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**ON ZAREMBA'S BOUNDARY
VALUE PROBLEM FOR HARMONIC
FUNCTIONS OF SMIRNOV CLASSES**

Abstract. Analogously to Smirnov classes of analytic functions, Smirnov classes of harmonic functions are introduced and mixed Zaremba's boundary value problem is studied in them, i.e., the problem of constructing a harmonic function when on a part of the boundary its values are given, while on the remaining part the values of its normal derivative.

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რეზიუმე. ანალიზური ფუნქციების სმირნოვის კლასების ანალოგიურად შემოღებულია ჰარმონიული ფუნქციების სმირნოვის კლასები და მათში განხილულია ზარემბას შერეული ამოცანა, ანუ ამოცანა ჰარმონიული ფუნქციის აკუბის შესახებ, როცა საზღვრის ერთ ნაწილზე მისი მნიშვნელობებია მოცემული, ხოლო დანარჩენ ნაწილზე – ნორმალური წარმოებულის მნიშვნელობები.

The Boundary value problems for harmonic functions of two variables are studied well enough under different assumptions for unknown functions (see, e.g., [1–6]). Because of the fact that such harmonic functions are real parts of analytic functions, the properties of the latter often play an important role in studying boundary value problems. One of the most interesting classes is the set of those functions, analytic in the given domain D , whose integral p -means along a sequence of curves converging to the domain boundary are uniformly bounded. These classes are generalizations of Hardy classes H^p and are called Smirnov classes $E^p(D)$ (see, e.g., [7], Ch. IX–X, [8]). The functions from these classes are representable for $p \geq 1$ by the Cauchy integral and possess a number of various important properties, so they are frequently encountered in the theory of functions. These properties are also preserved for harmonic functions from the class $e^p(D)$ which is composed of the real parts of functions from $E^p(D)$. It was that fact that evoked great interest in the theory of boundary value problems for harmonic functions from $e^p(D)$. The Dirichlet, Neumann and Riemann–Hilbert problems in domains with arbitrary piecewise smooth boundaries have been studied in [9–13]. Boundary value problems have also been considered in those classes which are defined analogously to Smirnov classes, or represent their generalizations ([14]–[15]).

Our aim is to consider in these classes the problem when some value of an unknown harmonic function is given on one part of the boundary and the value of its derivative in the direction of the normal is given on the supplementary part of the boundary. S. Zaremba [16] was the first who considered this problem, and that is why the problem is called after his name, Zaremba's problem (see, e.g., [17]). This problem is a particular case of the so-called mixed problems for elliptic equations (see [18], p. 16; references concerning these problems can be found on pages 201–202 therein). The simplest solution of Zaremba's problem under the assumption that the boundary function is differentiable along the whole boundary, is given in [19]. In [20] we can find an explicit solution of the problem for a half-plane when the boundary functions belong to the Hölder class. The more general problem $a \frac{\partial u}{\partial n} + bu = c$ (or $a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = d$) has been investigated thoroughly in [2], [3], [24], etc. However, in all those works the assumptions regarding the coefficients do not cover the problem of our interest.

In the present paper we pose and investigate the problem of Zaremba in a sufficiently wide weighted Smirnov class $e(\Gamma_{1p}(\rho_1), \Gamma'_{2q}(\rho_2))$, first in a circle (Sections 3⁰–7⁰) and then in domains bounded by Lyapunov curves (Section 8⁰). In Section 9⁰ we consider the mixed boundary value problem in a more narrow (than that mentioned above) class of harmonic functions.

1⁰. Some Definitions.

Let U be the unit circle $\{z : |z| < 1\}$ bounded by the circumference $\gamma = \{\tau : |\tau| = 1\}$, and let $\gamma_k = [a_k, b_k]$, $k = \overline{1, m}$, be arcs separately lying on it; note that the points $a_1, b_1, a_2, b_2, \dots, a_m, b_m$ follow each other in the

positive direction on γ . Let m_1 be an integer from the segment $[0, 2m]$. We denote by c_1, c_2, \dots, c_{2m} the points $a_k, b_k, k = \overline{1, m}$, taken arbitrarily and consider in the plane cut along the set of curves $\Gamma_1 = \bigcup_{k=1}^m \gamma_k$ the analytic functions

$$\Pi_1(z) = \sqrt{\prod_{k=1}^{m_1} (z - c_k)}, \quad \Pi_2(z) = \sqrt{\prod_{k=m_1+1}^{2m} (z - c_k)}, \quad (1)$$

where of the first we take an arbitrary branch and the second function we choose in such a way that the function

$$R(z) = \frac{\Pi_1(z)}{\Pi_2(z)} \quad (2)$$

decomposes in the neighborhood of the point $z = \infty$ as follows:

$$R(z) = z^{m-m_1} + A_1 z^{m-m_1+\dots}$$

(see [2], p. 277). For $z = t \in \Gamma_1$, under $\Pi_1(t), \Pi_2(t), R(t)$ we will mean the value which the corresponding function takes on the left of Γ_1 .

Let $q > 1$,

$$\omega(t) = \prod_{k=1}^{2m} |t - c_k|^{\alpha_k}, \quad -\frac{1}{q} < \alpha_k < \frac{1}{q'}, \quad q' = \frac{q}{q-1}, \quad (3)$$

and let Γ be a measurable set on γ . By $L^q(\Gamma; \omega)$ we denote the set of measurable on Γ (by the arc measure ds) functions f for which

$$\int_{\Gamma} |f(t)\omega(t)|^q ds < \infty.$$

Suppose $L^q(\Gamma) = L^q(\Gamma; 1)$, $L(\Gamma) \equiv L^1(\Gamma)$.

Next, let $[a'_k, b'_k]$ be the arcs lying on Γ_k (the point b'_k follows a'_k in the direction on γ_k from a_k to b_k). Denote

$$\Gamma_1 = \bigcup_{k=1}^m \gamma_k, \quad \tilde{\gamma} = \bigcup_{k=1}^m [a_k, a'_k] \bigcup_{k=1}^m [b'_k, b_k], \quad \Gamma_2 = \gamma \setminus \Gamma_1. \quad (4)$$

If $E \subset \gamma$, then we denote by χ_E the characteristic function of the set E . Moreover, suppose

$$\Theta(E) = \{\theta : 0 \leq \theta \leq 2\pi, e^{i\theta} \in E\}. \quad (5)$$

When E is a finite union of closed arcs on γ , by $A(E)$ we denote the set of functions $f(t) = f(e^{i\theta})$ absolutely continuous on $\Theta(E)$, i.e., the functions f for which for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $\cup(e^{i\alpha_k}, e^{i\beta_k})$ is any union of non-intersecting intervals from E satisfying $\sum(\beta_k - \alpha_k) < \delta$, then the inequality $\sum |f(e^{i\beta_k}) - f(e^{i\alpha_k})| < \varepsilon$ holds.

Let d_1, d_2, \dots, d_n be different from c_k points on γ ; note that d_1, d_2, \dots, d_{n_1} are located on $\Gamma_1 \setminus \tilde{\gamma}$, and d_{n_1+1}, \dots, d_n on Γ_2 . Assuming that $p \geq 1$, and

$q > 1$ we put

$$\omega_1(z) = \prod_{k=1}^{n_1} (z - d_k)^{\alpha_k}, \quad -\frac{1}{p} < \alpha_k < \frac{1}{p'}, \quad p' = \frac{p}{p-1}, \quad (6)$$

$$\omega_2(z) = \prod_{k=1}^{m_1} (z - c_k)^{\nu_k} \prod_{k=m_1+1}^{2m} (z - c_k)^{\lambda_k} \prod_{k=n_1+1}^n (z - d_k)^{\beta_k}, \quad (7)$$

$$-\frac{1}{q} < \nu_k < 0, \quad 0 \leq \lambda_k < \frac{1}{q'}, \quad -\frac{1}{q} < \beta_k < \frac{1}{q'},$$

where ω_1 and ω_2 are arbitrary branches of the functions which are analytic in the plane cut along γ , and for $p = 1$ we assume that $p' = \infty$, $\frac{1}{p'} = 0$.

2⁰. Classes $h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ and Some of Their Properties.

We say that a harmonic in the circle U function $u(z)$, $z = x + iy = re^{i\varphi}$ belongs to the class $h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p \geq 0$, $q \geq 0$, if

$$\sup_{0 < r < 1} \left[\int_{\Theta(\Gamma_1)} |u(re^{i\theta})\omega_1(re^{i\theta})|^p d\theta + \int_{\Theta(\Gamma_2)} \left(\left| \frac{\partial u}{\partial x}(re^{i\theta}) \right|^q + \left| \frac{\partial u}{\partial y}(re^{i\theta}) \right|^q \right) |\omega_2(re^{i\theta})|^q d\theta \right] < \infty. \quad (8)$$

For $\Gamma_1 = \gamma$, $\omega_1 = 1$ this class coincides with the class h_p (see, e.g., [7], p. 373). For $p = 0$, $\omega_2 = 1$, $\Gamma_2 = \gamma$ we get the class $h'_q(U)$ (see. [10], [11], p. 169).

For $\omega_1 = 1$, instead of $h(\Gamma_{1p}(1), \Gamma'_{2q}(\omega_2))$ we will write $h(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$.

Lemma 1. *The class $h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ coincides with the class of those harmonic in the circle U functions $u(re^{i\theta})$ for which*

$$\sup_{0 < r < 1} \left[\int_{\Theta(\gamma_1)} |u(re^{i\theta})\omega_1(re^{i\theta})|^p d\theta + \int_{\Theta(\Gamma_2)} \left| \sqrt{\left(\frac{\partial u}{\partial x}(re^{i\theta}) \right)^2 + \left(\frac{\partial u}{\partial y}(re^{i\theta}) \right)^2} \omega_2(re^{i\theta}) \right|^q d\theta \right] < \infty. \quad (9)$$

Proof. The validity of the lemma follows immediately if we apply to the second summand in (9) the inequality

$$\frac{1}{2^x} (a+b)^x \leq a^x + b^x \leq 2(a+b)^x, \quad a \geq 0, \quad b \geq 0, \quad x > 0, \quad (10)$$

assuming in it that $a = \left(\frac{\partial u}{\partial x} \right)^2$, $b = \left(\frac{\partial u}{\partial y} \right)^2$, $x = \frac{q}{2}$.

The validity of the inequality (10) for $a = 0$, $b = 0$ is obvious, and for $a + b > 0$ it follows from the equality $a^x + b^x = (a+b)^x \left[\left(\frac{a}{a+b} \right)^x + \left(\frac{b}{a+b} \right)^x \right]$, if we take into account that $0 \leq \frac{a}{a+b} \leq 1$ and $[\max(a, b)](a+b)^{-1} \geq \frac{1}{2}$. \square

Lemma 2. *If $u \in h(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, $p \geq 1$, $p \leq q$ then $u \in h_p$. In particular, if $u \in h(\Gamma_{11}, \Gamma'_{2q}(\omega_2))$, $q \geq 1$, then $u \in h_1$.*

Proof. Let $I(r) = \int_0^{2\pi} |u(re^{i\theta})|^p d\theta$.

We have

$$\begin{aligned} I(r) &= \int_{\Theta(\Gamma_1)} |u(re^{i\theta})|^p d\theta + \int_{\Theta(\Gamma_2)} \left| \int_0^r \frac{\partial u(re^{i\theta})}{\partial r} dr - u(0) \right|^p d\theta \leq \\ &\leq M_1 + 2^p \left(\int_{\Theta(\Gamma_2)} \left| \int_0^r \frac{\partial u}{\partial r} dr \right|^p d\theta + |u(0)|^p 2\pi \right) = \\ &= M_2 + 2^p \int_{\Theta(\Gamma_2)} \left| \int_0^r \frac{\partial u}{\partial r} dr \right|^p d\theta = M_2 + 2^p I_1(r). \end{aligned} \quad (11)$$

Since $\left| \frac{\partial u}{\partial r} \right| \leq \left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right|$,

$$I_1(r) = \int_{\Theta(\Gamma_2)} \left| \int_0^r \frac{\partial u}{\partial r} dr \right|^p d\theta \leq \int_{\Theta(\Gamma_2)} \left(\int_0^r \left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| dr \right)^p d\theta. \quad (12)$$

If $q = 1$, then $p = 1$, and $|\omega_2| = \prod_{k=1}^{m_1} |z - c_k|^{\nu_k} \prod_{k=n_1+1}^n |z - d_k|^{\beta_k}$, $-1 < \nu_k < 0$, $-1 < \beta_k \leq 0$ (since $q' = \infty$, $\lambda_k = 0$). Therefore the function $\frac{1}{|\omega_2|}$ is bounded.

By virtue of the above-said, it follows from (12) that

$$\begin{aligned} I_1(r) &= \int_{\Theta(\Gamma_2)} \int_0^r \left(\left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| \right) dr d\theta \leq \\ &\leq M_3 \int_0^r dr \int_{\Theta(\Gamma_2)} \left(\left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| \right) |\omega_2(re^{i\theta})| d\theta \max_{\Theta(\Gamma_2)} \frac{1}{|\omega_2(re^{i\theta})|} \leq \\ &\leq M_4 \sup_{0 < r < 1} \int_{\Theta(\Gamma_2)} \left(\left| \frac{\partial u}{\partial x}(re^{i\theta}) \right| + \left| \frac{\partial u}{\partial y}(re^{i\theta}) \right| \right) |\omega_2(re^{i\theta})| d\theta < \infty. \end{aligned}$$

This implies that $u \in h_1$.

Let now $q > 1$. Using Hölder's inequality, the expression (12) results in

$$I_1(r) \leq \int_{\Theta(\Gamma_2)} \left(\int_0^r \left(\left| \frac{\partial u}{\partial x} \right|^q + \left| \frac{\partial u}{\partial y} \right|^q \right) |\omega_2|^q dr \right)^{\frac{p}{q}} \left(\int_0^r \frac{dr}{|\omega_2|^{q'}} \right)^{\frac{p}{q'}} d\theta. \quad (13)$$

But

$$\frac{1}{|\omega_2|^{q'}} \leq \frac{M_5}{\prod_{k=m_1+1}^{2m} |re^{i\theta} - c_k|^{\frac{\lambda_k}{2} q'}} \frac{1}{\prod_{\beta_k > 0} |re^{i\theta} - d_k|^{\frac{\beta_k}{2} q'}}.$$

Suppose $\alpha = \sup_k(\lambda_k q', \beta_k q')$. Then from the assumptions (7) regarding λ_k, β_k it follows that $\alpha < 1$. Taking now into account the obvious inequality $|re^{i\theta} - c_k| \geq |1 - r|$, we obtain

$$\begin{aligned} \sup_{\Theta(\Gamma_2)} \left(\int_0^r \frac{dr}{|\omega_2(re^{i\theta})|^{q'}} \right)^{\frac{p}{q'}} &\leq \sup_k \sup_{\theta \in \Theta((\beta_k, a_{k+1}))} \left(\int_0^r \frac{dr}{|\omega_2(re^{i\theta})|^{q'}} \right)^{\frac{p}{q'}} \leq \\ &\leq \left[\int_0^1 \frac{M_6 dr}{(1-r)^\alpha} \right]^{\frac{p}{q'}} = M_7, \end{aligned} \tag{14}$$

$$M_6 = \left(\sup_k \frac{1}{|z_k - z_{k+1}|} \right)^{2m-1}, \quad \{z_k\} = \cup\{c_k\} \cup \{d_k\}.$$

Since $\frac{p}{q} \leq 1$, it follows from (13) that

$$\begin{aligned} I_1(r) &\leq M_8 \int_{\Theta(\Gamma_2)} \int_0^r \left(\left| \frac{\partial u}{\partial x} \right|^q + \left| \frac{\partial u}{\partial y} \right|^q \right) |\omega_2|^q d\theta = \\ &= M_8 \int_0^r \int_{\Theta(\Gamma_2)} \left(\left| \frac{\partial u}{\partial x} \right|^q + \left| \frac{\partial u}{\partial y} \right|^q \right) |\omega_2|^q d\theta. \end{aligned}$$

Consequently, by (8) we have $\sup_r I_1(r) < \infty$, and the expression (11) allows us to conclude that $u \in h_p$. \square

Corollary 1. *If $u \in h(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, $p \geq 1$, $q \geq 1$ then $u \in h_s$, where $s = \min(p, q)$.*

Indeed, if $q \geq p$, then in this case $s = p$ and, according to the lemma, $u \in h_p$. If $p > q$, then $s = q$.

We can easily verify that for $p_1 < p_2$ the embedding $h(\Gamma_{1p_2}, \Gamma'_{2q}(\omega_2)) \subset h(\Gamma_{1p_1}, \Gamma'_{2q}(\omega_2))$ is valid, and therefore the function $u(re^{i\theta})$, being of the class $h(\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, belongs to the class $h(\Gamma_{1q}, \Gamma'_{2q}(\omega_2))$ as well. But then u belongs to $h_q (= h_s)$, by our lemma.

Corollary 2. *If $u \in (\Gamma_{1p}, \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$, then $u(z)$ possesses for almost all $t \in \gamma$ the angular boundary values $u^+(t) = u^+(e^{i\theta}) \equiv u^+(\theta)$, $u^+ \in L^s(\gamma)$, $s = \min(p, q)$, and $u(z)$ is representable by the Poisson integral*

$$\begin{aligned} u(z) = u(re^{i\varphi}) &= \frac{1}{2\pi} \int_0^{2\pi} u^+(\theta) P(r, \theta - \varphi) d\theta, \\ P(r, x) &= \frac{1 - r^2}{1 + r^2 - 2r \cos x}. \end{aligned} \tag{15}$$

Indeed, under the adopted assumptions $s > 1$, and for the functions from h_s , $s > 1$, the statements of the lemma are valid (see, e.g., [7], Ch. IX).

Remark. If the function $u \in h(\Gamma_{11}, \Gamma'_{2q}(\omega_2))$, $q \geq 1$, then $u \in h_1$. Therefore it has almost everywhere on γ angular boundary values and is representable by the Poisson–Stieltjes integral (see [7], p. 374).

Lemma 3. *If $u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$, then there exists a number $\sigma > 1$ such that $u \in h_\sigma$. If v is the function conjugate harmonically to u , then $v \in h(\Gamma_{1p_1}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p_1 = \frac{p\sigma}{p+\sigma}$.*

Proof. By the conditions $\alpha_k \in (-\frac{1}{p}, \frac{1}{p'})$ (see (6)), it follows that $\frac{1}{\omega_1} \in H^{p'+\varepsilon}$, ($0 < \varepsilon < \frac{1}{\alpha} - p'$, $\alpha = \max \alpha_k$), where H^x is the class of Hardy (for the definition of H^x see, e.g., [8]). Therefore there exists $\eta (> 0)$ such that

$$\sup_{0 < r < 1} \int_{\Theta(\Gamma_1)} |u(re^{i\theta})|^{1+\eta} d\theta < \infty. \quad (16)$$

Thus $u \in h(\Gamma_{11+\eta}, \Gamma'_{2q}(\omega_2))$ and, according to Corollary 1 of Lemma 2, we conclude that $u \in h_\sigma$, $\sigma = \min(1 + \eta, q)$.

By the M. Riesz theorem (see [21] and also [8], p.54), it follows that $v \in h_\sigma$. Therefore the function $\phi(z) = u(z) + iv(z)$ belongs to the class H^σ .

Since $\omega_1 \in H^p$, the function $\phi(z)\omega_1(z)$ belongs to H^{p_1} , $p_1 = \frac{p\sigma}{p+\sigma}$. Consequently,

$$\sup_{0 < r < 1} \int_{\Theta(\Gamma_1)} |v(re^{i\theta})\omega_1(re^{i\theta})|^{p_1} d\theta < \infty. \quad (17)$$

Next, by the Riemann–Cauchy conditions, (8) implies that

$$\sup_{\tau} \int_{\Theta(\Gamma_2)} \left(\left| \frac{\partial v}{\partial x}(re^{i\theta}) \right|^q + \left| \frac{\partial v}{\partial y}(re^{i\theta}) \right|^q \right) |\omega_2(re^{i\theta})|^q d\theta < \infty,$$

which together with (17) yields $v \in h(\Gamma_{1p_1}(\omega_1), \Gamma'_{2q}(\omega_2))$. \square

Remark. The index p_1 in fact, can be replaced by the number $p_2 = \frac{p\alpha\sigma}{p\alpha+\sigma}$, where $p_\alpha = \min_{\alpha_k < 0} \frac{1}{|\alpha_k|}$, and now from (6) it follows that $p_\alpha > p_1$ and hence $p_2 > p_1$.

Corollary 1. *If $u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$ then $u(re^{i\varphi})$ is representable by the Poisson integral (15) with the function $u^+ \in L^\sigma(\gamma)$.*

Corollary 2. *If $u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$ and $\phi(z) = u(z) + iv(z)$, then $\phi \in H^\sigma$ and*

$$\sup_{0 < r < 1} \int_{\Theta(\Gamma_2)} |\Phi'(re^{i\theta})|^q |\omega_2(re^{i\theta})|^q d\theta < \infty. \quad (18)$$

Let now $\vec{n} = \vec{n}_\theta$ be an interior normal to the circumference γ at the point $t = e^{i\theta}$. We calculate the derivative of the function u along the vector \vec{n} at the point $re^{i\theta}$. Denote this derivative by $\frac{\partial u}{\partial \vec{n}}(re^{i\theta})$. Thus

$$\left(\frac{\partial u}{\partial \vec{n}} \right)(re^{i\theta}) = \frac{\partial u}{\partial \vec{n}_\theta}(re^{i\theta}) = \frac{\partial u}{\partial x}(re^{i\theta}) \cos(\vec{n}, x) +$$

$$+\left(\frac{\partial u}{\partial y}\right)(re^{i\theta})\cos(\vec{n}, y) = \frac{\partial u}{\partial x}(re^{i\theta})(-\sin\theta) + \left(\frac{\partial u}{\partial y}\right)(re^{i\theta})\cos\theta. \quad (19)$$

Sometimes, instead of $\left(\frac{\partial u}{\partial \vec{n}_\theta}\right)(re^{i\theta})$ we will write $\frac{\partial u}{\partial n}$, assuming that if that value is calculated at the point $re^{i\theta}$, then $n = \vec{n} = \vec{n}_\theta$.

If $\left(\frac{\partial u}{\partial x}\right)(re^{i\theta})$ and $\left(\frac{\partial u}{\partial y}\right)(re^{i\theta})$ have limits $\left(\frac{\partial u}{\partial x}\right)^+$ and $\left(\frac{\partial u}{\partial y}\right)^+$ as $r \rightarrow 1$, then we put

$$\left(\frac{\partial u}{\partial n}\right)^+(e^{i\theta}) = \left(\frac{\partial u}{\partial x}\right)^+(-\sin\theta) + \left(\frac{\partial u}{\partial y}\right)^+\cos\theta$$

(cf. [2], p. 243).

Lemma 4. *If $u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$ and u^+ is absolutely continuous on Γ_2 , then the function $\frac{\partial u}{\partial \varphi}(re^{i\varphi})$ has angular boundary values almost everywhere on Γ_2 and the equality*

$$\lim_{re^{i\varphi} \widehat{\rightarrow} e^{i\varphi_0}} \frac{\partial u}{\partial \varphi}(re^{i\varphi}) = \frac{\partial u^+}{\partial \varphi}(e^{i\varphi_0}), \quad e^{i\varphi_0} \in \Gamma_2$$

holds.

Proof. Since $p > 1$, $q > 1$, according to Corollary 1 of Lemma 2, the equality (15) holds. Therefore

$$\begin{aligned} \frac{\partial u}{\partial \varphi}(re^{i\varphi}) &= \frac{1}{2\pi} \int_{\Theta(\Gamma_1)} u^+(\theta) \frac{\partial}{\partial \varphi} P(r, \theta - \varphi) d\theta + \\ &+ \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} u^+(\theta) \frac{\partial}{\partial \varphi} P(r, \theta - \varphi) d\theta = \\ &= \frac{1}{2\pi} \int_{\Theta(\Gamma_1)} u^+(\theta) \frac{(1-r^2)2r \sin(\theta - \varphi)}{[1+r^2-2r \cos(\theta - \varphi)]^2} d\theta - \\ &- \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} u^+(\theta) \frac{\partial}{\partial \theta} P(r, \theta - \varphi) d\theta = \\ &= u_1(re^{i\varphi}) - u_2(re^{i\varphi}). \end{aligned}$$

Let $e^{i\varphi_0} \in \Gamma_2$. Then there exists a number $\delta > 0$ such that for $\theta \in \Theta(\Gamma_1)$ we have $\delta < |\theta - \varphi_0| < 2\pi - \delta$, and therefore

$$\lim_{re^{i\varphi} \widehat{\rightarrow} e^{i\varphi_0}} u_1(re^{i\varphi}) = 0. \quad (20)$$

As regards $u_2(re^{i\varphi})$, the density u^+ in the integral which represents this function is absolutely continuous, and hence partial integration is quite admissible here. As a result, we obtain

$$u_2(re^{i\varphi}) = u^+(\theta)P(r, \theta - \varphi)|_{\Gamma_2} - \int_{\Theta(\Gamma_2)} \frac{\partial u^+}{\partial \theta} P(r, \theta - \varphi) d\theta.$$

Passing in this equality to the limit, we get

$$\lim_{re^{i\varphi} \xrightarrow{\widehat{}} e^{i\varphi_0}} u_2(re^{i\varphi}) = -\frac{\partial u^+}{\partial \theta}(e^{i\varphi_0}),$$

which together with (20) provides us with the equality being proved. \square

Lemma 5. *If $u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$, $p > 1$, $q > 1$ and u^+ is absolutely continuous on Γ_2 , then the function $\left(\frac{\partial u}{\partial \varphi}\right)^+$ belongs to the class $L^q(\Gamma_2; \omega_2)$.*

Proof. Since

$$\frac{\partial u}{\partial \varphi}(re^{i\varphi}) = -\frac{\partial u}{\partial x}r \sin \varphi + \frac{\partial u}{\partial y}r \cos \varphi,$$

we have

$$\left|\frac{\partial u}{\partial \varphi}\right|^q \leq 2^q \left(\left|\frac{\partial u}{\partial x}\right|^q + \left|\frac{\partial u}{\partial y}\right|^q\right),$$

and from (8) it follows that

$$\int_{\Theta(\Gamma_2)} \left|\frac{\partial u}{\partial \varphi}\right|^q |\omega_2|^q d\theta \leq 2^q \left(\int_{\Theta(\Gamma_2)} \left|\frac{\partial u}{\partial x}\right|^q + \left|\frac{\partial u}{\partial y}\right|^q\right) |\omega_2|^q d\theta < M_8.$$

Using Lemma 4, for almost all $t = e^{i\varphi_0} \in \Gamma_2$ we obtain

$$\lim_{re^{i\varphi} \xrightarrow{\widehat{}} e^{i\varphi_0}} \frac{\partial u}{\partial \varphi}(re^{i\varphi}) = -\frac{\partial u^+}{\partial \theta}\omega_2(e^{i\varphi_0}).$$

According to Fatou's lemma, the last equality and the previous inequality allow us to conclude that

$$\int_{\Theta(\Gamma_2)} \left|\frac{\partial u^+}{\partial \theta}\omega_2(e^{i\theta})\right|^q d\theta = \int_{\Theta(\Gamma_2)} \left|\frac{\partial u^+}{\partial \theta}\omega_2\right|^q d\theta \leq M_8. \quad \square$$

Corollary. *Under the conditions of the lemma, the function $\left(\frac{\partial u}{\partial \varphi}\right)^+$ (or, what comes to the same thing, the function $\frac{\partial u^+}{\partial \varphi}$) belongs to the class $L(\Gamma_2)$.*

Indeed, by the conditions (7) regarding the weight ω_2 we find that $\frac{1}{\omega_2} \in L^q(\Gamma_2)$, and the statement of the corollary follows from the equality $\frac{\partial u^+}{\partial \varphi} = \left(\frac{\partial u^+}{\partial \varphi}\omega_2\right) \frac{1}{\omega_2}$.

Lemma 6. *Let*

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)P(r, \theta - \varphi)d\theta,$$

where $f \in L^p(\Gamma_1 \setminus \tilde{\gamma}, (\omega_1))$, $p > 1$, $f \in A(\Theta(\Gamma_2 \cup \tilde{\gamma}))$, $f' \in L^q(\Gamma_2; \omega_2)$, $q > 1$, Then

$$u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2)).$$

Proof. In the circle U we consider the function

$$\begin{aligned} \phi(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f_1(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta + \\ &+ \frac{1}{2\pi} \int_0^{2\pi} f_2(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta = \phi_1(z) + \phi_2(z), \end{aligned}$$

where $f_1(\theta) = \chi_{\Gamma_1 \setminus \bar{\gamma}}(\theta)f(\theta)$, $f_2(\theta) = \chi_{\Gamma_2 \cup \bar{\gamma}}(\theta)f(\theta)$ and by the condition $f' \in L^q(\Gamma_2; \omega_2)$ we have $f'_2 \in L^q(\Gamma_2; \omega_2)$.

Because of the fact that the Cauchy type integral in the case of a circumference belongs to the set $\bigcap_{\delta < 1} H^\delta$ (see, e.g., [8], p.39), we can easily show that $\phi_j \in \bigcap_{\delta < 1} H^\delta$, $j = 1, 2$.

Performing partial integration, we write ϕ'_2 in the form

$$\begin{aligned} \phi'_2(z) &= \frac{1}{2\pi} \int_0^{2\pi} f_2(\theta) \frac{2e^{i\theta} d\theta}{(e^{i\theta} - z)^2} = \frac{1}{\pi i} \frac{f_2(\theta)}{e^{i\theta} - z} \Big|_{\Gamma_2 \cup \bar{\gamma}} - \frac{1}{\pi i} \int_0^{2\pi} \frac{f'_2(\theta) d\theta}{e^{i\theta} - z} = \\ &= \sum_{k=1}^m \frac{1}{\pi i} \left(\frac{f(b'_k)}{b'_k - z} - \frac{f(a'_k)}{a'_k - z} \right) - \frac{1}{\pi i} \int_0^{2\pi} \frac{f'_2(\theta) d\theta}{e^{i\theta} - z}. \end{aligned} \quad (21)$$

Density of the latter integral belongs to $L^q(\gamma; \omega_2)$ and the Cauchy singular integral of such a function belongs to $L^q(\gamma; \omega_2)$ (see, e.g., [23], p.79).

Since $\omega_2(z) \in H^q$, the function

$$\frac{\omega_2(z)}{\pi i} \int_0^{2\pi} \frac{f'_2(\theta) d\theta}{e^{i\theta} - z}$$

belongs to H^η for some $\eta > 0$, and the limit function $\left(\frac{\omega_2(z)}{\pi i} \int_0^{2\pi} \frac{f'_2(\theta) d\theta}{e^{i\theta} - z} \right)^+$, as is just said, belongs to $L^q(\gamma)$. But then the function itself belongs to H^q (by virtue of the well-known Smirnov theorem which states that if $F(z) \in H^\eta$ and $F^+ \in L^{\eta+\varepsilon}(\gamma)$, $\varepsilon > 0$, then $F \in H^{\varepsilon+\eta}$ (see, e.g., [7], p. 393)). Consequently,

$$\sup_{0 < \rho < 1} \int_0^{2\pi} \left| \omega_2(\rho e^{i\theta}) \int_0^{2\pi} \frac{f'_2(\theta)}{e^{i\theta} - \rho e^{i\varphi}} \right|^q d\varphi < \infty.$$

As far as $a'_k, b'_k \in \overline{\Gamma_2}$ on the basis of (21), we can conclude that

$$\sup_{0 < \rho < 1} \int_{\Theta(\Gamma_2)} \left| \phi'_2(\rho e^{i\theta}) \right|^q |\omega_2(\rho e^{i\theta})|^q d\theta < \infty. \quad (22)$$

The function f_2 on $\Gamma_2 \cup \tilde{\gamma}$ is absolutely continuous and its derivative belongs to $L^q(\Gamma_2; \omega_2)$. Since $\frac{1}{\omega_2} \in L^{q'+\varepsilon}(\gamma)$, $\varepsilon > 0$, the function $f_2' \in L^{1+\eta}(\Gamma_2 \cup \tilde{\gamma})$, $\eta > 0$. Therefore it is not difficult to establish that f satisfies Hölder's condition. On the basis of that fact, $\phi_2(z)$ may have at the end points a'_k, b'_k only logarithmic singularity (see [2], p.75). Hence on Γ_1 we have

$$|\phi_2(z)\omega_1(z)| \leq M_g |\omega_1(z)| \left(\sum_{k=1}^m |\ln|z - a'_k|| + |\ln|z - b'_k|| \right).$$

Moreover, even if the points d_k in the product (6) coincide with some points a'_k, b'_k , we will have

$$\sup_{|z|<1} \int_0^{2\pi} |\omega_1(z)|^p \left(\sum_{k=1}^m |\ln|z - a'_k|| + |\ln|z - b'_k|| \right)^p d\theta < \infty.$$

Consequently,

$$\sup_{0<\rho<1} \int_{\Theta(\Gamma_1)} |\phi_2(\rho e^{i\theta})\omega_1(\rho e^{i\theta})|^p d\theta < \infty,$$

which together with (22) yields

$$\begin{aligned} & \sup_{0<r<1} \left[\int_{\Theta(\Gamma_1)} |\phi_2(re^{i\theta})\omega_1(re^{i\theta})|^p d\theta + \right. \\ & \left. + \int_{\Theta(\Gamma_2)} |\phi_2'(re^{i\theta})\omega_2(re^{i\theta})|^q d\theta \right] < \infty. \end{aligned} \quad (23)$$

Density of the integral ϕ_1 belongs to $L^p(\Gamma_1; \omega_1)$, and thus, as above, for ϕ_2' we establish that

$$\sup_{0<r<1} \int_{\Theta(\Gamma_1)} |\phi_1(re^{i\theta})\omega_1(re^{i\theta})|^p d\theta < \infty.$$

The inequality

$$\sup_{0<r<1} \int_{\Theta(\Gamma_2)} |\phi_1'(re^{i\theta})\omega_2(re^{i\theta})|^q d\theta < \infty$$

is obvious because the distance from $\Gamma_1 \setminus \tilde{\gamma}$ to Γ_2 is positive and $\omega_2 \in H^q$.

The last two inequalities result in

$$\sup \left[\int_{\Theta(\Gamma_1)} |\phi_1(re^{i\theta})\omega_1(re^{i\theta})|^p d\theta + \int_{\Theta(\Gamma_2)} |\phi_1'(re^{i\theta})\omega_2(re^{i\theta})|^q d\theta \right] < \infty. \quad (24)$$

Since $u(re^{i\varphi}) = \operatorname{Re} \phi(re^{i\theta})$, we have

$$|u(re^{i\varphi})\omega_1(re^{i\varphi})| \leq |\phi_1(re^{i\varphi})\omega_1(re^{i\varphi})| + |\phi_2(re^{i\varphi})\omega_2(re^{i\varphi})|,$$

and

$$\left| \frac{\partial u}{\partial x} \right| \leq |\phi'_1| + |\phi'_2|, \quad \left| \frac{\partial u}{\partial y} \right| \leq |\phi'_1| + |\phi'_2|.$$

Taking into account the above-said and also the inequalities (23) and (24), we conclude that $u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$. \square

3⁰. Statement of the Mixed Problem and Scheme of Its Solution.

Let $\Gamma_1, \Gamma_2, \tilde{\gamma}$ be the sets defined in Section 1⁰, and let ω_1, ω_2 be given by the equalities (6)–(7). Consider the following boundary value problem: find a harmonic function in the class $h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ such that (i) its angular boundary values coincide almost everywhere on $\Gamma_1 \setminus \tilde{\gamma}$ with the given function $f, f \in L^p(\Gamma_1 \setminus \tilde{\gamma}; \omega_1)$; (ii) the boundary values on $\Gamma_2 \cup \tilde{\gamma}$ form an absolutely continuous function on $\tilde{\gamma}$ and $u^+ = \psi, \psi' \in L^q(\tilde{\gamma}; \omega_2)$; (iii) the boundary value of the normal derivative on Γ_2 coincides with the function $g, g \in L^q(\Gamma_2; \omega_2)$.

Thus it is required to find the function u satisfying the following conditions:

$$\left\{ \begin{array}{l} \Delta u = 0, \quad u \in h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2)), \quad p > 1 \quad q > 1, \\ u^+|_{\Gamma_1 \setminus \tilde{\gamma}} = f, \quad f \in L^p(\Gamma_1 \setminus \tilde{\gamma}; \omega_1); \quad u^+ A(\Gamma_2 \cup \tilde{\gamma}), \\ u^+|_{\tilde{\gamma}} = \psi, \quad \psi \in A(\tilde{\gamma}), \quad \psi' \in L^q(\tilde{\gamma}; \omega_2); \\ \left(\frac{\partial u}{\partial n} \right)^+|_{\Gamma_2} = g, \quad g \in L^q(\Gamma_2; \omega_2). \end{array} \right. \quad (25)$$

Here we present a brief scheme of solving the problem formulated above.

If a solution u of the problem (25) does exist, then according to Lemma 3 and its Corollary 1, this solution belongs to the class $h_\sigma, \sigma > 1$ and it is representable by the equality (15) with the function $u^+, u^+ \in L^\sigma(\gamma), \sigma > 1$. The function v conjugate harmonically to the function u also belongs to h_σ , and

$$v(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} u^+(\theta) Q(r, \theta - \varphi) d\theta, \quad Q(r, x) = \frac{2r \sin x}{1 + r^2 - 2r \cos x}$$

(see, e.g., [8], p. 54). Moreover, since

$$\frac{\partial u}{\partial n}(re^{i\varphi}) = -\frac{\partial u}{\partial r}(re^{i\varphi})$$

and

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \varphi},$$

we find that

$$\frac{\partial u}{\partial n}(re^{i\varphi}) = -\frac{1}{r} \frac{\partial v}{\partial \varphi}(re^{i\varphi}).$$

Taking this into account, we obtain

$$\begin{aligned} \frac{\partial u}{\partial n} &= -\frac{1}{2\pi r} \int_0^{2\pi} u^+ \frac{\partial}{\partial \varphi} Q(r, \theta - \varphi) d\theta = -\frac{1}{2\pi r} \int_{\Theta(\Gamma_1)} u^+ \frac{\partial}{\partial \varphi} Q(r, \theta - \varphi) d\theta - \\ &\quad - \frac{1}{2\pi r} \int_{\Theta(\Gamma_2)} u^+ \frac{\partial}{\partial \varphi} Q(r, \theta - \varphi) d\theta = v_1(re^{i\varphi}) + v_2(re^{i\varphi}). \end{aligned} \quad (26)$$

For $e^{i\varphi} \in \Gamma_2$, in the integral with $v_1(re^{i\varphi})$ we can pass to the limit under the integral sign, i.e.,

$$\lim_{r \rightarrow 1} v_1(re^{i\varphi}) = -\frac{1}{2\pi} \int_{\Theta(\Gamma_1)} u^+(\theta) \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}}.$$

We write v_2 in the form

$$\begin{aligned} v_2(re^{i\varphi}) &= -\frac{1}{2\pi r} \int_{\Theta(\Gamma_2)} u^+(\theta) \frac{\partial}{\partial \varphi} Q(r, \theta - \varphi) d\theta = \\ &= \frac{1}{2\pi r} \int_{\Theta(\Gamma_2)} u^+(\theta) \frac{\partial}{\partial \theta} Q(r, \theta - \varphi) d\theta. \end{aligned}$$

In this integral, u^+ is absolutely continuous on $Q(\Gamma_2)$, and hence partial integration is quite admissible here. This yields

$$v_2(re^{i\varphi}) = \frac{1}{2\pi r} u^+(\theta) Q(r, \theta - \varphi) |_{\Gamma_2} - \frac{1}{2\pi r} \int_{Q(\Gamma_2)} \frac{\partial u^+}{\partial \theta} Q(r, \theta - \varphi) d\theta.$$

Passing to the limit as $r \rightarrow 1$ and using the property of the integral with the kernel $Q(r, x)$ (see [8], p. 62), we obtain

$$\begin{aligned} \lim_{r \rightarrow 1} v_2(re^{i\varphi}) &= \frac{1}{2\pi} u^+(\theta) \operatorname{ctg} \frac{\theta - \varphi}{2} |_{\Gamma_2} - \\ &\quad - \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} \frac{\partial u^+}{\partial \theta} \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta, \quad e^{i\varphi} \in \Gamma_2. \end{aligned}$$

If we take into account the above-obtained expression for limiting values v_1 and v_2 , then from (26) for $e^{i\varphi} \in \Gamma_2$ we get

$$\begin{aligned} \left(\frac{\partial u}{\partial n} \right)^+(e^{i\varphi}) &= -\frac{1}{2\pi} \int_{\Theta(\Gamma_1)} u^+ \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}} + \frac{1}{2\pi} u^+(\theta) \operatorname{ctg} \frac{\theta - \varphi}{2} |_{\Gamma_2} - \\ &\quad - \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} \frac{\partial u^+}{\partial \varphi} \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta. \end{aligned}$$

As far as u is a solution of the problem (25), we arrive at the equality

$$\begin{aligned} & \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} \frac{\partial u^+}{\partial \theta} \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta = \\ & = -g(e^{i\varphi}) - \frac{1}{2\pi} \int_{\Theta(\Gamma_1 \setminus \tilde{\gamma})} f(\theta) \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}} - \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \psi(\theta) \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}} + \\ & + \frac{1}{2\pi} \sum_{k=1}^m \left[\psi(a_{k+1}) \operatorname{ctg} \frac{\alpha_{k+1} - \varphi}{2} - \psi(b_k) \operatorname{ctg} \frac{\beta_k - \varphi}{2} \right], \quad e^{i\varphi} \in \Gamma_2, \end{aligned}$$

where $a_k = e^{i\alpha_k}$, $b_k = e^{i\beta_k}$, $a_{m+1} = a_1$.

Thus if $u(re^{i\varphi})$ is a solution of the problem (25), then the function $\frac{\partial u^+}{\partial \theta}$ belongs to $L^q(\Gamma_2; \omega_2)$ (by Lemma 5) and is a solution of the integral equation

$$\frac{1}{2\pi} \int_{\Theta(\Gamma_2)} \frac{\partial u^+}{\partial \theta} \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta = \mu(\varphi), \quad e^{i\varphi} \in \Gamma_2, \quad (27)$$

where

$$\begin{aligned} \mu(\varphi) & = -g(e^{i\varphi}) - \frac{1}{2\pi} \int_{\Theta(\Gamma_1 \setminus \tilde{\gamma})} f(\theta) \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}} - \\ & - \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \psi(\theta) \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}} + \\ & + \frac{1}{2\pi} \sum_{k=1}^m \left[\psi(a_{k+1}) \operatorname{ctg} \frac{\alpha_{k+1} - \varphi}{2} - \psi(b_k) \operatorname{ctg} \frac{\beta_k - \varphi}{2} \right]. \quad (28) \end{aligned}$$

Let us show that under the adopted assumptions regarding the functions f , ψ and g , the function in the right-hand side of the equality (27) belongs to $L^q(\Gamma_2; \omega_2)$.

Indeed, we have

$$\begin{aligned} & \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \psi(\theta) \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}} = -\psi(\theta) \operatorname{ctg} \frac{\theta - \varphi}{2} \Big|_{\tilde{\gamma}} + \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \frac{\partial \psi}{\partial \theta} \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta = \\ & = -\frac{1}{2\pi} \sum_{k=1}^m \left[\psi(b_k) \operatorname{ctg} \frac{\beta_k - \varphi}{2} - \psi(b'_k) \operatorname{ctg} \frac{\beta'_k - \varphi}{2} \right] - \\ & - \frac{1}{2\pi} \sum_{k=1}^m \left[\psi(a'_k) \operatorname{ctg} \frac{\alpha'_k - \varphi}{2} - \psi(a_k) \operatorname{ctg} \frac{\alpha_k - \varphi}{2} \right] + \\ & + \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \frac{\partial \psi}{\partial \theta} \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta, \quad a'_k = e^{i\alpha'_k}, \quad b'_k = e^{i\beta'_k}. \end{aligned}$$

Here the last summand can be written as

$$\tilde{\psi}_1(\varphi) = \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \frac{\partial \psi}{\partial \theta} \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \chi_{\tilde{\gamma}}(\theta) \frac{\partial \psi}{\partial \theta} \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta.$$

Since $\frac{\partial \psi}{\partial \theta} \in L^q(\tilde{\gamma}; \omega_2)$, the function $\psi_1(\theta) = \chi_{\tilde{\gamma}}(\theta) \frac{\partial \psi}{\partial \theta}$ belongs to $L^q(\gamma; \omega_2)$. But then the function $\tilde{\psi}_1$ likewise belongs to $L^q(\gamma; \omega_2)$ (see, e.g., [24], p.24). Inserting the expression for $\tilde{\psi}_1$ into (28), we obtain

$$\begin{aligned} \mu(\varphi) = & -g(e^{i\varphi}) + \sum_{k=1}^m \left[\psi(a'_k) \operatorname{ctg} \frac{\alpha'_k - \varphi}{2} - \psi(b'_k) \operatorname{ctg} \frac{\beta'_k - \varphi}{2} \right] - \\ & - \frac{1}{2\pi} \int_{\Theta(\Gamma_1 \setminus \tilde{\gamma})} f(\theta) \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}} - \tilde{\psi}_1(\varphi). \end{aligned}$$

By virtue of what has been said about $\tilde{\psi}_1$, we can conclude that $\mu \in L^q(\Gamma_2; \omega_2)$.

Consequently, (27) is the singular integral equation with respect to $\frac{\partial u^+}{\partial \theta}$ in the class $L^q(\Gamma_2; \omega_2)$, where Γ_2 is the union of the arcs (b_k, a_{k+1}) , $k = \overline{1, m}$, $a_{m+1} = a_1$ and ω_2 is given by the equality (7). This equation can easily be reduced to the equation with the Cauchy kernel; the latter has been solved in [24] for a particular case with the weight ω_2 .

On the basis of the results obtained in [24] (pp. 35-46; see also [23], pp. 104-108), we will be able to find conditions for the solvability of the above equation and to construct its solution in the case of more general weights (below, see the conditions (32) regarding ν_k and λ_k).

All this will be done in Section 4⁰. We will prove there that the equation (27) is, undoubtedly, solvable for $m_1 \leq m$ and the solution contains an arbitrary polynomial of order $r - 1 = m - m_1 - 1$. If, however, $m_1 > m$, then it is solvable provided that $m_1 - m$ integral conditions are fulfilled (see the equalities (57) below). Solutions, if they exist, are written out in quadratures.

Having known $\frac{\partial u^+}{\partial \theta}$, we can find the values u^+ on Γ_2 to within constant summands B_k , $k = \overline{1, m}$, on (b_k, a_{k+1}) . The condition of absolute continuity of the functions u^+ on $\Gamma_2 \cup \tilde{\gamma}$ results in the equalities $u^+(a_k) = \psi(a_k)$ and $u^+(b_k) = \psi(b_k)$. This allows us to get conditions for the solvability of the problem (25) and to find specific values for B_k . As a result, we find the values of u^+ on the whole circumference γ and construct solutions of the problem (25).

4⁰. On the Inversion Formula of the Cauchy Singular Integral in Lebesgue Classes with Power Weight.

Let $\mathcal{L}_k = (A_k, B_k)$, $k = \overline{1, m}$ be the arcs lying separately on the oriented Lyapunov curve \mathcal{L} , and let m_1 be an integer from the segment $[0, 2m]$. We denote by C_1, C_2, \dots, C_{2m} the points A_k, B_k , $k = \overline{1, m}$ taken arbitrarily in

any order. Let $q > 1$,

$$\rho(t) = \prod_{k=1}^{2m} |t - C_k|^{\alpha_k}, \quad \Gamma = \bigcup_{k=1}^m \mathcal{L}_k,$$

and $f \in L(\Gamma) \cap L^q(\Gamma; \rho)$. Consider the Cauchy singular integral

$$(S_\Gamma f)(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\zeta) d\zeta}{\zeta - t}, \quad t \in \Gamma.$$

As is known, the Cauchy singular operator $S_\Gamma : f \rightarrow S_\Gamma f$ is continuous in the space $L^q(\Gamma; \rho)$ if and only if

$$-\frac{1}{q} < \alpha_k < \frac{1}{q'} \tag{29}$$

(see, e.g., [23], [25]).

Consider now in $L^q(\Gamma; \rho)$ the integral equation

$$S_\Gamma \varphi = f. \tag{30}$$

If we wish a solution φ to be "bounded" in the neighborhood of the points c_1, c_2, \dots, c_{m_1} and "free" in the neighborhood of the remaining points c_{m_1+1}, \dots, c_{2m} , we assume that

$$\rho(t) = \prod_{k=1}^{m_1} |t - C_k|^{\nu_k} \prod_{k=m_1+1}^{2m} |t - C_k|^{\lambda_k}, \quad -\frac{1}{q} < \nu_k < 0, \quad 0 \leq \lambda_k < \frac{1}{q'}. \tag{31}$$

Under these conditions S_Γ acts from $L^q(\Gamma; \rho)$ to $L^q(\Gamma; \rho)$ and thus we assume that $f \in L^q(\Gamma; \rho)$.

Suppose (following [2], p.279, or [23], p.104) that

$$(U_\Gamma \varphi)(t) = \frac{R(t)}{\pi i} \int_\Gamma \frac{1}{R(\zeta)} \frac{\varphi(\zeta)}{\zeta - t} d\zeta,$$

where $R(z)$ is the function defined by the equalities (2)–(1).

It is easy to verify that the operator U_Γ is continuous in $L^q(\Gamma; \rho)$ if and only if the operator

$$U_{\Gamma, \rho} : \varphi \rightarrow U_{\Gamma, \rho}(\varphi), \quad U_{\Gamma, \rho}(\varphi) = \frac{R(t)\rho(t)}{\pi i} \int_\Gamma \frac{1}{R(\zeta)\rho(\zeta)} \frac{\varphi(\zeta) d\zeta}{\zeta - t}$$

is continuous in $L^q(\Gamma)$.

Since

$$|R(t)\rho(t)| = \prod_{k=1}^{m_1} |t - C_k|^{\frac{1}{2} + \nu_k} \prod_{k=m_1+1}^{2m} |t - C_k|^{\lambda_k - \frac{1}{2}},$$

it follows from (29) that for the operator $U_{\Gamma, \rho}$ to be continuous in $L^q(\Gamma)$, it is necessary and sufficient that

$$-\frac{1}{q} < \frac{1}{2} + \nu_k < \frac{1}{q'}, \quad -\frac{1}{q} < \lambda_k - \frac{1}{2} < \frac{1}{q'}.$$

By virtue of the above said and together with the conditions (31) we can conclude that:

– if the weight ρ is given by the equality (31), then the operator U_Γ is continuous in $L^q(\gamma; \rho)$, if

$$-\frac{1}{q} < \nu_k < \min\left(0; \frac{1}{q'} - \frac{1}{2}\right), \quad \max\left(0; \frac{1}{2} - \frac{1}{q}\right) \leq \lambda_k < \frac{1}{q'}. \quad (32)$$

In particular,

(a) if $1 < q \leq 2$, then

$$-\frac{1}{q} < \nu_k < \frac{q-2}{2q}; \quad 0 \leq \lambda_k < \frac{1}{q'}, \quad (32_1)$$

(b) if $q > 2$, then

$$-\frac{1}{q} < \nu_k < 0, \quad \frac{q-2}{2q} \leq \lambda_k < \frac{1}{q'}. \quad (32_2)$$

Taking this statement into account and following the reasoning from [23] (pp.106-108), we establish the following

Theorem A. *If for the weight*

$$\rho(t) = \prod_{k=1}^{m_1} |t - C_k|^{\nu_k} \prod_{k=m_1+1}^{2m} |t - C_k|^{\lambda_k}$$

the conditions (32) are fulfilled, then for $m_1 \leq m$ the equation (30) is solvable in $L^q(\Gamma; \rho)$, and all its solutions are given by the equality

$$\varphi(t) = (U_\Gamma f)(t) + R(t)P_{r-1}(t), \quad r = m - m_1,$$

where $P_{r-1}(t)$ is an arbitrary polynomial of order $r - 1$, and if $m_1 = m$, then $P_{r-1} = 0$. However, if $m_1 > m$, then the equation (30) is solvable the if and only if the conditions

$$\int_{\Gamma} t^k R(t) f(t) dt = 0, \quad k = 0, \overline{l-1}, \quad l = m_1 - m, \quad (33)$$

are fulfilled and if they are fulfilled we obtain the unique solution given by the equality $\varphi = U_\Gamma f$.

Remark 1. The condition that L_k , $k = \overline{1, m}$, lie on a Lyapunov curve has been adopted for the sake of simplicity (this condition is sufficient for applications; Theorem A will be used below in case L is a circumference). Theorem A remains also valid in the case of curves Γ for which the operator S_Γ is continuous in the space $L^q(\Gamma; \rho)$ with all the power weights ρ satisfying the condition (29).

Remark 2. It is not difficult to see that Theorem A remains valid if instead of the weight ρ we take the weight

$$r(t) = \rho(t) \prod_{k=1}^n |t - D_k|^{\alpha_k}, \quad -\frac{1}{q} < \alpha_k < \frac{1}{q'},$$

where D_k are arbitrary points on Γ , different from the ends L_k .

Remark 3. In [23] and [24], the equation (30) is solved for the particular case in which $\nu_k = -\frac{1}{2q}$ and $\lambda_k = \frac{1}{2q}$.

5⁰. Determination of Function $\frac{\partial u^+}{\partial \theta}$ on Γ_2 .

To apply Theorem A to the equation (27), it is necessary to make use of the fact that for $\tau = e^{i\theta}$, $t = e^{i\varphi}$ we have

$$\frac{d\tau}{\tau - t} = \left(\frac{1}{2} \operatorname{ctg} \frac{\theta - \varphi}{2} + \frac{i}{2} \right) d\theta, \quad (34)$$

and we write the equation (27) in the form

$$\frac{1}{\pi i} \int_{\Gamma_2} \frac{\partial u^+}{\partial \theta} \frac{d\tau}{\tau - e^{i\varphi}} - \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} \frac{\partial u^+}{\partial \theta} d\theta = i\mu(\varphi).$$

Putting here $\frac{1}{2\pi} \int_{\Gamma_2} \frac{\partial u^+}{\partial \theta} = a$, we obtain

$$\frac{1}{\pi i} \int_{\Gamma_2} \frac{\partial u^+}{\partial \theta} \frac{d\tau}{\tau - e^{i\varphi}} = i\mu(\varphi) + a. \quad (35)$$

Since u^+ is the boundary value of the solution of the problem (25),

$$a = \frac{1}{2\pi} \sum_{k=1}^m [\psi(a_{k+1}) - \psi(b_k)], \quad a_{m+1} = a_1, \quad (36)$$

and finally we have

$$\frac{1}{\pi i} \int_{\Gamma_2} \frac{\partial u^+}{\partial \theta} \frac{d\tau}{\tau - e^{i\varphi}} = i\mu(\varphi) + \frac{1}{2\pi} \sum_{k=1}^m [\psi(a_{k+1}) - \psi(b_k)], \quad (37)$$

which, according to Theorem A, allows us to conclude that for $m_1 \leq m$,

$$\frac{\partial u^+}{\partial \theta} = W_{\Gamma_2}(e^{i\theta}) = \frac{R(e^{i\theta})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)} \frac{d\tau}{\tau - e^{i\theta}} + R(e^{i\theta})P_{r-1}(e^{i\theta}), \quad (38)$$

$$r = m - m_1, \quad \tau = e^{i\alpha},$$

and for $m_1 > m$, the equation (37) (and hence the equation (27)) is solvable if and only if the conditions

$$\int_{\gamma_2} \frac{i\mu(\tau) + a}{R(\tau)} \tau^k d\tau = 0, \quad k = \overline{0, l-1}, \quad l = m_1 - m, \quad (39)$$

are fulfilled. If these conditions are fulfilled, then

$$\frac{\partial u^+}{\partial \theta} = \frac{R(e^{i\theta})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) d\tau}{R(\tau)(\tau - e^{i\theta})} \quad (38_1)$$

(here we have taken into account the fact that for $m_1 > m$ the equality $\frac{1}{\pi i} \int_{\Gamma_2} \frac{1}{R(\tau)} \frac{d\tau}{\tau - e^{i\theta}} = 0$ holds (see [23], p. 105)).

If W_{Γ_2} is the function defined by the equality (38), then

$$\frac{1}{\pi i} \int_{\Gamma_2} W_{\Gamma_2}(\tau) \frac{d\tau}{\tau - e^{i\varphi}} = i\mu(\varphi) + a,$$

and hence from (34) we find that

$$\begin{aligned} \frac{1}{2\pi} \int_{\Gamma_2} W_{\Gamma_2}(e^{i\theta}) \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta &= \\ &= i \left(\frac{1}{\pi i} \int_{\Gamma_2} W_{\Gamma_2} \frac{d\tau}{\tau - e^{i\varphi}} - \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} W_{\Gamma_2}(e^{i\theta}) d\theta \right) = \\ &= -\mu(\varphi) + i \left(a - \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} W_{\Gamma_2}(e^{i\theta}) d\theta \right). \end{aligned}$$

Consequently, the function $W_{\Gamma_2}(e^{i\theta})$ is a solution of the equation (27) if and only if

$$\frac{1}{2\pi} \int_{\Theta(\Gamma_2)} W_{\Gamma_2}(e^{i\theta}) d\theta = a,$$

i.e. if

$$\frac{1}{2\pi i} \int_{\Theta(\Gamma_2)} \left[\frac{R(e^{i\theta})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)} \frac{d\tau}{\tau - e^{i\theta}} + R(e^{i\theta}) P_{r-1}(e^{i\theta}) \right] d\theta = a. \quad (40)$$

If the conditions (40) are fulfilled, then

$$\frac{1}{2\pi} \int_{\Theta(\Gamma_2)} W_{\Gamma_2}(e^{i\theta}) \operatorname{ctg} \frac{\theta - \varphi}{2} d\theta = \mu(\varphi).$$

As far as we are interested only in real solutions of the equation (27), this means that for $m_1 \leq m$ the solution is the function

$$\begin{aligned} W_{\Gamma_2}^*(e^{i\theta}) &= \operatorname{Re} W_{\Gamma_2}(e^{i\theta}) = \\ &= \operatorname{Re} \left[\frac{R(e^{i\theta})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)} \frac{d\tau}{\tau - e^{i\theta}} + R(e^{i\theta}) P_{r-1}(e^{i\theta}) \right], \end{aligned} \quad (41)$$

while for $m_1 > m$, if the conditions (39) are fulfilled, such solution is the function

$$W_{\Gamma_2}^*(e^{i\theta}) = \operatorname{Re} \left[\frac{R(e^{i\theta})}{\pi i} \int_{\Gamma_2} \frac{\mu(\tau) + a}{R(\tau)} \frac{d\tau}{\tau - e^{i\theta}} \right]. \quad (41_1)$$

Thus we have proved the following

Lemma 7. *If the functions f , ψ , g , from the boundary conditions (25) are such that the function μ defined by means of these functions (by the equality (28)) and the constant a defined by the equality (36) satisfy the condition*

(40), then the function $W_{\Gamma_2}^*$ given by the equality (41) is the solution of the equation (27) for $m_1 \leq m$. If, however, $m_1 > m$, and along with (40) the conditions (39) are fulfilled, then the solution of the equation under consideration is the function given by the equality (41₁). In both cases $W_{\Gamma_2}^*$ provides us with the sought for value of $\frac{\partial u^+}{\partial \bar{\theta}}$ on Γ_2 .

6⁰. Solution of Homogeneous Mixed Problem.

In this case $f = 0, \psi = 0, g = 0$ and hence $\mu = 0, a = 0$.

First, let $m_1 > m$. Then the conditions (39) are fulfilled and by (38₁) we have $\frac{\partial u^+}{\partial \bar{\theta}} = 0$ on Γ_2 . Thus $u^+(\theta) = A_k, e^{i\varphi} \in [b_k, a_{k+1}]$ where A_k are the real constants, and only the functions

$$u(re^{i\varphi}) = \sum_{k=1}^m A_k \int_{\Theta([b_k, a_{k+1}])} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} d\theta, \quad e^{i\varphi} \in \gamma, \quad (42)$$

may be a solution of the homogeneous problem (25).

The condition $u^+ \in A(\Gamma_2 \cup \tilde{\gamma})$ implies that for $e^{i\varphi} \in [b_k, a_{k+1}]$, we have $A_k = \psi(b_k^-) = 0, k = \overline{1, m}$. Hence for $m_1 > m$, the homogeneous problem (25) has only the zero solution.

If $m_1 \leq m$, then the conditions (40) take the form

$$\int_{\Theta(\Gamma_2)} R(e^{i\theta})P_{r-1}(e^{i\theta})d\theta = 0, \quad (40_1)$$

and if these conditions are fulfilled, then

$$u^+(e^{i\theta}) = \int_{\beta_1}^{\theta} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re}[R(e^{i\alpha})P_{r-1}(e^{i\alpha})]d\alpha + A_k, \quad e^{i\theta} \in (b_k, a_{k+1}). \quad (43)$$

It is not difficult to see that u^+ is absolutely continuous on $\Gamma_2 \cup \tilde{\gamma}$ if and only if

$$\int_{\beta_1}^{\alpha_{k+1}} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re}[R(e^{i\alpha})P_{r-1}(e^{i\alpha})]d\alpha + A_k = 0$$

$$k = \overline{1, m}, \quad e^{i\alpha_k} = a_k, \quad e^{i\beta_k} = b_k, \quad (44)$$

$$\int_{\beta_1}^{\beta_k} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re}[R(e^{i\alpha})P_{r-1}(e^{i\alpha})]d\alpha + A_k = 0.$$

The above conditions are compatible only if

$$\int_{\beta_k}^{\alpha_{k+1}} \operatorname{Re}[R(e^{i\alpha})P_{r-1}(e^{i\alpha})]d\alpha = 0. \quad (45)$$

If these conditions are fulfilled, then

$$A_k = - \int_{\beta_k}^{\alpha_{k+1}} \operatorname{Re}[R(e^{i\alpha})P_{r-1}(e^{i\alpha})]d\alpha. \quad (46)$$

Let us find for which polynomials P_{r-1} the conditions (40₁) (and hence the conditions (45)) are fulfilled.

Suppose $R(t) = R_1(t) + iR_2(t)$, and let

$$P_{r-1}(t) = \sum_{j=0}^{r-1} (x_j + iy_j)t^j, \quad t = e^{i\theta}. \quad (47)$$

Then the conditions (45) take the form

$$\begin{cases} \sum_{j=0}^{r-1} \int_{\beta_k}^{\alpha_{k+1}} [x_j R_1(e^{i\theta}) \cos j\theta - y_j R_2(e^{i\theta}) \sin j\theta] d\theta = 0, \\ \sum_{j=0}^{r-1} \int_{\beta_k}^{\alpha_{k+1}} [x_j R_2(e^{i\theta}) \cos j\theta + y_j R_1(e^{i\theta}) \sin j\theta] d\theta = 0. \end{cases} \quad k = \overline{1, m}, \quad r = m - m_1, \quad (48)$$

Thus we have obtained a linear system with respect to the unknowns $x_0, x_0, x_{r-1}, y_0, y_1, \dots, y_{r-1}$. The determining matrix of the system (48) is the matrix

$$A = (a_{ij})_{i,j=\overline{1,m}},$$

$$a_{ij} = \begin{cases} \int_{\beta_i}^{\alpha_{i+1}} R_1(\theta) \cos(j-1)\theta d\theta, & i = \overline{1, m}, \quad j = \overline{1, r}, \\ - \int_{\beta_i}^{\alpha_{i+1}} R_2(\theta) \sin(j-r-1)\theta d\theta, & i = \overline{1, m}, \quad j = \overline{r+1, 2r}, \\ \int_{\beta_i}^{\alpha_{i+1}} R_2(\theta) \cos(j-1)\theta d\theta, & i = \overline{m+1, 2m}, \quad j = \overline{1, r}, \\ \int_{\beta_i}^{\alpha_{i+1}} R_1(\theta) \sin(j-r-1)\theta d\theta, & i = \overline{m+1, 2m}, \quad j = \overline{r+1, 2r}. \end{cases} \quad (49)$$

The matrix A has $2m$ rows and $2r = 2(m - m_1)$ columns.

Let

$$\nu = \operatorname{rank} A. \quad (50)$$

(Obviously, $\nu \leq 2r$). The system (48) has a solution which depends on $2r - \nu$ real constants. Inserting the values of these solutions in the expression P_{r-1} from (47) and substituting the obtained polynomials into (41), we define $W_{\Gamma_2}^*$. Next, from (46) we can find the values of A_k . Having obtained $W_{\Gamma_2}^*$

and A_k , by the formula (43) we can find u^+ on Γ_2 . Let

$$u^+(\theta) = \begin{cases} 0, & e^{i\theta} \in \Gamma_1, \\ \int_{\beta_1}^{\theta} \chi_{\Theta(\Gamma_2)}(\alpha) W_{\Gamma_2}^*(\theta) d\theta + A_k, & e^{i\theta} \in \Gamma_2. \end{cases} \quad (51)$$

Then the formula (15) provides us with a possible solution. By Lemma 6, the obtained function belongs to $h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$.

We can easily verify the fulfilment of all boundary conditions. Thus we have proved the following

Theorem 1. *For $m_1 > m$, the homogeneous boundary value problem (25) has only the zero solution. For $m_1 \leq m$, the problem has the solution depending on $2(m - m_1) - \nu$ arbitrary real constants, where ν is the rank of the matrix A defined by (50) in which R_1 and R_2 are, respectively, the real and imaginary parts of the function $R(t)$ given by the equality (2). All the solutions are represented by the Poisson integral (15) with density u^+ given by (51) in which $W_{\Gamma_2}^*$ is defined by the formula (41) with the polynomial P_{r-1} whose coefficients $x_j + iy_j$ are defined from the system (48) and the constants A_k by the equality (46).*

7⁰. Solution of Problem (25).

We will now proceed to the investigation of the non-homogeneous problem.

Towards this end, we construct a particular solution.

As is seen, for $m_1 \leq m$, the function $\frac{\partial u^+}{\partial \theta}$ on Γ_2 is defined by the equality (41), while when $m_1 > m$ and the conditions (39) are fulfilled, then this function is defined by the equality (41₁) (see Lemma 7). Taking in these formulas $P_{r-1} \equiv 0$ and integrating with respect to θ , we find u^+ on Γ_2 ,

$$u^+(\theta) = W_{\Gamma_2}(\theta) = \int_{\beta_1}^{\theta} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re} \left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)(\tau - e^{i\alpha})} d\tau \right] d\alpha + B_k, \quad e^{i\theta} \in (b_k, a_{k+1}). \quad (52)$$

This function on Γ_2 belongs to $L^q(\Gamma_2; \omega_2)$ for any $\mu \in L^q(\Gamma_2; \omega_2)$ if $m_1 \leq m$. However, if $m_1 > m$, then u^+ belongs to $L^q(\Gamma_2; \omega_2)$ only for those μ for which the equalities (39) are valid.

Consider on γ the function

$$u^+(\theta) = \begin{cases} \psi(e^{i\theta}), & e^{i\theta} \in \tilde{\gamma}, \\ f(e^{i\theta}), & e^{i\theta} \in \Gamma_1 \setminus \tilde{\gamma}, \\ W_{\Gamma_2}(\theta), & e^{i\theta} \in \Gamma_2. \end{cases} \quad (53)$$

The functions $u(re^{i\varphi})$ constructed by the formula (15) with that density are candidates for being a particular solutions of the problem (25). It is not difficult to notice that only absolute continuity of the function $u^+(\theta)$ on $\Gamma_2 \cup \tilde{\gamma}$ needs checking, hence it is necessary and sufficient that this

function be continuous (since it is absolutely continuous on the parts $\tilde{\gamma}$ and Γ_2 , coincides on $\tilde{\gamma}$ with ψ , $\psi \in A(\tilde{\gamma})$ and on Γ_2 it has the form (52)).

Thus it is necessary to fulfill the conditions

$$W_{\Gamma_2}(\alpha_{k+1}) = \psi(a_{k+1}), \quad W_{\Gamma_2}(\beta_k) = \psi(b_k), \quad (54)$$

i.e. we have to choose in (52) the constants B_k such that

$$\int_{\beta_1}^{\alpha_{k+1}} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re} \left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)(\tau - e^{i\alpha})} d\tau \right] d\alpha + B_k = \psi(a_{k+1}), \quad (55)$$

$$k = \overline{1, m},$$

$$\int_{\beta_1}^{\beta_k} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re} \left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)(\tau - e^{i\alpha})} d\tau \right] d\alpha + B_k = \psi(b_k). \quad (56)$$

The numbers B_k in the equalities (55) are defined uniquely. Next, for (56) to be valid, it is necessary and sufficient that the conditions

$$\begin{aligned} & \int_{\beta_k}^{\alpha_{k+1}} \operatorname{Re} \left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Theta(\Gamma_2)} \frac{i\mu(\tau) + a}{R(\tau)(\tau - e^{i\alpha})} d\tau \right] d\alpha = \\ & = \psi(a_{k+1}) - \psi(b_k), \quad k = \overline{1, m}, \end{aligned} \quad (57)$$

be fulfilled. Moreover, the condition (40) must also be fulfilled, but as far as $\Gamma_2 = \cup(b_k, a_{k+1})$, the equalities (57) imply that (40) is fulfilled.

If the conditions (57) are fulfilled, and if the equalities (39) are fulfilled for $m > m_1$, then the solution of the problem (25) is the function

$$\begin{aligned} u^*(re^{i\varphi}) &= \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \psi(e^{i\theta}) P(r, \theta - \varphi) d\theta + \\ &+ \frac{1}{2\pi} \int_{\Theta(\Gamma_1 \setminus \tilde{\gamma})} f(e^{i\theta}) P(r, \theta - \varphi) d\theta + \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} W_{\Gamma_2}(\theta) P(r, \theta - \varphi) d\theta, \end{aligned} \quad (58)$$

where $W_{\Gamma_2}(\theta)$ is defined by the equality (52) in which the constants B_k are calculated by virtue of (55), i.e.,

$$B_k = \psi(\alpha_{k+1}) - \int_{\beta_1}^{\alpha_{k+1}} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re} \left[\frac{R(t)}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)(\tau - e^{i\alpha})} d\tau \right] d\alpha. \quad (59)$$

The consequence of our reasoning in Sections 4⁰–7⁰ is the following

Theorem 2. *Let U be the unit circle with the boundary γ , $\Gamma_1 = \bigcup_{k=1}^m [a_k, b_k]$ be the union of the arcs $\gamma_k = (a_k, b_k)$, $a_k = e^{i\alpha_k}$, $b_k = e^{i\beta_k}$ lying separately on γ , $\tilde{\gamma} = \bigcup_{k=1}^m [a_k, a'_k] \bigcup_{k=1}^m [b'_k, b_k]$, where $[a'_k, b'_k]$ are the arcs on γ_k , $a'_k = e^{i\alpha_k}$, $b'_k = e^{i\beta'_k}$, $\Gamma_2 = \gamma \setminus \Gamma_1$. Moreover, let c_1, c_2, \dots, c_{2m} be the points*

$a_1, \dots, a_m, b_1, \dots, b_m$ taken arbitrarily, m_1 be an integer satisfying $0 \leq m_1 \leq 2m$, R, ω_1, ω_2 be the functions defined respectively by the equalities (2), (6), (7) and let the conditions (32) be fulfilled.

For the problem (25) to be solvable:

(I) if $m_1 \leq m$, it is necessary and sufficient that the conditions

$$\begin{aligned} & \int_{\beta_k}^{\alpha_{k+1}} \operatorname{Re} \left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Theta(\Gamma_2)} \frac{i\mu(\tau) + a}{R(\tau)(\tau - e^{i\alpha})} d\tau \right] d\alpha = \\ & = \psi(e^{i\alpha_{k+1}}) - \psi(e^{i\beta_k}), \quad k = \overline{1, m}, \end{aligned} \quad (57)$$

be fulfilled, where

$$\begin{aligned} \mu(\varphi) = & -g(e^{i\varphi}) + \frac{1}{2\pi} \sum_{k=1}^m \left[\psi(e^{i\alpha_{k+1}}) \operatorname{ctg} \frac{\alpha_{k+1} - \varphi}{2} - \psi(e^{i\beta_k}) \operatorname{ctg} \frac{\beta_k - \varphi}{2} \right] - \\ & - \frac{1}{2\pi} \int_{\Theta(\Gamma_1 \setminus \tilde{\gamma})} f(\theta) \frac{d\theta}{2 \sin^2 \frac{2\theta - \varphi}{2}} - \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \psi(\theta) \frac{d\theta}{2 \sin^2 \frac{\theta - \varphi}{2}}, \end{aligned} \quad (29)$$

$$a = \frac{1}{2\pi} \sum_{k=1}^m [\psi(e^{i\alpha_{k+1}}) - \psi(e^{i\beta_k})]; \quad (36)$$

(II) if $m_1 > m$, it is necessary and sufficient that the condition (57) and

$$\int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)} \tau^k d\tau = 0, \quad k = \overline{0, l-1}, \quad l = m_1 - m,$$

be fulfilled;

(III) if the above-given conditions are fulfilled, then the solution of the problem is given by the equality

$$u(re^{i\varphi}) = u^*(re^{i\varphi}) + u_0(re^{i\varphi}),$$

where

$$\begin{aligned} u^*(re^{i\varphi}) = & \frac{1}{2\pi} \int_{\Theta(\tilde{\gamma})} \psi(\theta) P(r, \theta - \varphi) d\theta + \frac{1}{2\pi} \int_{\Theta(\Gamma_1 \setminus \tilde{\gamma})} f(e^{i\theta}) P(r, \theta - \varphi) d\theta + \\ & + \frac{1}{2\pi} \int_{\Theta(\Gamma_2)} W_{\Gamma_2}(\theta) P(r, \theta - \varphi) d\theta. \end{aligned}$$

Here

$$\begin{aligned} W_{\Gamma_2}(\theta) = & \int_{\beta_1}^{\theta} \chi_{\Theta(\Gamma_2)}(\alpha) \left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)(\tau - e^{i\alpha})} d\tau \right] d\alpha + B_k \quad (52) \\ & r = m - m_1, \end{aligned}$$

$$B_k = \psi(e^{i\alpha_{k+1}}) - \int_{\beta_1}^{\alpha_{k+1}} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re} \left[\frac{R(e^{i\alpha})}{\pi i} \int_{\Gamma_2} \frac{i\mu(\tau) + a}{R(\tau)(\tau - e^{i\alpha})} d\tau \right] d\alpha, \quad (59)$$

and

$$u_0(z) = \begin{cases} 0, & \text{for } m_1 > m, \\ \frac{1}{2\pi} \int_0^{2\pi} W_{\Gamma_2}^*(\theta) P(r, \theta - \varphi) d\theta, \\ W_{\Gamma_2}^*(\theta) = \int_{\beta_1}^{\theta} \chi_{\Theta(\Gamma_2)}(\alpha) \operatorname{Re}[R(e^{i\alpha}) P_{r-1}(e^{i\alpha})] d\alpha + A_k, \\ e^{i\theta} \in (b_k, a_{k+1}) \text{ for } m_1 \leq m \end{cases}$$

where A_k are defined by (46); if, however, $m_1 < m$, then $P_{r-1}(e^{i\theta}) = \sum_{j=0}^{r-1} (x_j + iy_j) e^{ij\theta}$ is the polynomial whose coefficients $x_j, y_j, j = \overline{0, r-1}$ are solutions of the system (48). Note that if $\nu = \operatorname{rank} A$, where A is the matrix given by the equality (49), then among the numbers $x_0, x_1, \dots, x_{r-1}, y_0, y_1, \dots, y_{r-1}$, there are $2(m - m_1) - \nu$ arbitrary parameters.

8⁰. A Mixed Problem in Domains with Lyapunov Curves.

Let D be a simply connected finite domain bounded by a simple oriented Lyapunov curve L . Let $\mathcal{L}_k = (A_k, B_k), k = \overline{1, m}$, be arcs lying separately on that curve. Moreover, let $[A'_k, B'_k]$ be arcs lying on \mathcal{L}_k . We denote

$$L_1 = \bigcup_{k=1}^m \mathcal{L}_k, \quad \tilde{L} = \bigcup_{k=1}^m [A_k, A'_k] \bigcup_{k=1}^m [B'_k, B_k], \quad L_2 = L \setminus L_1. \quad (60)$$

Let $z = z(w)$ be a conformal mapping of the unit circle U onto the domain D and let $w = w(t)$ be an inverse mapping. Suppose

$$\Gamma_1 = w(L_1), \quad \tilde{\gamma} = w(\tilde{L}), \quad \Gamma_2 = w(L_2), \\ \Gamma_j(r) = \{w : w = re^{i\theta}, \theta \in \Theta(\Gamma_j)\}, \quad L_j(r) = z(\Gamma_j(r)).$$

Let C_1, C_2, \dots, C_{2m} be the points of $A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m$ taken arbitrarily, and let D_1, D_2, \dots, D_n be points on L are different from C_k ; the points D_1, D_2, \dots, D_{n_1} lie on L_1 , and the points D_{n_1+1}, \dots, D_n on L_2 .

Suppose

$$\rho_1(z) = \prod_{k=1}^{n_1} (z - D_k)^{\alpha_k}, \quad -\frac{1}{p} < \alpha_k < \frac{1}{p'}, \quad (61)$$

$$\rho_2(z) = \prod_{k=1}^{m_1} (z - C_k)^{\nu_k} \prod_{k=m_1+1}^{2m} (z - C_k)^{\lambda_k} \prod_{k=n_1+1}^n (z - D_k)^{\beta_k}, \quad -\frac{1}{q} < \beta_k < \frac{1}{q'}. \quad (62)$$

We say that the function $u(z), z = x + iy$, harmonic in the domain D belongs to the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ if

$$\sup_{0 < r < 1} \left[\int_{L_1(r)} |u(z)\rho_1(z)|^p |dz| + \int_{L_2(r)} \left(\left| \frac{\partial u}{\partial x} \right|^q + \left| \frac{\partial u}{\partial y} \right|^q \right) |\rho_2(z)|^q |dz| \right] < \infty. \quad (63)$$

Lemma 8. *If $U(z) = U(x, y) \in L(L_{1p}(\rho_1), L'_{2q}(\rho_2))$, then the function $u(w) = U(z(w)) = U(x(\xi, \eta), y(\xi, \eta)) = u(\xi, \eta)$ belongs to the class*

$$h(\Gamma_{1p}(\rho_1(z(w))) \sqrt[p]{|z'(w)|}, \Gamma'_{2q}(\rho_2(z(w))) \sqrt[q]{|z'(w)|}).$$

Indeed, taking into account the Cauchy–Riemann conditions, we can easily verify that $\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial \xi}\right)^2 + \left(\frac{\partial u}{\partial \eta}\right)^2$. Hence the validity of the lemma follows from (9) if we transform the variable z by the equality $z = z(w)$ and take into account that (9) is equivalent to (8).

Consequently, transforming $z = z(w)$, from the function of the class $e(L_{1p}(\rho_1), L'_{2q}(\rho_2))$ we obtain the function of the class $h(L_{1p}(w_1), L'_{2q}(w_2))$, where

$$\omega_1(w) = \rho_1(z(w)) \sqrt[p]{|z'(w)|}, \tag{64}$$

$$\omega_2(w) = \rho_2(z(w)) \sqrt[q]{|z'(w)|}. \tag{65}$$

If $f(z)$ is the function defined on a finite union of arcs $E = L_2 \cup \tilde{L}$, and $z = z(s)$ is the equation of the curve L with respect to the arc abscissa s , then we say that it is absolutely continuous on E , if the function $f(z(s))$ is absolutely continuous on the set $\{s : z(s) \in E\}$, and we write $f \in A(L_2 \cup \tilde{L})$.

Lemma 9. *If $f(z) \in A(L_2 \cup \tilde{L})$, then the function $f(z(\tau))$, where $z(\tau)$ is the restriction on γ of the conformal mapping $z(w)$ of the circle \overline{U} on \overline{D} , belongs to $A(\Gamma_2 \cup \tilde{\gamma})$, and vice versa, if $\varphi(w) \in A(\Gamma_2 \cup \tilde{\gamma})$, then $\varphi(w(t)) \in A(L_2 \cup \tilde{L})$.*

This lemma is the consequence of the fact that under the conformal mapping $z = z(w)$ of the circle onto finite domain which is bounded by a simple rectifiable curve, the functions $z = z(e^{i\theta})$ and $w = w(z(s))$ are absolutely continuous with respect to the arguments θ and s , respectively (see, e.g., [7], pp.405-407).

Let us consider the following mixed boundary value problem: find in the domain D the function U , satisfying the following conditions:

$$\begin{cases} \Delta U = 0, & U \in e(L_{1p}(\rho_1), L'_{2q}(\rho_2)), \quad p > 1, \quad q > 1. \\ U^+|_{L_1 \setminus \tilde{L}} = F, & F \in L^p(L_1 \setminus \tilde{L}; \rho_1); \quad U^+ \in A(L_2 \cup \tilde{L}), \\ U^+|_{\tilde{L}} = \Psi, & \Psi' \in L^q(\tilde{L}; \rho_2); \quad \left(\frac{\partial U}{\partial n}\right)^+|_{L_2} = G, \quad G \in L^q(L_2; \rho_2). \end{cases} \tag{66}$$

Since L is the Lyapunov curve, the functions $|z'(w)|$ and $|w'(z)|$ are continuous respectively in \overline{U} and \overline{D} and different from zero. If we put $z(C_k) = c_k$, then such will be the functions $\frac{z(w) - z(c_k)}{w - c_k}$, $k = \overline{1, m}$. Therefore the classes $h(\Gamma_{1p}(\omega_1), \Gamma'_{2q}(\omega_2))$ with the weights (64) and (65) coincide with the same class with the weights defined by the equalities (6) and (7). Taking into account Lemma 9 and putting

$$f(t) = F(z(t)), \quad \psi(t) = \Psi(z(t)), \quad g(t) = G(z(t)), \quad t \in \gamma, \tag{67}$$

we easily see that problem (66) is equivalent to the problem (25) with the boundary data (67). Applying to the latter Theorem 2, we state that the following theorem is valid.

Theorem 3. *If L is the Lyapunov curve, ρ_1 and ρ_2 are the weight functions defined by the equalities (61) and (62) in which ν_k and λ_k satisfy the conditions*

$$-\frac{1}{q} < \nu_k < \min\left(0; \frac{1}{q'} - \frac{1}{2}\right), \quad \max\left(0; \frac{1}{2} - \frac{1}{q}\right) \leq \lambda_k < \frac{1}{q'},$$

then for the solution U of the problem (66) the statements (I)-(III) of Theorem 2, in which f , ψ , g are given by the equalities (67), are valid.

9⁰. The Mixed Boundary Value Problem in the Class $e'_q(D; \omega)$.

Let D be the domain bounded by a simple Lyapunov curve L , and let $\omega = \omega_2$ be the weight function given by the equality (7) with the values ν_k and λ_k for which the conditions (32) are fulfilled. Consider the mixed boundary value problem in the class $e'_q(D, \omega)$ (somewhat narrower than the class $e(L_{1p}(\rho_1), L'_{2q}(\omega))$):

$$e'_q(D; \omega) = \left\{ u : \Delta u = 0, \quad \sup_{0 < r < 1} \int_{\Gamma_r} \left(\left| \frac{\partial u}{\partial x} \right|^q + \left| \frac{\partial u}{\partial y} \right|^q \right) |\omega^q(z)| |dz| \right\} < \infty,$$

where Γ_r is the of the image circumference of radius r under the conformal mapping of U onto D .

If $q > 1$, and v is the function, harmonically conjugate to u , $u \in e'_q(D)$, then by the Cauchy–Riemann conditions we have $v \in e'_q(D)$. Therefore if $\phi(z) = u(z) + iv(z)$, then $\omega(z)\phi'(z) \in E^q(D)$. But then $\phi(z)$ is continuous in \bar{D} , absolutely continuous on L (see, e.g., [7], p.395), and $\phi'(t) \in L^q(L; \omega)$.

Consider the problem for a circle.

Let $\Gamma_1 = \bigcup_{k=1}^m \gamma_k$, where $\gamma_k = (a_k, b_k)$ are arcs lying separately on the unit circumference γ , $\Gamma_2 = \gamma \setminus \Gamma_1$. It is required to find a function u for which

$$\begin{cases} \Delta u = 0, & u \in e'_q(U; \omega), \quad q > 1, \quad u^+ \in C(\gamma) \\ u^+|_{\Gamma_1} = \psi, & \psi \in A(\Gamma_1), \quad \psi' \in L^q(\Gamma_1; \omega); \\ \left(\frac{\partial u}{\partial n} \right)^+|_{\Gamma_2} = g, & g \in L^q(\Gamma_2; \omega). \end{cases} \quad (68)$$

It can be easily verified that if u is a solution of the problem (68), then it is represented by the formula (15) in which $u^+ \in A(\gamma)$.

Reasoning just in the same way as in Sections 4⁰ – 7⁰, we arrive at the conclusion that the following theorem is valid.

Theorem 4. *For the problem (68) the statements (I)-(III) of Theorem 2, in which we assume that $\tilde{\gamma} = \Gamma_1$ and*

$$\mu(\varphi) = g(e^{i\varphi}) + \frac{1}{2\pi} \int_{\Theta(\Gamma_1)} \psi(e^{i\theta}) \operatorname{ctg} \frac{\theta - \varphi}{2} d\varphi +$$

$$+ \sum_{k=1}^m \left[\psi(b_k) \operatorname{ctg} \frac{\beta_k - \varphi}{2} - \psi(a_{k+1}) \operatorname{ctg} \frac{\alpha_{k+1} - \varphi}{2} \right]$$

are valid.

For the domain D bounded by the Lyapunov curve in the class $e'_q(D; \omega)$, the statement analogous to Theorem 4 is true.

Note that general boundary value problems, including mixed type problems, have been considered by many authors in different classes of functions (see, e.g., [26–28]).

REFERENCES

1. R. COURANT AND D. HILBERT., Methods of mathematical physics. Vol. II: Partial differential equations. (Vol. II by R. Courant.) *Interscience Publishers (a division of John Wiley & Sons), New York-London*, 1962.
2. N. MUSKHELISHVILI, Singular integral equations, boundary value problems and some of their applications to mathematical physics. 3rd ed. (Russian) *Nauka, Moscow*, 1968.
3. A. V. BITSADZE, Boundary value problems for second order elliptic equations. (Russian) *Nauka, Moscow*, 1966.
4. V. G. MAZ'YA, S. A. NAZAROV AND V. A. PLAMENEVSKIĬ, Asymptotic behavior of solutions of elliptic boundary value problems under singular perturbations of the domain. *Tbilis. Gos. Univ., Inst. Prikl. Mat., Tbilisi*, 1981.
5. V. MAZ'YA, Boundary integral equations. *Analysis*, IV, 127–222, *Encyclopaedia Math. Sci.*, 27, *Springer, Berlin*, 1991.
6. M. DAUGE, Elliptic boundary value problems in corner domains. *Lecture Notes in Math.* 1341. *Springer-Verlag, Berlin-Heidelberg-New York*, 1997.
7. G. M. GOLUZIN, Geometric theory of functions of a complex variable. (Russian) *Nauka, Moscow*, 1966.
8. P. DUREN, Theory of H^p spaces. *Pure and Applied Mathematics*, 38 *Academic Press, New York-London*, 1970.
9. V. KOKILASHVILI AND V. PAATASHVILI, On the Riemann–Hilbert problem in the domain with a nonsmooth boundary. *Georgian Math. J.* **4**(1997), No. 3, 279–302.
10. V. KOKILASHVILI AND V. PAATASHVILI, Neumann problem in a class of harmonic functions in domains with a piecewise Lyapunov boundary. *Mem. Differential Equations Math. Phys.* **12**(1997), 114–121.
11. G. KHUSKIVADZE, V. KOKILASHVILI AND V. PAATASHVILI, Boundary value problems for analytic and harmonic functions in domains with non-smooth boundaries. Applications to conformal mapping. *Mem. Differential Equations Math. Phys.* **14**(1998), 1–195.
12. Z. MESHVELIANI AND V. PAATASHVILI, On Smirnov classes of harmonic functions, and the Dirichlet problem. *Proc. A. Razmadze Math. Inst.* **123**(2000), 61–91.
13. R. DUDUCHAVA AND B. SILBERMANN, Boundary value problems in domains with peaks. *Mem. Differential Equations Math. Phys.* **21**(2000), 1–122.
14. Z. MESHVELIANI AND V. PAATASHVILI, On the Dirichlet problem in Smirnov classes of harmonic functions. *Proc. A. Razmadze Math. Inst.* **126**(2001), 59–64.
15. Z. MESHVELIANI, The Neumann problem in domain with piecewise smooth boundaries in weight classes of harmonic Smirnov type functions. *Proc. A. Razmadze Math. Inst.* **126**(2001), 37–52.
16. S. ZAREMBA, Sur un probleme mixte relatif a l'equation de Laplace. *Bull. de l'Academie des sciences de Cracovie, Classe des sciences mathematiques et naturelles serie A*, 1910, 313–344.

17. S. ZARGARYAN AND V. MAZ'YA, Singularities of solutions of a system of equations of potential theory for Zaremba's problem. (Russian) *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **1983**, 43–48.
18. K. MIRANDA, Partial differential equation of elliptic type. (Translation into Russian) *IL, Moscow*, 1957.
19. M. LAVRENT'EV AND B. SHABAT, Methods of the theory of functions of a complex variable. (Russian) *Nauka, Moscow*, 1966.
20. E. I. OBOLASHVILI, The Fourier transform and its application in elasticity theory. (Russian) *Metsniereba, Tbilisi*, 1979.
21. M. RIESZ, Sur les fonctions conjuguées. *Math. Z.* **27**(1927), No. 2, 218–244.
22. A. ZYGMUND, Trigonometric series: Vols. I, II. Second edition, reprinted with corrections and some additions. *Cambridge University Press, London-New York*, 1968.
23. B. KHVEDELIDZE, The method of Cauchy type integrals for discontinuous boundary value problems of the theory of holomorphic functions of one complex variable. (Russian) *Itogi Nauki i Tekhniki, Sovremen. Probl. Mat.* **7**(1975), 5–162. *J. Sov. Math.* **7**(1977), 309–414.
24. B. KHVEDELIDZE, Linear discontinuous boundary value problems of function theory, singular integral equations, and some of their applications. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **23**(1956), 3–158.
25. S. FREYDKIN, On the unboundedness of a singular operator. (Russian) *Materialy Nauchn. Conferen. Kishinev. Univ. (Sect. Estestv. i Eksperem. Nauk)*, 84–86, *Kishinev*, 1970.
26. M. VISHIK, On boundary value problems for systems of elliptic differential equations and on stability of their solutions. (Russian) *Dokl. Akad. Nauk SSSR* **81**(1952), No. 4, 645–648.
27. M. VISHIK AND G. ESKIN, General problems with discontinuous boundary conditions. (Russian) *Dokl. Akad. Nauk SSSR* **158**(1964), No. 1, 25–28.
28. O. CHKADUA, The Dirichlet, Neumann and mixed boundary value problems with domains containing closed cuspidal edges. *Math. Nachr.* **189**(1998), 61–105.

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