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ON THE FREDHOLM PROPERTY OF LINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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Let n_1 and n_2 be natural numbers, $-\infty < a < a_0 < b_0 < b < \infty$, $A_{ik} \in BV_{\rm loc}(]a,b[;\mathbb{R}^{n_i\times n_k})$, $f_i \in BV_{\rm loc}(]a,b[;\mathbb{R}^{n_i})$, $c_i \in \mathbb{R}^{n_i}$ $(i,k = 1,2)$, and $\ell_i : BV([a,b];\mathbb{R}^{n_1}) \times BV([a_0,b_0];\mathbb{R}^{n_2}) \to \mathbb{R}^{n_i}$ $(i = 1,2)$ be linear bounded operators.

Consider the linear generalized ordinary differential system

$$dx_i(t) = dA_{i1}(t)\cdot x_1(t) + dA_{i2}(t)\cdot x_2(t) + df_i(t) \quad (i = 1,2) \tag{1}$$

with the boundary conditions

$$\ell_i(x_1,x_2) = c_i \quad (i = 1,2). \tag{2}$$

It is known (see, e.g., [1], [4]) that if $A_{ik} \in BV([a,b];\mathbb{R}^{n_i\times n_k})$, $f_i \in BV([a,b];\mathbb{R}^{n_i})$ $(i = 1,2)$, then the problem (1), (2) has the Fredholm property, i.e. this problem is uniquely solvable if and only if its corresponding homogeneous problem

$$dx_i(t) = dA_{i1}(t)\cdot x_1(t) + dA_{i2}\cdot x_2(t) \quad (i = 1,2), \tag{1_0}$$

$$\ell_i(x_1,x_2) = 0 \quad (i = 1,2) \tag{2_0}$$

has only the trivial solution. In the case where the system (1) has singularities at the points a and b, i.e. $A_{ik} \notin BV([a,b];\mathbb{R}^{n_i\times n_k})$, $f_i \notin BV([a,b];\mathbb{R}^{n_i})$ for some $i,k \in \{1,2\}$, the question on the Fredholm property of the problem (1), (2) needs additional investigation. Such is the case we consider in the present paper. The similar problem has been studied in [3] for systems of linear ordinary differential equations with singularities.

Throughout the paper we use the following notation and definitions.

$\mathbb{R} =]-\infty,\infty[$; $[a,b]$ and $]a,b[$ are, respectively, closed and open intervals from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i=1}^{n}\sum_{j=1}^{m}|x_{ij}|;$$

$O_{n\times m}$ is the zero $n\times m$-matrix.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} is the matrix inverse to X; $\det X$ is the determinant of X; I_n is the identity $n\times n$-matrix.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $X : [a,b] \to \mathbb{R}^{n\times m}$ is a matrix function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i = 1,\dots,n;\ j = 1,\dots,m)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a_0) = 0$, $v(x_{ij})(t) = \bigvee\limits_{a_0}^{t}(x_{ij})$ for $a_0 < t \le b$, $v(x_{ij})(t) = -\bigvee\limits_{t}^{a_0}(x_{ij})$ for $a \le t < a_0$;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits at the point $t \in [a,b]$ $(X(a-) = X(a),\ X(b+) = X(b))$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

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$BV([a,b];\mathbb{R}^{n\times m})$ is the Banach space of all matrix functions $X:[a,b]\to\mathbb{R}^{n\times m}$ such that $\bigvee\limits_a^b(X)<\infty$ with the norm

$$\|X\|_V=\|X(a_0)\|+\bigvee_a^b(X);$$

$BV_{\mathrm{loc}}(]a,b[;\mathbb{R}^{n\times m})$ is the set of all matrix functions $X:]a,b[\to\mathbb{R}^{n\times m}$ such that $\bigvee\limits_c^d(X)<\infty$ for $a<c<d<b$.

$s_j:BV_{\mathrm{loc}}(]a,b[;\mathbb{R})\to BV_{\mathrm{loc}}(]a,b[;\mathbb{R})$ $(j=0,1,2)$ are the operators defined, respectively, by

$$s_1(g)(a_0)=s_2(g)(a_0)=0;$$

$$s_1(g)(t)=\begin{cases}-\sum\limits_{t<\tau\le a_0}d_1g(\tau) & \text{for } a<t<a_0,\\ \sum\limits_{a_0<\tau\le t}d_1g(\tau) & \text{for } a_0<t<b;\end{cases}$$

$$s_2(g)(t)=\begin{cases}-\sum\limits_{t\le\tau<a_0}d_2g(\tau) & \text{for } a<t<a_0,\\ \sum\limits_{a_0\le\tau<t}d_2g(\tau) & \text{for } a_0<t<b;\end{cases}$$

$$s_0(g)(t)=g(t)-s_1(g)(t)-s_2(g)(t)\quad\text{for}\quad a<t<b.$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\le s<t\le b$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_0(g)(\tau)+\sum_{s<\tau\le t}x(\tau)\,d_1g(\tau)+\sum_{s\le\tau<t}x(\tau)\,d_2g(\tau),$$

where $\int\limits_{]s,t[}x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stiltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $s_0(g)$ (if $s=t$, then $\int\limits_s^t x(\tau)\,dg(\tau)=0$).

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_s^t x(\tau)\,ds_1(g)(\tau)-\int_s^t x(\tau)\,ds_2(g)(\tau)\quad\text{for}\quad a\le s\le t\le b.$$

.

If $G=(g_{ik})_{i,k=1}^{l,n}\in BV([a,b];\mathbb{R}^{l\times n})$ and $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau)=\Big(\sum_{k=1}^n\int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\Big)_{i,j=1}^{l,m}\quad\text{for}\quad a\le s\le t\le b.$$

$\mathcal{A}:BV_{\mathrm{loc}}(]a,b[;\mathbb{R}^{n\times n})\times BV_{\mathrm{loc}}(]a,b[;\mathbb{R}^{n\times m})\to BV_{\mathrm{loc}}(]a,b[;\mathbb{R}^{n\times m})$ is the operator defined by

$$\mathcal{A}(X,Y)(c)=O_{n\times m},$$

$$\mathcal{A}(X,Y)(t)=Y(t)-Y(c)+\sum_{c<\tau\le t}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}d_1Y(\tau)-$$

$$-\sum_{c\le\tau<t}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}d_2Y(\tau)\quad\text{for}\quad c<t<b,$$

$$\mathcal{A}(X,Y)(t)=Y(c)-Y(t)+\sum_{t<\tau\le c}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}d_1Y(\tau)-$$

$$-\sum_{t\le\tau<c} d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}d_2Y(\tau) \quad \text{for} \quad a<t<c,$$

where $c=(a+b)/2$.

$\mathcal{F}_0 : BV([a,b];\mathbb{R})\times BV_{\rm loc}(]a,b[;\mathbb{R})\to BV_{\rm loc}(]a,b[;\mathbb{R})$ is the operator defined by

$$\mathcal{F}_0(f,g)(t)=\int\limits_{a_0}^{t}(v(f)(\tau)-v(f)(a))(v(f)(b)-v(f)(\tau))ds_0(v(g))(\tau)+$$

$$+\int\limits_{a_0}^{t}(v(f)(\tau-)-v(f)(a))(v(f)(b)-v(f)(\tau-))ds_1(v(g))(\tau)+$$

$$+\int\limits_{a_0}^{t}(v(f)(\tau+)-v(f)(a))(v(f)(b)-v(f)(\tau+))ds_2(v(g))(\tau) \quad \text{for} \quad a<t<b.$$

By a solution of the system (1) we understand a vector function $x=(x_i)_{i=1}^2$, $x_i\in BV_{\rm loc}(]a,b[;\mathbb{R}^{n_i})$ $(i=1,2)$ such that

$$x_i(t)-x_i(s)=\int\limits_s^t dA_{i1}(\tau)\cdot x_1(\tau)+\int\limits_s^t dA_{i2}(\tau)\cdot x_2(\tau)+f_i(t)-f_i(s)$$
$$\text{for} \quad a<s\le t<b \quad (i=1,2).$$

A solution $x=(x_i)_{i=1}^2$ of the system (1) is said to be a solution of the problem (1), (2) if $x_1\in BV([a,b];\mathbb{R}^{n_1})$, $x_2\in BV([a_0,b_0];\mathbb{R}^{n_2})$ and the equalities (2) are fulfilled.

It will be assumed that

$$\det(I_{n_i}+(-1)^j d_jA_{ii}(t))\ne 0 \quad \text{for} \quad t\in]a,b[\quad (i,j=1,2).$$

For every $i\in\{1,2\}$, by X_i we denote the fundamental matrix of the system

$$dx_i(t)=dA_{ii}(t)\cdot x_i(t),$$

satisfying the condition

$$X_i(c)=I_{n_i}.$$

By means of the mapping

$$x_i(t)=X_i(t)y_i(t) \quad (i=1,2)$$

the problems (1), (2) and (1_0), (2_0) are reduced, respectively, to the problems

$$dy_i(t)=dA_i(t)\cdot y_{3-i}(t)+d\widetilde{f}_i(t) \quad (i=1,2), \tag{3}$$

$$\widetilde{l}_i(y_1,y_2)=c_i \quad (i=1,2) \tag{4}$$

and

$$dy_i(t)=dA_i(t)\cdot y_{3-i}(t) \quad (i=1,2), \tag{3_0}$$

$$\widetilde{l}_i(y_1,y_2)=c_i \quad (i=1,2), \tag{4_0}$$

where

$$A_i(t)=\int\limits_c^t X_i^{-1}(\tau)d\mathcal{A}(A_{ii},A_{i3-i})(\tau)\cdot X_{3-i}(\tau) \quad \text{for} \quad t\in]a,b[\quad (i=1,2),$$

$$\widetilde{f}_i(t)=\int\limits_c^t X_i^{-1}(\tau)d\mathcal{A}(A_{ii},A_i)(\tau) \quad \text{for} \quad t\in]a,b[\quad (i=1,2)$$

and

$$\widetilde{l}_i(y_1,y_2)=\ell_i(X_1y_1,X_2y_2) \quad (i=1,2).$$

Let $A_i(t) \equiv (a_{ijk}(t))_{j,k=1}^{n_i,n_{3-i}}$ $(i=1,2)$ and $\widetilde{f}_i(t) \equiv (\widetilde{f}_{ik}(t))_{k=1}^{n_i}$ $(i=1,2)$.

Theorem. *Let $\ell_i : BV([a,b];\mathbb{R}^{n_1}) \times BV([a_0,b_0];\mathbb{R}^{n_2}) \to \mathbb{R}^{n_i}$ $(i=1,2)$ be the linear bounded operators and*

$$\bigvee_a^b(A_{11}) + \bigvee_a^b(A_1) + \bigvee_a^b(f_1) < \infty. \tag{5}$$

Let, moreover,

$$1 + (-1)^m \sum_{i=1}^{n_2} \sum_{j,k=1}^{n_1} v(a_{2ij})(t)|d_m a_{1ki}(t)| \neq 0 \quad \textit{for} \quad (-1)^m(t-a_0) < 0 \quad (m=1,2), \tag{6}$$

$$\mathcal{F}_0(a_{1ki}, a_{2ij})(b-) - \mathcal{F}_0(a_{1ki}, a_{2ij})(a+) < \infty \quad (i=1,\dots,n_2;\ \ k,j=1,\dots,n_1) \tag{7}$$

and

$$\mathcal{F}_0(a_{1ki}, \widetilde{f}_{2i},)(b-) - \mathcal{F}_0(a_{1ki}, \widetilde{f}_{2i})(a+) < \infty \quad (i=1,\dots,n_2;\ \ k=1,\dots,n_1). \tag{8}$$

Then the boundary value problem (1), (2) *has a unique solution if and only if its corresponding homogeneous problem* (1_0), (2_0) *has only the trivial solution. On the other hand, the problem* (1_0), (2_0) *has only the trivial solution if and only if the problem* (3_0), (4_0) *has only the trivial solution.*

The proof of this theorem is based on the Fredholm alternative theorem for the operator equations belonging to S. M. Nikol'skiĭ (see, e.g., [2], Ch. XIII, §5).

Consider now the case where the boundary value conditions (2) are of the form

$$\sum_{k=1}^{m} [B_{1ik} x_1(t_{1ik}) + B_{2ik} x_2(t_{2ik})] = c_i \quad (i=1,2), \tag{9}$$

where $B_{jik} \in \mathbb{R}^{n_i \times n_k}$ $(i,j=1,2; k=1,\dots,m)$.

Corollary. *Let $t_{1ik} \in [a,b]$, $t_{2ik} \in]a,b[$ $(i=1,2;\ k=1,\dots,m)$ and the conditions* (5)–(8) *be fulfilled. Then the boundary value problem* (1), (9) *has a unique solution if and only if the system* (1_0) *has only the trivial solution satisfying the boundary value condition*

$$\sum_{k=1}^{m} [B_{1ik} x_1(t_{1ik}) + B_{2ik} x_2(t_{2ik})] = 0 \quad (i=1,2). \tag{9_0}$$

On the other hand, the problem (1_0), (9_0) has only the trivial solution if and only if the system (3_0) has only the trivial solution satisfying the condition

$$\sum_{k=1}^{m} [B_{1ik} X_1(t_{1ik}) y_1(t_{1ik}) + B_{2ik} X_2(t_{2ik}) y_2(t_{2ik})] = 0 \quad (i=1,2).$$

It is evident that if the conditions of Theorem are fulfilled, then the Fredholm property is true for the problem (3), (4) as well.

Finally, we note that the interest in the theory of generalized ordinary differential equations has, to a considerable extent, been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from the unified viewpoint.

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