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ON THE SOLVABILITY OF NONLINEAR OPERATOR EQUATIONS IN A BANACH SPACE

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Let $\mathcal{B}$ be a Banach space with a norm $\|\cdot\|_{\mathcal{B}}$ and $h : \mathcal{B} \to \mathcal{B}$ be a completely continuous nonlinear operator. In this paper, we give theorems on the existence of a solution of the operator equation

$$x = h(x), \tag{1}$$

which generalize the results of [1]–[4] concerning the solvability of boundary value problems for systems of nonlinear functional differential equations.

The use will be made of the following notation.

Θ is the zero element of the space $\mathcal{B}$.

$\overline{D}$ is the closure of the set $D \subset \mathcal{B}$.

$\mathcal{B} \times \mathcal{B} = \{(x, y) : \ x \in \mathcal{B}, \ y \in \mathcal{B}\}$ is the Banach space with the norm

$$\|(x, y)\|_{\mathcal{B}\times\mathcal{B}} = \|x\|_{\mathcal{B}} + \|y\|_{\mathcal{B}}.$$

$\Lambda(\mathcal{B} \times \mathcal{B})$ is the set of completely continuous operators $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that:

(i) $g(x, \cdot) : \mathcal{B} \to \mathcal{B}$ is a linear operator for every $x \in \mathcal{B}$;

(ii) for any x and $y \in \mathcal{B}$ the equation

$$z = g(x, z) + y$$

has a unique solution z and

$$\|z\|_{\mathcal{B}} \le \gamma \|y\|,$$

where γ is a positive constant, independent of x and y.

$\Lambda_0(\mathcal{B} \times \mathcal{B})$ is the set of completely continuous operators $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that:

(i) $g(x, \cdot) : \mathcal{B} \to \mathcal{B}$ is a linear operator for any $x \in \mathcal{B}$;

(ii) the set

$$\big\{g(x, y) : \ x \in \mathcal{B}, \ \|y\|_{\mathcal{B}} \le 1\big\}$$

is relatively compact;

(iii) $y \notin \overline{\{g(x, y) : \ x \in \mathcal{B}\}}$ for $y \in \mathcal{B}$ and $y \ne \Theta$.

Let $g_0 \in \Lambda_0(\mathcal{B} \times \mathcal{B})$. We say that a linear bounded operator $\overline{g} : \mathcal{B} \to \mathcal{B}$ belongs to the set $\mathcal{L}_g$ if there exists a sequence $x_k \in \mathcal{B}$ $(k = 1, 2, \dots)$ such that

$$\lim_{k\to\infty} g(x_k, y) = \overline{g}(y) \ \text{ for } \ y \in \mathcal{B}.$$

Along with $\mathcal{B}$, we consider a partially ordered Banach space $\mathcal{B}_0$ in which the partial order is generated by a cone $\mathcal{K}$, i.e., for any u and $v \in \mathcal{B}_0$, it is said that u does not exceed v, and is written $u \le v$ if $v - u \in \mathcal{K}$.

A linear operator $\eta : \mathcal{B}_0 \to \mathcal{B}_0$ is said to be *positive* if it transforms the cone $\mathcal{K}$ into itself.

An operator $\nu : \mathcal{B} \to \mathcal{B}_0$ is said to be *positively homogeneous* if $\nu(\lambda x) = \lambda \nu(x)$ for $\lambda \ge 0$, $x \in \mathcal{B}$.

By $r(\eta)$ we denote the spectral radius of the operator η.

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Lemma 1. $\Lambda_0(\mathcal{B}\times\mathcal{B})\subset\Lambda(\mathcal{B}\times\mathcal{B})$.

Theorem 1 (A priori boundedness principle). *Let there exist an operator $g\in\Lambda(\mathcal{B}\times\mathcal{B})$ and a positive constant ρ_0 such that for any $\lambda\in]0,1[$ an arbitrary solution of the equation*

$$x=(1-\lambda)g(x,x)+\lambda h(x)$$

admits the estimate

$$\|x\|_{\mathcal{B}}\le\rho_0. \tag{2}$$

Then the equation (1) *is solvable.*

Corollary 1. *Let there exist a linear completely continuous operator $g:\mathcal{B}\to\mathcal{B}$ and a positive constant ρ_0 such that the equation*

$$y=g(y)$$

has only a trivial solution, and for any $\lambda\in]0,1[$ an arbitrary solution of the equation

$$x=(1-\lambda)g(x)+\lambda h(x)$$

admits the estimate (2). *Then the equation* (1) *is solvable.*

On the basis of Lemma 1 and Theorem 1 we prove the following theorem.

Theorem 2. *Let there exist an operator $g\in\Lambda_0(\mathcal{B}\times\mathcal{B})$, a partially ordered Banach space $\mathcal{B}_0$ with a cone $\mathcal{K}$ and positively homogeneous continuous operators μ and $\nu:\mathcal{B}\to\mathcal{K}$ such that*

$$\mu(y)-\nu(y-z)\notin\mathcal{K}\ \text{ for }\ y\neq\Theta,\ \ z\in\overline{\{g(x,y):\ x\in\mathcal{B}\}}$$

and

$$\nu\big(h(x)-g(x,x)-h_0(x)\big)\le\mu(x)+\mu_0(x)\ \text{ for }\ x\in\mathcal{B}, \tag{3}$$

where $h_0:\mathcal{B}\to\mathcal{B}$ and $\mu_0:\mathcal{B}\to\mathcal{K}$ satisfy the conditions

$$\lim_{\|x\|_{\mathcal{B}}\to\infty}\frac{\|h_0(x)\|_{\mathcal{B}}}{\|x\|_{\mathcal{B}}}=0,\qquad \lim_{\|x\|_{\mathcal{B}}\to\infty}\frac{\|\mu_0(x)\|_{\mathcal{B}_0}}{\|x\|_{\mathcal{B}}}=0. \tag{4}$$

Then the equation (1) *is solvable.*

Corollary 2. *Let there exist an operator $g\in\Lambda_0(\mathcal{B}\times\mathcal{B})$, a partially ordered Banach space $\mathcal{B}_0$ with a cone $\mathcal{K}$, a positively homogeneous operator $\nu:\mathcal{B}\to\mathcal{K}$ and a linear bounded positive operator $\eta:\mathcal{B}_0\to\mathcal{K}$ such that*

$$r(\eta)<1,$$

$\|\nu(x)\|_{\mathcal{B}_0}>0$ *for $x\neq\Theta$ and*

$$\nu\big(h(x)-g(x,x)-h_0(x)\big)\le\eta(\nu(x))+\mu_0(x)\ \text{ for }\ x\in\mathcal{B},$$

where $h_0:\mathcal{B}\to\mathcal{B}$ and $\mu_0:\mathcal{B}\to\mathcal{K}$ are operators satisfying (4). *Then the equation* (1) *is solvable.*

Corollary 3. *Let there exist an operator $g\in\Lambda_0(\mathcal{B}\times\mathcal{B})$ such that*

$$\lim_{\|x\|_{\mathcal{B}}\to 0}\frac{\|h(x)-g(x,x)\|_{\mathcal{B}}}{\|x\|_{\mathcal{B}}}=0. \tag{5}$$

Then the equation (1) *is solvable.*

Theorem 3. *Let the space $\mathcal{B}$ be separable. Let, moreover, there exist an operator $g\in\Lambda_0(\mathcal{B}\times\mathcal{B})$, a partially ordered Banach space $\mathcal{B}_0$ with a cone $\mathcal{K}$, and positively homogeneous continuous operators μ and $\nu:\mathcal{B}\to\mathcal{K}$ such that for every $\overline{g}\in\mathcal{L}_g$ the inequality*

$$\nu(y-\overline{g}(y))\le\mu(y)$$

has only a trivial solution and the condition (3) *is fulfilled, where $h_0:\mathcal{B}\to\mathcal{B}$ and $\mu_0:\mathcal{B}\to\mathcal{K}$ are operators satisfying* (4). *Then the equation* (1) *is solvable.*

Corollary 4. *Let the space $\mathcal{B}$ be separable, let there exist an operator $g \in \Lambda_0(\mathcal{B} \times \mathcal{B})$ such that the condition* (5) *hold, and let for every $\overline{g} \in \mathcal{L}_g$ the equation*

$$y = \overline{g}(y)$$

have only a trivial solution. Then the equation (1) *is solvable.*

Theorem 1 implies a priori boundedness principles proved in [1] and [4], while Theorems 2 and 3 imply the Conti–Opial type theorems proved in [2] and [3].

We give one more application of Theorem 1 concerning the existence of an ω-periodic solution of the functional differential equation

$$u^{(n)}(t) = f(u)(t) + f_0(t). \tag{6}$$

Here $n \geq 1$, $\omega > 0$, $f_0 \in L_\omega$, $f : C_\omega \to L_\omega$ is a continuous operator, C_ω is the space of continuous ω-periodic functions $u : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|u\|_{C_\omega} = \max\big\{|u(t)| :\ 0 \leq t \leq \omega\big\}$$

and L_ω is the space of integrable on $[0, \omega]$ ω-periodic functions $v : \mathbb{R} \to \mathbb{R}$ with the norm

$$\|v\|_{L_\omega} = \int_0^\omega |v(t)|\, dt.$$

By an ω-periodic solution of the equation (6) we understand an ω-periodic function $u : \mathbb{R} \to \mathbb{R}$ which is absolutely continuous together with $u^{(i)}$ $(i = 1, \dots, n-1)$ and almost everywhere on $\mathbb{R}$ satisfies the equation (6).

On the basis of Corollary 1 we prove the following theorem.

Theorem 4. *Let there exist $q \in L_\omega$, $\sigma \in \{-1, 1\}$ and a positive constant ρ such that*

$$0 \leq \sigma f(x)(t) \operatorname{sgn} x(t) \leq q(t) \ \text{ for } \ x \in C_\omega, \ \ t \in \mathbb{R},$$

and for any $x \in C_\omega$, satisfying the inequality

$$|x(t)| > \rho \ \text{ for } \ t \in \mathbb{R},$$

the condition

$$\int_0^\omega f(x)(t)\, dt \neq 0$$

is fulfilled. Let, moreover,

$$\int_0^\omega f_0(t)\, dt = 0. \tag{7}$$

Then the equation (6) *has at least one solution.*

As an example, consider the differential equation

$$u^{(n)}(t) = \sum_{k=1}^{m} f_k(t)\, \frac{|u(\tau_k(t))|^{\lambda_k} \operatorname{sgn} u(\tau_k(t))}{1 + |u(\tau_k(t))|^{\mu_k}} + f_0(t), \tag{8}$$

where

$$f_k \in L_\omega \ \ (k = 0, \dots, n), \quad \mu_k \geq \lambda_k > 0 \ \ (k = 1, \dots, n),$$

and $\tau_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \dots, n)$ are measurable functions such that the fraction

$$\frac{\tau_k(t + \omega) - \tau_k(t)}{\omega}$$

is an integral number for any $t \in \mathbb{R}$ and $k \in \{1, \dots, n\}$.

Corollary 5. *Let there exist a number $\sigma \in \{-1, 1\}$ such that*

$$\sigma f_k(t) \geq 0 \ \ for \ \ t \in \mathbb{R} \ \ (k = 1, \dots, n)$$

and

$$\sigma \sum_{k=1}^{n} \int_{0}^{\omega} f_k(t)\, dt > 0.$$

Let, moreover, the condition (7) *hold. Then the equation* (8) *has at least one ω-periodic solution.*

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