], \quad e_4 < t < e_5, \\
g_6(t) = E, \quad f_6(t) &= [0, 0], \quad e_5 < t < e_6 < t < +\infty,
\end{aligned}$$

where E is the unit matrix.

Characteristic exponents for $\omega'(t)$ and $z'(t)$ at the points A_j , $j = \overline{1, 7}$, are defined in the following manner: $A_j[\alpha_{1j}^*; \alpha_{2j}^*]$, $j = \overline{1, 7}$, $A_1[-1/2; \beta - 1]$, $A_2[0; 0]$, $A_3[1/2 - \beta; 0]$, $A_4[-1/2; -1/2]$, $A_5[1/2; -1/2]$, $A_6 = [2; 0]$, $A_7[3; 2]$. $A_5[1/2; -1/2]$ is the removable singular point.

Let us introduce a new unknown vector by the formula

$$\Phi'(t) = \chi_{03}(t)\Phi_3(t), \quad -\infty < t < +\infty, \quad (3.2)$$

where

$$\chi_{03}(t) = \sqrt{(t - e_5)(t - e_5)^{-1}} > 0, \quad t > e_5. \quad (3.2_1)$$

We renumerate the angular points of the boundary $l(w)$ of the domain $s(w)$ and the corresponding points along the axis and introduce for them the following notation: $B_j[\alpha_{1j}; \alpha_{2j}]$, $t = a_j$, $e_j = a_j$, $j = \overline{1, 6}$, $a_6 = e_7 = \infty$. We fix the points $t = a_j$, $j = \overline{1, 5}$ as: $a_1 = -b$, $a_2 = -a$, $a_3 = 0$, $a_4 = a$, $a_5 = b$.

For the points $B_j[\alpha_{1j}; \alpha_{2j}]$, $j = \overline{1, 6}$, the characteristic exponents α_{kj} , $k = \overline{1, 6}$, and the corresponding matrices $\theta_j^+ = \theta_j$, $\bar{\theta}_1^+ = \theta_1^-$, $j = \overline{1, 6}$ have the form

$$\begin{aligned}
&B_1[-1/2; \beta - 1], \quad B_2[0; 0], \quad B_3[1/2 - \beta; 0] \\
&B_4[-1; -1], \quad B_5[2; 0], \quad B_6[3; 2], \quad (3.3) \\
\theta_1 &= (-1) \begin{pmatrix} i, & 0 \\ 0, & \exp(i\pi\beta) \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 1, & 0 \\ \pi i, & 1 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} \exp(i\pi\beta), & 0 \\ 0, & 1 \end{pmatrix}, \\
\theta_4 &= (-1) \begin{pmatrix} 1, & 0 \\ i\pi, & 1 \end{pmatrix}, \quad \theta_5 = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}.
\end{aligned}$$

Now we write the Fuchs class equation analogously to (2.3), where α_{kj} are defined by (3.3).

To construct the matrix $\chi(t)$, it is necessary to construct the local matrices for the points $t = a_j$, $j = \overline{1, 6}$, $a_6 = \infty$. We fix $t = a_j$, $j = \overline{1, 5}$, as follows: $t = a_1 = -b$, $t = a_2 = -a$, $t = a_3 = 0$, $t = a_4 = a$, $t = a_5 = b$.

Having constructed the matrix $\chi(t)$, we have

$$\chi(t) = T\Theta_5(t), \quad a_5 < t < +\infty, \quad \chi(t) = T^*\Theta_4(t), \quad a_4 < t < a_5,$$

where $T^* = TT_4$, $p^* = pp_4 + qr_4$, $q^* = pq_4 + qs_4$, $r^* = rp_4 + sr_4$, $s^* = rq_4 + ss_4$.

$$\begin{aligned}
\chi(t) &= T^*\theta_4\Theta_4^*(t), & a_3 < t < a_4, \\
\chi(t) &= T^*\theta_4T_3\Theta_3(t), & a_3 < t < a_4, \\
\chi(t) &= T^*\theta_4T_3\theta_3\Theta_3^*(t), & a_2 < t < a_3, \\
\chi(t) &= T^*\theta_4T_3\theta_3T_2\Theta_2(t), & a_2 < t < a_3, \\
\chi(t) &= T^*\theta_4T_3\theta_3T_2\theta_2\Theta_2^*(t), & a_1 < t < a_2, \\
\chi(t) &= T^*\theta_4T_3\theta_3T_2\theta_2T_1\Theta_1(t), & a_1 < t < a_2, \\
\chi(t) &= T^*\theta_4T_3\theta_3T_2\theta_2T_1\theta_1\Theta_1^*(t), & -\infty < t < a_1, \\
\chi(t) &= T^*\theta_4T_3\theta_3T_2\theta_2T_1\theta_1T_{-\infty}\Theta_{\infty}(t), & -\infty < t < a_1, \\
\chi(t) &= TT_{+\infty}\Theta_{\infty}(t), & a_5 < t < +\infty.
\end{aligned} \tag{3.31}$$

The transition matrices from one singular point to the neighboring are defined as

$$\begin{aligned}
\Theta_j^*(t) &= T_{j-1}\Theta_{j-1}(t), & j = \overline{2, 6}, \\
\Theta_1^*(t) &= T_{-\infty}\Theta_{\infty}(t), & \Theta_5(t) = T_{+\infty}\Theta_{\infty}(t).
\end{aligned} \tag{3.32}$$

The system of equations (1.10₁), (1.45) and (1.46) for the problem under consideration is of the form

$$\begin{aligned}
\sum_{j=1}^5 c_j &= 0, & \sum_{j=1}^5 [\alpha_{1j}\alpha_{2j} + a_j c_j] - 6 &= 0, \\
\sum_{j=1}^5 [2\alpha_{1j}\alpha_{2j} + c_j a_j] a_j - p_{1\infty}\alpha_{2\infty} &= 0,
\end{aligned} \tag{3.4}$$

where

$$p_{1\infty} = \sum_{j=1}^5 (1 - \alpha_{1j} - \alpha_{2j}) a_j,$$

and the equation for the point $t = a_5$ has the form

$$q_{25} + q_{15}^2 + q_{15}p_{15} = 0, \tag{3.5}$$

where

$$\begin{aligned}
p_{15} &= \sum_{k=1, k \neq 5}^5 (1 - \alpha_{1k} - \alpha_{2k})(a_5 - a_k)^{-1}, \\
q_{0j} &= \alpha_{1j}\alpha_{2j}, & q_1 &= c_j, & n &= 0, 1, \\
q_{n5} &= (-1)^{n-2} \sum_{k=1, k \neq 5} [\alpha_{1k}\alpha_{2k}(n-1) + c_k(a_5 - a_k)](a_5 - a_k)^{-n}, & n &= \overline{2, \infty}.
\end{aligned}$$

The matrix equations, respectively, for the points a_j , $j = 4, 3, 2, 1$, have the form

$$\begin{aligned}
t = a_4 : T^*\theta_4 &= g_3 T^*\bar{\theta}_4, \\
t = a_3 : T^*\theta_4 T_3 \theta_3 &= g_2 T^*\bar{\theta}_4 T_3 \bar{\theta}_3, \\
t = a_2 : T^*\theta_4 T_3 \theta_3 T_2 \theta_4 &= g_1 T^*\bar{\theta}_4 T_3 \bar{\theta}_3 T_2 \bar{\theta}_2, \\
t = a_1 : T^*\theta_4 T_3 \theta_3 T_2 \theta_2 T_1 \theta_1 &= T^*\bar{\theta}_4 T_3 \bar{\theta}_3 T_2 \bar{\theta}_2 T_1 \bar{\theta}_1.
\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$t = a_4 : q^* = 0, \tag{3.7}$$

$$p^* + \pi s^* = 0; \tag{3.8}$$

$$t = a_3 : r^* p_3 + s^* r_3 = 0, \tag{3.9}$$

$$r^* q_3 \sin(\pi\beta) + s^* [\pi q_3 \cos(\pi\beta) + s_3 \sin(\pi\beta)] = 0; \tag{3.10}$$

$$t = a_2 : p_3 q_2 \sin(\pi\beta) + q_3 s_2 = 0, \tag{3.11}$$

$$p_3 [p_2 \sin(\pi\beta) - \pi q_2 \cos(\pi\beta)] + q_3 r_2 = 0; \tag{3.12}$$

$$t = a_1 : p_2 q_1 + q_2 s_1 = 0, \tag{3.13}$$

$$p_1 = 0. \tag{3.14}$$

The compatibility conditions for the systems (3.9), (3.10) and (3.11), (3.12) have, respectively, the form

$$\operatorname{tg}(\pi\beta) \det T_3 + \pi q_3 p_3 = 0, \tag{3.15}$$

$$\operatorname{tg}(\pi\beta) \det T_2 - \pi q_2 s_2 = 0. \tag{3.16}$$

The system (3.7), (3.8) and (3.9) allows us to determine p/s , r/s and q/s :

$$r/s = -[r_3 s_4 + r_4 p_3] / [p_3 p_4 + r_3 q_4], \tag{3.17}$$

$$p/s = -(s_4/q_4) / (q/s), \tag{3.18}$$

$$q/s = -\{(p/s)(p_4/r_4) + \pi[(r/s)(q_4/s_4) + (s_4/r_4)]\}.$$

From the system (3.4) and (3.5) we can define the parameters c_j , $j = \overline{1, 4}$, which are the functions of the parameters $\pi\beta$, c_5 , a , b . The parameter β is given beforehand. To determine the parameters c_5 , a , b we have the system (3.15), (3.11), (3.16), (3.13) and (3.14). The difference between the number of equations and that of unknown parameters c_5 , a , b is equal to two. But were the coordinates of the vertex A_6 known, the above-mentioned difference would be equal to three. From the system (3.10)–(3.16) we select the system (3.14)–(3.16), define the elements of the matrices T_j , $j = \overline{1, 4}$, from (3.3₂) and substitute them in (3.6)–(3.16). Then we solve the system (3.14)–(3.16) with respect to c_5 , a and b and substitute the obtained values in (3.4)–(3.17). Thus we define the parameters c_j , a_j , p/s and r/s . It remains now to define the parameters e_1 , Q and s for which below we will get a system of equations.

Further, following the reasoning of Sections 1 and 2, we define the matrix $\chi(t)$ along the axis t . We have

$$\begin{aligned}
\chi(t) &= \begin{pmatrix} p, & q \\ r, & s \end{pmatrix} \Theta_5(t), \quad a_5 < t < +\infty, \\
\chi(t) &= TT_{+\infty} \Theta_\infty(t), \quad a_5 < t < +\infty, \\
\chi(t) &= \begin{pmatrix} p^*, & 0 \\ r^*, & s^* \end{pmatrix} \Theta_5(t), \quad a_4 < t < a_5, \\
\chi(t) &= (-1) \begin{pmatrix} p^*, & 0 \\ r^* + i\pi s^*, & s^* \end{pmatrix} \Theta_4^*(t), \quad t < a_4, \\
\chi(t) &= (-1) \begin{pmatrix} p_3, & q_3 \\ -ir_3, & q_3 \exp(-i\pi\beta/\sin(\pi\beta)) \end{pmatrix} \Theta_3(t), \quad a_3 < t < a_4, \\
\chi(t) &= (-1)p^* \exp(-i\pi\beta) \begin{pmatrix} ip_3, & q_3 \exp(i\pi\beta) \\ p_3, & q_3/\sin(\pi\beta) \end{pmatrix} \Theta_3^*(t), \quad a_2 < t < a_3, \\
\chi(t) &= (-1)p^* p_3 \cos(\pi\beta) \begin{pmatrix} ip_2 + \pi q_2, & iq_2 \\ \pi q_2 \exp(-i\pi\beta)/\sin(\pi\beta), & 0 \end{pmatrix} \Theta_2(t), \quad a_2 < t < a_3, \\
\chi(t) &= (-1)p^* p_3 \cos(\pi\beta) \begin{pmatrix} ip_2, & iq_2 \\ \pi q_2 \exp(-i\pi\beta)/\sin(\pi\beta), & 0 \end{pmatrix} \Theta_2^*(t), \quad a_1 < t < a_2, \\
\chi(t) &= (-1)p^* p_3 q_3 \cos(\pi\beta) \begin{pmatrix} ir_1, & 0 \\ 0, & \pi q_1 \exp(-i\pi\beta)/\sin(\pi\beta) \end{pmatrix} \Theta_1(t), \quad a_1 < t < a_2, \\
\chi(t) &= (-1)p^* p_3 q_2 \cos(\pi\beta) \begin{pmatrix} r_1, & 0 \\ 0, & -\pi q_1/\sin(\pi\beta) \end{pmatrix} \Theta_1^*(t), \quad -\infty < t < a_1.
\end{aligned} \tag{3.19}$$

The equations (3.19) allow us to define linearly independent solutions $u_1(t)$ and $u_2(t)$ of the Fuchs class equation. Thus we have

$$\begin{aligned}
u_1(t) &= pu_{15}(t) + qu_{25}(t), \quad a_5 < t < +\infty, \\
u_2(t) &= ru_{15}(t) + su_{25}(t), \quad a_5 < t < +\infty; \\
u_1(t) &= pu_{15}^*(t) + qu_{25}^*(t), \quad a_4 < t < a_5, \\
u_2(t) &= ru_{15}^*(t) + su_{25}^*(t), \quad a_4 < t < a_5; \\
u_1(t) &= p^*(t)u_{14}(t), \quad a_4 < t < a_5, \\
u_2(t) &= r^*u_{14}(t) + s^*u_{24}(t), \quad a_4 < t < a_5; \\
u_1(t) &= (-1)p^*u_{14}^*(t), \quad a_3 < t < a_4, \\
u_2(t) &= (-1)[(r^* + i\pi s^*)u_{14}^*(t) + s^*u_{24}^*(t)], \quad a_3 < t < a_4; \\
u_1(t) &= (-1)p^*[p_3u_{13}(t) + q_3u_{23}(t)], \quad a_3 < t < a_4, \\
u_2(t) &= (-1)p^*[-ip_3u_{13}(t) + q_3 \exp(-i\pi\beta)/\sin(\pi\beta)u_{23}(t)], \quad a_3 < t < a_4; \\
u_1(t) &= (-1)p^*[ip_3 \exp(-i\pi\beta)u_{13}^*(t) + q_3u_{23}^*(t)], \quad a_2 < t < a_3, \\
u_2(t) &= (-1)p^* \exp(-i\pi\beta)\{p_3u_{13}^*(t) + [q_3/\sin(\pi\beta)] \cdot u_{23}^*(t)\}, \quad a_2 < t < a_3; \\
u_1(t) &= (-1)p^* \cos(\pi\beta)[(ip_2 + \pi q_2)u_{12}(t) + iq_2u_{22}(t)], \quad a_2 < t < a_3,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
u_2(t) &= (-1)\pi p^* p_3 q_2 \operatorname{ctg}(\pi\beta) \exp(-i\pi\beta) u_{12}(t), \quad a_2 < t < a_3; \\
u_1(t) &= (-i)p^* p_3 \cos(\pi\beta) (p_2 u_{12}^*(t) + q_2 u_{22}^*(t)), \quad a_1 < t < a_2, \\
u_2(t) &= (-1)p^* p_3 \operatorname{ctg}(\pi\beta) \pi q_2 \exp(-i\pi\beta) u_{12}^*(t), \quad a_1 < t < a_2; \\
u_1(t) &= (-i)p^* p_3 q_2 r_1 \cos(\pi\beta) u_{11}(t), \quad a_1 < t < a_2, \\
u_2(t) &= (-1)\pi p_3 q_2 q_1 \operatorname{ctg}(\pi\beta) \exp(-i\pi\beta) u_{21}(t), \quad a_1 < t < a_2; \\
u_1(t) &= (-1)p^* p_3 q_2 r_1 \cos(\pi\beta) u_{11}^*(t), \quad -\infty < t < a_1, \\
u_2(t) &= \pi p^* p_3 q_2 q_1 \operatorname{ctg}(\pi\beta) u_{21}^*(t), \quad -\infty < t < a_1.
\end{aligned}$$

The components of the vector $\Phi'(t) = [\omega'(t), z'(t)]$ are defined as follows:

$$\begin{aligned}
d\omega(t) &= \chi_{03}(t) u_1(t) dt, \quad -\infty < t < +\infty, \\
dz(t) &= \chi_{03}(t) u_2(t) dt, \quad -\infty < t < +\infty.
\end{aligned} \tag{3.21}$$

Integrating (3.21) within (a_j, t) , $j = \overline{1, 6}$, we obtain

$$z(t) = (-1)\pi p_3 q_2 q_1 \operatorname{ctg}(\pi\beta) \exp(-i\pi\beta) \int_{a_1}^t \chi_{03}(t) u_{21}(t) dt + L, \tag{3.22}$$

$$\omega(t) = (-i)p^* p_3 q_2 r_1 \cos(\pi\beta) \int_{a_1}^t \chi_{03}(t) u_{11}(t) dt - H_2, \tag{3.23}$$

$$z(t) = (-1)\pi p^* p_3 q_2 q_1 \operatorname{ctg}(\pi\beta) \exp(-i\pi\beta) \int_{a_1^*}^t \chi_{03}(t) u_{12}^*(t) dt + z(a_1^*), \tag{3.24}$$

$$\begin{aligned}
\omega(t) &= (-i)p^* p_3 \cos(\pi\beta) \left\{ p_2 \int_{a_1^*}^t \chi_{03}(t) u_{12}^*(t) dt + \right. \\
&\quad \left. + q_2 \int_{a_1^*}^t \chi_{03}(t) u_{22}^*(t) dt \right\} - H_2 + i\psi(a_1^*),
\end{aligned} \tag{3.25}$$

where

$$a_1^* = (a_j + a_{j+1})/2, \quad j = \overline{1, 4}, \tag{3.26}$$

$$z(t) = (-1)\pi p^* p_3 q_2 \operatorname{ctg}(\pi\beta) \exp(-i\pi\beta) \int_{a_2}^t \chi_{03}(t) u_{12}(t) dt + z(a_2), \tag{3.27}$$

$$\begin{aligned}
\omega(t) &= (-1)p^* \cos(\pi\beta) [(ip_2 + \pi q_2) \int_{a_2}^t \chi_{03}(t) u_{12}(t) dt + \\
&\quad + iq_2 \int_{a_2}^t \chi_{03}(t) u_{22}(t) dt] - H_2 + iQ',
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
z(t) = & (-1)p^* \exp(-ip\beta) \left[p_3 \int_{a_2^*}^t \chi_{03}(t) u_{13}^*(t) dt + \right. \\
& \left. + [q_3 / \sin(\pi\beta)] \int_{a_2^*}^t \chi_{03}(t) u_{23}^*(t) dt \right] + z(a_2^*), \tag{3.29}
\end{aligned}$$

$$\begin{aligned}
\omega(t) = & (-1)p^* [ip_3 \exp(-i\pi\beta) \int_{a_2^*}^t \chi_{03}(t) u_{13}^*(t) dt + \\
& q_3 \int_{a_2^*}^t \chi_{03}(t) u_{23}^*(t) dt] + \omega(a_2^*), \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
z(t) = & (-1)p^* \left[-ip_3 \int_{a_3}^t \chi_{03}(t) u_{13}(t) dt + \right. \\
& \left. + q_3 \exp(-i\pi\beta) / \sin(\pi\beta) \int_{a_3}^t \chi_{03}(t) u_{23}(t) dt \right] + z(a_3), \tag{3.31}
\end{aligned}$$

$$\begin{aligned}
\omega(t) = & (-1)p^* \left[p_3 \int_{a_3}^t \chi_{03}(t) u_{13}(t) dt + \right. \\
& \left. + q_3 \int_{a_3}^t \chi_{03}(t) u_{23}(t) dt \right] + \omega(a_3), \tag{3.32}
\end{aligned}$$

$$\begin{aligned}
z(t) = & (-1) \left[(r^* + i\pi s^*) \int_{a_3^*}^t \chi_{03}(t) u_{14}^*(t) dt + \right. \\
& \left. + s^* \int_{a_3^*}^t \chi_{03}(t) u_{24}^*(t) dt \right] + z(a_3^*), \tag{3.33}
\end{aligned}$$

$$\omega(t) = (-1)p^* \int_{a_3^*}^t \chi_{03}(t) u_{14}^*(t) dt + \omega(a_3^*), \tag{3.34}$$

$$z(t) = ir^* \int_{a_4}^t \tilde{\chi}_{03}(t) u_{14}(t) dt + is^* \int_{a_4}^t \tilde{\chi}_{03}(t) u_{24}(t) dt + z(a_4), \tag{3.35}$$

$$\omega(t) = ip^* \int_{a_4}^t \tilde{\chi}_{03}(t) u_{14}(t) dt + \omega(a_4), \quad (3.36)$$

where

$$\chi_{03}(t) = i\tilde{\chi}_{03}(t), \quad a_4 < t < e_5, \quad (3.37)$$

$$z(t) = r \int_{e_5}^t \chi_{03}(t) u_{15}^*(t) dt + s \int_{e_5}^t \chi_{03}(t) u_{25}^*(t) dt, \quad (3.38)$$

$$\omega(t) = r \int_{e_5}^t \chi_{03}(t) u_{15}^*(t) dt + q \int_{e_5}^t \chi_{03}(t) u_{25}^*(t) dt - H_1, \quad (3.39)$$

$$z(t) = r \int_{e_6}^t \chi_{03}(t) u_{15}(t) dt + s \int_{e_6}^t \chi_{03}(t) u_{25}(t) dt + z(e_6), \quad (3.40)$$

$$\omega(t) = p \int_{e_6}^t \chi_{03}(t) u_{15}(t) dt + q \int_{e_6}^t \chi_{03}(t) u_{25}(t) dt + \omega(e_6). \quad (3.41)$$

From (3.22) and (3.23) we can define $z(a_1^*)$ and $\omega(a_1^*)$. Thus we get

$$x(a_1^*) = (-1)\pi p_3 q_2 q_1 \cos^2(\pi\beta) / \sin(\pi\beta) \int_{a_1}^{a_1^*} \chi_{03}(t) u_{21}(t) dt + L, \quad (3.42)$$

$$y(a_1^*) = \pi p_3 q_2 q_1 \cos(\pi\beta) \int_{a_1}^{a_1^*} \chi_{03}(t) u_{21}(t) dt, \quad (3.43)$$

$$\varphi(a_1^*) = -H_2, \quad \psi(a_1^*) = -p^* p_3 q_2 r_1 \cos(\pi\beta) \int_{a_1}^{a_1^*} \chi_{03}(t) u_{11}(t) dt - H_2. \quad (3.44)$$

Considering (3.24) and (3.25) for $t = a_2$, we find that

$$x(a_2) = (-1)\pi p^* p_3 q_2 q_1 \cos^2(\pi\beta) / \sin(\pi\beta) \int_{a_1^*}^{a_2} \chi_{03}(t) u_{12}^*(t) dt + x(a_1^*), \quad (3.45)$$

$$y(a_2) = H_2 = \pi p^* p_3 q_2 q_1 \cos(\pi\beta) \int_{a_1^*}^{a_2} \chi_{03}(t) u_{12}^*(t) dt + y(a_1^*), \quad (3.46)$$

$$\varphi(a_2) = -H_2, \quad Q' = (-1)p^* p_3 \cos(\pi\beta) \left\{ p_2 \int_{a_1^*}^{a_2} \chi_{03}(t) u_{12}^*(t) dt + \right.$$

$$+ q_2 \int_{a_1^*}^{a_2} \chi_{03}(t) u_{22}^*(t) dt \Big\} + \psi(a_1^*). \quad (3.47)$$

Recall that the coordinates $[x(t), y(t)]$ in the interval (a_1, a_2) satisfy the boundary condition $y(t) = -\operatorname{tg}(\pi\beta)[x(t) - L]$.

The Equations (3.27) and (3.28) provide us with $z(a_2^*)$ and $\omega(a_2^*)$,

$$x(a_2^*) = (-1)\pi p^* p_3 q_2 \operatorname{ctg}(\pi\beta) \cos(\pi\beta) \int_{a_2}^{a_2^*} \chi_{03}(t) u_{12}(t) dt + x(a_2), \quad (3.48)$$

$$y(a_2^*) = \pi p^* p_3 q_2 \cos(\pi\beta) \int_{a_2}^{a_2^*} \chi_{03}(t) u_{12}(t) dt + y(a_2), \quad (3.49)$$

$$\varphi(a_2^*) = (-1)\pi p^* q_2 \cos(\pi\beta) \int_{a_2}^{a_2^*} \chi_{03}(t) u_{12}(t) dt - H_2, \quad (3.50)$$

$$\begin{aligned} \psi(a_2^*) = & (-1)p^* \cos(\pi\beta) \left[p_2 \int_{a_2}^{a_2^*} \chi_{03}(t) u_{12}(t) dt + \right. \\ & \left. + q_2 \int_{a_2}^{a_2^*} \chi_{03}(t) u_{22}(t) dt \right] + Q'. \end{aligned} \quad (3.51)$$

With the help of (3.29) and (3.30) we obtain

$$\begin{aligned} x(a_3) = & (-1)p^* \cos(\pi\beta) \left[p_3 \int_{a_2^*}^{a_3} \chi_{03}(t) u_{13}^*(t) dt + \right. \\ & \left. + (q_3 / \sin(\pi\beta)) \int_{a_2^*}^{a_3} \chi_{03}(t) u_{23}^*(t) dt \right] + x(a_2^*), \end{aligned} \quad (3.52)$$

$$\begin{aligned} y(a_3) = & p^* \sin(\pi\beta) \left[p_3 \int_{a_2^*}^{a_3} \chi_{03}(t) u_{13}^*(t) dt + \right. \\ & \left. + (q_3 / \sin(\pi\beta)) \int_{a_2^*}^{a_3} \chi_{03}(t) u_{23}^*(t) dt \right] + y(a_2^*), \end{aligned} \quad (3.53)$$

$$\varphi(a_3) = (-1) \left[p_3 \sin(\pi\beta) \int_{a_2^*}^{a_3} \chi_{03}(t) u_{13}^*(t) dt + \right.$$

$$+q_3 \int_{a_2^*}^{a_3} \chi_{03}(t) u_{23}^*(t) dt \Big] + \varphi(a_2^*), \quad (3.54)$$

$$Q = \psi(a_3) = (-1)p^* p_3 \cos(\pi\beta) \int_{a_2^*}^{a_3} \chi_{03}(t) u_{13}^*(t) dt + \psi(a_2^*). \quad (3.55)$$

Using (3.30) and (3.32), we define $z(a_3^*)$ and $\omega(a_3^*)$, and find that

$$x(a_3^*) = (-1)p^* q_3 \operatorname{ctg}(\pi\beta) \int_{a_3}^{a_3^*} \chi_{03}(t) u_{23}(t) dt + x(a_3), \quad (3.56)$$

$$y(a_3^*) = p^* \left[p_3 \int_{a_3}^{a_3^*} \chi_{03}(t) u_{13}(t) dt + q_3 \int_{a_3}^{a_3^*} \chi_{03}(t) u_{23}(t) dt \right] + y(a_3), \quad (3.57)$$

$$\psi(a_3^*) = \psi(a_3) = Q,$$

$$\varphi(a_3^*) = (-1)p^* \left[p_3 \int_{a_3}^{a_3^*} \chi_{03}(t) u_{13}(t) dt + q_3 \int_{a_3}^{a_3^*} \chi_{03}(t) u_{23}(t) dt \right] + \varphi(a_3). \quad (3.58)$$

By means of (3.33) and (3.34) we define $z(a_4)$ and $\omega(a_4)$. We have

$$x(a_4) = (-1) \left[r^* \int_{a_3^*}^{a_4} \chi_{03}(t) u_{14}^*(t) dt + s^* \int_{a_3^*}^{a_4} \chi_{03}(t) u_{24}^*(t) dt \right] + x(a_3^*) = 0, \quad (3.59)$$

$$y(a_4) = H_1 = (-1)\pi s^* \int_{a_3^*}^{a_4} \chi_{03}(t) u_{14}^*(t) dt + y(a_3^*), \quad (3.60)$$

$$\varphi(a_4) = -H_1, \quad H_1 = p^* \int_{a_3}^{a_4} \chi_{03}(t) u_{14}^*(t) dt - \varphi(a_3^*). \quad (3.61)$$

From (3.35) and (3.36) we define $z(e_5)$ and $\omega(e_5)$, and get

$$z(e_5) = 0, \quad r^* \int_{a_4}^{e_5} \tilde{\chi}_{03}(t) u_{14}(t) dt + s^* \int_{a_4}^{e_5} \tilde{\chi}_{03}(t) u_{24}(t) dt + H_1 = 0, \quad (3.62)$$

$$\varphi(e_5) = -H_1, \quad \psi(e_5) = p^* \int_{a_4}^{e_5} \tilde{\chi}_{03}(t) u_{14}(t) dt + Q = 0, \quad (3.63)$$

By virtue of (3.38) and (3.39) we define $z(e_6)$ and $\omega(e_6)$, and obtain

$$x(e_6) = r \int_{e_5}^{e_6} \chi_{03}(t) u_{15}^*(t) dt + s \int_{e_5}^{e_6} \chi_{03}(t) u_{25}^*(t) dt, \quad (3.64)$$

$$\psi(e_6) = 0, \quad y(e_6) = 0, \quad (3.65)$$

$$\varphi(e_6) = p \int_{e_5}^{e_6} \chi_{03}(t) u_{15}^*(t) dt + q \int_{e_5}^{e_6} \chi_{03}(t) u_{25}^*(t) dt - H_1. \quad (3.66)$$

Thus we have obtained the system of equations (3.46), (3.47), (3.53), (3.55), (3.62) and (3.63) with respect to the unknown parameters s , e_5 , Q' , Q and $y(a_3)$. Of the above-given system only the system (3.46) and (3.62) depends on the parameters e_5 and s . This system makes it possible to find e_5 and s . We substitute the obtained values in all the above-mentioned equations. Next, using the formula (3.47), we define Q' , and then by the formula (3.55) we find Q . We substitute the formula (3.53) allows us to find $y(a_3)$, and the formulas (3.64) and (3.66) provide us with the unknown parameters of the problem of filtration. Finally, by virtue of the formulas (3.31)–(3.34), we can find the parametric equations of unknown parts of the boundaries $l(z)$ and $l(\omega)$ of the domains $s(z)$ and $s(\omega)$.

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