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A. Tsitskishvili

CONNECTION BETWEEN SOLUTIONS OF THE SCHWARZ NONLINEAR DIFFERENTIAL EQUATION AND THOSE OF THE PLANE PROBLEMS FILTRATION

Abstract. In the present paper, using linearly independent solutions of the Fuchs class linear differential equation which contains a term with the first order derivative of the unknown function, we propose effective methods for solving both the Schwarz nonlinear equation, whose right-hand side is a doubled invariant of the Fuchs class linear differential equation, and the plane problems of filtration with partially unknown boundaries. The modulus of the difference of the characteristic numbers of the Fuchs class linear differential equation for every singular point is equal to the corresponding (divided by π) angle at the vertex of a circular polygon. For the first time it is shown that the coefficients at the poles of second order of the doubled invariant of the Fuchs class linear differential equation and those on the right-hand side of the Schwarz equation coincide completely.

Relying on the property mentioned above, we suggest simpler methods of solving the problems of the theory of stationary motion of incompressible liquid in a porous medium with partially unknown boundaries than those described by us earlier for the solution of the same problems.

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რეზიუმე. საძიებელი ფუნქციის პირველი რიგის წარმოებულის შემცველ წევრიანი ფუქსის კლასის წრფივი დიფერენციალური განტოლების (ფკწდგ) წრფივად დამოუკიდებელი ამონახსნების გამოყენებით ნაშრომში მოცემულია ამონახსნების აგების ეფექტური მეთოდები შვარცის ისეთი არაწრფივი დიფერენციალური განტოლებისათვის, რომლის მარჯვენა მხარე ფკწდგ-ს გაორკეცებული ინვარიანტია, აგრეთვე ფილტრაციის თეორიის ნაწილობრივ უცნობ საზღვრიანი ამოცანებისათვის. ფკწდგ-ს მახასიათებელი რიცხვების სხვაობის აბსოლუტური მნიშვნელობა ტოლია წრიული მრავალკუთხედის შესაბამის წვეროსთან მდებარე (π-ზე გაყოფილი) კუთხის სიდიდისა.

პირველადაა ნაჩვენები, რომ შვარცის განტოლების მარჯვენა მხარეში მოთავსებული გამოსახულების მეორე რიგის პოლუსის კოეფიციენტები ზუსტად ემთხვევა ფკწდგ-ს გაორკეცებული ინვარიანტის მეორე რიგის პოლუსის შესაბამის კოეფიციენტებს.

ზემოთ აღნიშნულ თვისებაზე დაყრდნობით, ნაშრომში მოცემულია ფილტრაციის თეორიის ნაწილობრივ უცნობ საზღვრიანი ამოცანების ამონახსნების აგების უფრო მარტივი მეთოდები, ვიდრე ჩვენ ადრინდელ შრომებშია მოყვანილი.

1. On the Connection Between Solutions of the Fuchs Class Linear Differential Equation of General Type and the Nonlinear Schwarz Differential Equation

The filtration theory uses analytic function $w(z) = u - iv$, $z = x + iy$, where $w(z)$ the is complex velocity, and u and $(-v)$ are its components satisfying the Cauchy-Riemann conditions [1–6].

Let on the plane $w = u - iv$ a simply connected domain $s(w)$ be given with the boundary $l(w)$ consisting, in the general case, of circular arcs of different radii. Such a domain is called a circular polygon. By A_k, $k = \overline{1,m}$, we denote the angular points of the boundary $l(w)$ and by $\pi\nu_k$, $k = \overline{1,m}$, the interior angles, respectively. In the general case it can be assumed that $-2 \le \nu_i \le 2$, [1–31].

We seek for an analytic function $w(\zeta)$ which maps conformally the half-plane $\operatorname{Im}(\zeta) \ge 0$ of the plane $\zeta = t + i\tau$ onto the domain $s(w)$ with the boundary $l(w)$. Denote by $t = a_k$, $k = \overline{1,m}$, the points of the axes $t = a_k$, $k = \overline{1,m}$, of the plane $\zeta = t + i\tau$ which are mapped respectively into the points A_k, $k = \overline{1,m}$, where $-\infty < a_1 < a_2 < \cdots < a_m < +\infty$. The point $t = \infty$ is assumed to be mapped into a nonangular point of the boundary $l(w)$ of $s(w)$, which may lie between the points A_m and A_1, although one can consider as well the case in which $t = \infty$ is mapped into an angular point A_k.

Using the linear-fractional transformation, we can map A_mA_1, the arc of the circumference of the boundary $l(w)$ of $s(w)$, onto a straight line or onto a part of a straight line parallel to or coinciding with the real axis $v = 0$.

For the sake of brevity, without restriction of generality, from the very beginning we assume that the side A_mA_1 of $l(w)$ is parallel to or coincides with the axis $v = 0$. Therefore the function $w(\zeta)$ can always be extended analytically through the intervals $-\infty < t < a_1$, $a_m < t < +\infty$ to the lower half-plane $\operatorname{Im}(\zeta) < 0$. Throughout the paper it will be assumed that if $\zeta \in \operatorname{Re}\zeta$, then $\zeta = t$.

The unknown function $w(\zeta)$ must satisfy the well-known Schwarz equation [12–17],

$$\{w, \zeta\} \equiv w'''(\zeta)/w'(\zeta) - 1,5[w''(\zeta)/w'(\zeta)]^2 = R(\zeta), \tag{1.1}$$

$$R(\zeta) = \sum_{k=1}^{m} \{0,5(1-\nu_k^2)(\zeta - a_k)^{-2} + c_k(\zeta - a_k)^{-1}\}, \tag{1.2}$$

where a_k and c_k, $k = \overline{1,m}$, are unknown real parameters to be defined later on.

The expansion of the function $R(\zeta)$ in the neighborhood of the point $t = \infty$ in terms of the powers of $1/\zeta$ yields

$$R(\zeta) = \sum_{n=1}^{\infty} N_n \zeta^{-n}.$$

The coefficients N_k, $k = 1, 2, 3$, must satisfy the conditions

$$N_1 = \sum_{k=1}^{m} c_k = 0, \quad N_2 = \sum_{k=1}^{m} [a_k c_k + 0,5(1-\nu_k^2)] = 0, \\ N_3 = \sum_{k=1}^{m} [a_k^2 c_k + a_k(1-\nu_k^2)] = 0, \tag{1.3}$$

because the point $\zeta = \infty$ is the image of a nonangular point of the boundary $l(w)$ [12–16].

According to the Riemann theorem, three of the parameters $t = a_k$, $k = \overline{1,m}$, can be chosen arbitrarily and fixed. From the system of equations (1.3) the parameters c_1, c_2 and c_3 in the system of equations (1.3) can be expressed in terms of the remaining a_k and c_k. Consequently, the number of unknown parameters a_k and c_k is equal to $2(m-3)$.

By substitution $w'(\zeta) = 1/[u(\zeta)]^2$, the equation (1.1) can be reduced to the linear Fuchs class equation

$$u''(\zeta) + 0,5R(\zeta)u(\zeta) = 0. \tag{1.4}$$

By means of linearly independent particular solutions of (1.4) $u_1(\zeta)$ and $u_2(\zeta)$ with the Wronskian $u_1(\zeta)u_2'(\zeta) - u_2(\zeta)u_1'(\zeta) = 1$, we can construct the general solution of (1.1) as follows:

$$w(\zeta) = [Au_1(\zeta) + Bu_2(\zeta)]/[Cu_1(\zeta) + Du_2(\zeta)], \tag{1.5}$$

where A, B, C, D with $AD - BC = 1$ are the integration constants of the equation (1.1).

The general solution (1.5) of the equation (1.1), along with the $2(m-3)$ essential parameters a_k, c_k, $k = \overline{1,m}$, depends in the general case on three unknown complex parameters A, B, C, D with $AD - BC = 1$, i.e. on six real parameters. Thus the number of unknown parameters is equal to $2m$.

The equation of the boundary $l(w)$ of $s(w)$ can be written as

$$w(\zeta) = [\overline{w(\zeta)}B_0 + iD_0]/[-iA_0\overline{w(\zeta)} + \overline{B}_0], \quad \zeta \in l(w), \tag{1.6}$$

where $w = u - iv$, $\overline{w} = u + iv$, $B_0 = (C_0^* + iB_0^*)/2$, $\overline{B}_0 = (C_0^* - iB_0^*)/2$, A_0, B_0^*, C_0^*, and D_0 are given real piecewise constant functions which, without restriction of generality, satisfy the condition $B_0\overline{B}_0 - A_0D_0 = 1$.

The coordinates of the centers (u_0, v_0) and the radii of the circumferences (1.6) can be determined as follows:

$$u_0 = -B_0^*/[2A_0], \quad V_0 = -C_0^*/[2A_0], \\ R_0 = \sqrt{[(B_0^*)^2 + (C_0^*)^2 - 4A_0D_0]/A_0^2}. \tag{1.7}$$

Suppose that we have constructed linearly independent solutions u_1^* and $u_2^*(\zeta)$ with the Wronskian $u_1^*(\zeta)(u_2^*(\zeta))' - (u_1^*(\zeta))'u_2^*(\zeta) = 1$. Then $w(\zeta) = u_1^*(\zeta)/u_2^*(\zeta)$,

$$u_1^*(\zeta)/u_2^*(\zeta) = [B_0\overline{u_1^*}(\zeta) + iD_0\overline{u_2^*}(\zeta)]/[-iA_0\overline{u_1^*}(\zeta) + \overline{B}_0\overline{u_2^*}(\zeta)]. \tag{1.8}$$

The methods of constructing $w(\zeta)$ in the general case have been described in our works [25–31].

The differentiation of (1.8) yields

$$1/[u_2^*(\zeta)]^2 = 1/[-iA_0\overline{u_1^*}(\zeta) + \overline{B}_0\overline{u_2^*}(\zeta)]^2. \tag{1.9}$$

The equalities (1.6)–(1.9) imply that

$$u_1^*(\zeta) = \pm[B_0\overline{u_1^*}(\zeta) + iD_0\overline{u_2^*}(\zeta)], \quad u_2^*(\zeta) = \pm[-iA_0\overline{u_1^*}(\zeta) + \overline{B}_0\overline{u_2^*}(\zeta)]. \tag{1.10}$$

In [24], we have proved the equality (1.10) in somewhat different way. The signs + and − are fixed uniquely by means of the boundary conditions.

Let us consider the Fuchs class second order differential equation [14–16]

$$v''(\zeta) + p(\zeta)v'(\zeta) + q(\zeta)v(\zeta) = 0, \tag{1.11}$$

where

$$\begin{aligned} p(\zeta) &= \sum_{j=1}^{m}[1 - (\alpha_{1j} + \alpha_{2i})](\zeta - a_i)^{-1}, \\ q(\zeta) &= \sum_{j=1}^{m}[\alpha_{1j}\alpha_{2i}(\zeta - a_j)^{-2} + c_j^*(\zeta - a_j)^{-1}]. \end{aligned} \tag{1.12}$$

For the points $t = a_j$, $j = \overline{1, m}$, $t = \infty$ to be regular singular points, it is necessary and sufficient that $p(\zeta)$ and $q(\zeta)$ have the form (1.12) and the parameters c_j^*, $j = \overline{1, m}$, satisfy the condition [11–20]

$$M_1 = \sum_{k=1}^{m} c_k^* = 0. \tag{1.13}$$

Suppose that the parameters a_j, α_{kj}, c_j^*, $k = 1, 2$, $j = \overline{1, m}$, are real and $t = a_j$, $j = \overline{1, m}$, are the same as in (1.2). Using the linearly independent particular solutions (1.1) $v_1(\zeta)$ and $v_2(\zeta)$, we construct the general solution of the Schwarz equation

$$w(\zeta) = [A_1w_1(\zeta) + B_1]/[C_1w_1(\zeta) + D_1], \tag{1.14}$$

where $w_1(\zeta) = v_1(\zeta)/v_2(\zeta)$ is a particular solution of the Schwarz equation with the right-hand side equal to

$$\{w, \zeta\} = 2q(\zeta) - p'(\zeta) - 0,5[p(\zeta)]^2, \tag{1.15}$$

and A_1, B_1, C_1, D_1, $A_1D_1 - B_1C_1 \neq 0$ are the integration constants of (1.14).

The Wronskian for (1.11) has the form

$$v_{1j}(\zeta)v'_{2j}(\zeta) - v'_{1j}(\zeta)v_{2j}(\zeta) = c_{*j}\prod_{j=1}^{m}(\zeta - a_j)^{\alpha_{1j}+\alpha_{2j}-1}. \tag{1.16}$$

The paper [14, p. 300] states that for reducing the right-hand side of (1.15) to the function $R(\zeta)$ appearing in (1.2), we have to choose two functions $p(\zeta)$ and $q(\zeta)$ due to which the problem becomes indeterminate. In

[14] the author considers the linear second order equation of general type. But if one takes (1.11), where α_{1j}, α_{2j}, $j=\overline{1,(m+1)}$, satisfy the conditions

$$\begin{gathered}\alpha_{1j}-\alpha_{2j}=\nu_i,\quad j=\overline{1,m},\quad \alpha_{1(m+1)}-\alpha_{2(m+1)}=1,\quad t=a_{m+1}=\infty,\\ \alpha_{1(m+1)}=3,\quad \alpha_{2(m+1)}=2,\quad \sum_{k=1}^{m}[1-(\alpha_{1j}+\alpha_{2i})]=6,\end{gathered}\tag{1.17}$$

then the right-hand side of (1.15) is, as it can be directly verified, represented in the form

$$\begin{gathered}\{w,\zeta\}=2q(\zeta)-p'(\zeta)-0,5[p(\zeta)]^2=\\ =\sum_{j=1}^{m}\{0,5[1-(\alpha_{1i}-\alpha_{2i})^2](\zeta-a_j)^{-2}+c_j^{**}(\zeta-a_j)^{-1}\},\end{gathered}\tag{1.18}$$

where

$$c_j^{**}=2c_j^{*}-\beta_j\sum_{k=1,k\neq j}^{m}\beta_k(a_j-a_k)^{-1},\quad \beta_k=1-(\alpha_{1k}+\alpha_{2k}),\quad k=\overline{1,m}.\tag{1.19}$$

Since $\alpha_{1j}-\alpha_{2j}=\nu_i$, $j=\overline{1,m}$, the coefficients at $(\zeta-a_j)^{-2}$ in (1.2) and (1.18) coincide.

The expansion of the function $2q(\zeta)-p'(\zeta)-0,5[p(\zeta)]^2$ in the neighborhood of the point $\zeta=\infty$ into the series with respect to the powers $1/\zeta$ results in

$$2q(\zeta)-p'(\zeta)-0,5[p(\zeta)]^2=\sum_{k=1}^{m}M_k^{*}\zeta^{-k}.\tag{1.20}$$

The point $\zeta=\infty$ is not a branching point of (1.11), therefore the conditions

$$\begin{gathered}M_1^{*}\equiv\sum_{j=1}^{m}c_j^{**}=0,\quad M_2^{*}=\sum_{k=1}^{m}[a_kc_k^{**}+0,5(1-\nu_k^2)]=0,\\ M_3^{*}=\sum_{k=1}^{m}[a_k^2c_k^{**}+a_k(1-\nu_k^2)]=0\end{gathered}\tag{1.21}$$

must be fulfilled.

The condition $M_1^{*}=0$ coincides with (1.13). Below we will see that the last two equations of (1.21) can be obtained in somewhat different, natural way.

As is known, an equation of the type (1.11) can be reduced to an equation of the type (1.4). The expression $q(\zeta)-0,5p(\zeta)-0,25[p(\zeta)]^2$ is, in a certain sense, an invariant of (1.4) [23, p. 243]. Indeed, using the substitution $v(\zeta)=\exp[-0,5\int p(\zeta)d\zeta]v_0(\zeta)$ [23], we reduce the equation (1.11) to the type

$$v_0''(\zeta)+(q(\zeta)-0,25p^2(\zeta)-0,5p'(\zeta))v_0(\zeta)=0.\tag{1.22}$$

If the characteristics α_{1j}, α_{2j}, $j = \overline{1,m}$, of equation (1.11) satisfy the conditions $\alpha_{1j} + \alpha_{2j} = 1$, $j = \overline{1,m}$, then $p'(\zeta) = 0$, $p(\zeta) = 0$ and hence $R(\zeta) = 2q(\zeta)$, $2c_j^* = c_j$, $j = \overline{1,m}$.

The parameters α_{1j} and α_{2j} in the case of the equation (1.1) are defined by the equalities $\alpha_{1j} = 0,5(1+\nu_j)$, $\alpha_{2j} = 0,5(1-\nu_j)$, $\alpha_{1j} + \alpha_{2j} = 1$, $\alpha_{1j} - \alpha_{2j} = \nu_j$, $j = \overline{1,m}$.

In (1.6) there take place indeterminate constants c_{*j}, $j = \overline{1,m}$, which can be defined by the equality (1.16).

Indeed, if we divide both sides of the equality (1.16), by $(\zeta - a_j)^{\alpha_{1j}+\alpha_{2j}-1}$ and then pass to the limit $\zeta \to a_j$, we will get a system of equations for determination of c_{*j}, $j = \overline{1,m}$.

Note here that the equalities (1.10) can be generalized even in the case where $\alpha_{1j} + \alpha_{2j} \neq 1$.

2. Solution of Plane Problems of Filtration with Partially Unknown Boundaries

Consider some plane problems of the theory of stationary motion of incompressible liquid in a porous medium subjected to the Darcy law. The porous medium is assumed to be undeformable, isotropic and homogeneous [1–7].

The plane of the liquid motion coincides with the plane of the complex variable $z = x + iy$. In the domain $s(z)$ with the boundary $l(z)$ we seek for a complex potential $w(z) = \varphi(x,y) + i\psi(x,y)$, where $\varphi(x,y)$ and $\psi(x,y)$ are, respectively, the velocity potential and the stream function which satisfies the boundary conditions given below. The functions $\varphi(x,y)$ and $\psi(x,y)$ are connected by means of the Cauchy-Riemann conditions. If the analytic function $\omega(z)$ is found, then due to the dependencies [1–7]

$$\begin{gathered} \varphi(x,y) = -k(p/\gamma + y) + c, \quad w(z) = u - iv, \\ u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}, \end{gathered} \tag{2.1}$$

where p is the hydrodynamic pressure, γ is the specific weight of the liquid, u and v are the vector components of filtration velocity, $\omega'(z) \equiv w(z)$ is the complex velocity, k is the coefficient of filtration, and c is an arbitrary constant, all the characteristics of the filtration stream can be found, namely: filtration velocity, pressure head, pressure, liquid discharge for filtration and unknown parts of the boundary $l(z)$ of $s(z)$ [1–7; 24–31]. Below we shall consider the reduced complex potential $\omega(z)$, the complex potential divided by the coefficient of filtration. Next we assume that the boundary $l(z)$ of $s(z)$ is a simple, piecewise analytic contour consisting of a finite number of unknown depression curves, segments of straight lines, half-lines and straight lines. The domains $s(z)$, $\omega(z)$ and $w(z) = \omega'(z)$ may be bounded or unbounded. In particular, if the boundary $l(z)$ has no depression curves, then the domain $s(z)$ turns into a linear polygon.

In the domain $s(z)$ we have to find an analytic function $\omega(z) = \varphi(x, y) + i\psi(x, y)$ which must satisfy the boundary conditions [1–7]

$$a_{k1}\varphi(x,y)+a_{k2}\psi(x,y)+a_{k3}x+a_{k4}y=f_k, \quad k=1,2, \quad (x,y)\in l(z), \tag{2.2}$$

where a_{kj}, f_k, $j=\overline{1,4}$, are given piecewise constant real functions.

Before we proceed to solution of the basic problem of filtration, we can determine the boundary $l(w)$ of the domains $s(w)$ and also a part of the boundary $l(\omega)$ of $s(\omega)$ [1–7].

Using the functions $\omega(z)$ and $\omega'(z) = d\omega(z)/dz$, the domain $s(z)$ with the boundary $l(z)$ is mapped conformally respectively onto the domain $s(\omega)$ and $s(w)$ with the boundaries $l(\omega)$ and $l(w)$, where the domain $s(w)$ is a circular polygon with the boundary $l(w)$ consisting of a finite number of circular arcs, segments of straight lines, half-lines and straight lines.

If we take arbitrarily any part of the boundary $l(z)$ of $s(z)$ and differentiate (2.2) along that part of the boundary $l(z)$ with respect to the parameter s, where s is the arc length of the curve, we get

$$(a_{11}u - a_{12}v + a_{13})\cos(x,s) + (a_{11}v + a_{12}u + a_{14})\cos(y,s) = 0, \tag{2.3}$$

$$(a_{21}u - a_{22}v + a_{23})\cos(x,s) + (a_{21}v + a_{22}u + a_{24})\cos(y,s) = 0, \tag{2.4}$$

where $dx/ds = \cos(x,s)$ and $dy/ds = \cos(y,s)$.

For the system (2.3) and (2.4) have a nontrivial solution with respect to dx/ds and dy/ds, it is necessary and sufficient that the determinant of the system at the given part of the boundary be equal to zero,

$$A_{11}(u^2+v^2) + A_{12}u + A_{13}v + A_{14} = 0. \tag{2.5}$$

The coefficients a_{kj}, $k=1,2$, $j=\overline{1,4}$, are given by (2.2), and therefore coefficients A_{11}, A_{12}, A_{13}, and A_{14} are fixed.

The equation (2.5) can be written in the complex form

$$w = [B\overline{w} + i2A_{14}][-2iA_{11}\overline{w} + \overline{B}]^{-1}, \tag{2.6}$$

where $w = u - iv$, $\overline{w} = u + iv$, $B = A_{13} + iA_{12}$, $\overline{B} = A_{13} - iA_{12}$,

$$A_{11} = a_{11}a_{22} - a_{21}a_{12}, \quad A_{12} = a_{11}a_{24} - a_{21}a_{14} + a_{13}a_{22} - a_{23}a_{12},$$
$$A_{13} = a_{14}a_{22} - a_{24}a_{12} + a_{13}a_{21} - a_{23}a_{11}, \quad A_{14} = a_{13}a_{24} - a_{23}a_{14}. \tag{2.7}$$

The coordinates (u_*, v_*) of the center and the radius R_* of the circumference (2.5) for the chosen by us part of the boundary $l(w)$ are defined as follows:

$$\begin{gathered} u_* = -A_{12}/[2A_{11}], \quad v_* = -A_{13}/[2A_{11}], \\ R_* = \frac{1}{2}\sqrt{[A_{12}/A_{11}]^2 + [A_{13}/A_{11}]^2 - 4A_{14}/A_{11}}. \end{gathered} \tag{2.8}$$

We can require the condition $B\overline{B} - 4A_{11}A_{14} \neq 0$, but not the condition $B\overline{B} - 4A_{11}A_{14} = 1$ because the parameters a_{kj}, $k=1,2$, $j=\overline{1,4}$, are fixed by the condition (2.2).

For solving the problems of filtration, one usually introduces the plane $\zeta = t + i\tau$ and maps conformally the half-plane $\mathrm{Im}(\zeta) > 0$ onto the domains

$s(z)$, $s(\omega)$ and $s(w)$. We denote the conformally mapping functions respectively by $z(\zeta)$, $\omega(\zeta)$ and $w(\zeta) = \omega'(\zeta)/z'(\zeta)$, where $d\omega(\zeta)/d\zeta = \omega'(\zeta)$ and $dz(\zeta)/d\zeta = z'(\zeta)$. B_k, $k = \overline{1,n}$, denote angular points of the boundary $l(z)$, $l(\omega)$ and $l(w)$ of the domains $s(z)$, $s(\omega)$ and $s(w)$ which will be met at least on one of the above-mentioned boundaries $l(z)$, $l(\omega)$ and $l(w)$, as a result of a circuit in the positive direction. By $t = e_k$, $k = \overline{1,n}$, we denote the points of the t-axis of the plane ζ which are mapped, respectively, into the points B_k, $k = \overline{1,n}$, where $-\infty < e_1 < e_2 < \cdots < e_n < +\infty$. The point $t = e_{n+1} = \infty$ is mapped into the nonangular point which lies on some part of the boundary B_nB_1.

The boundary values of the functions $z(\zeta)$, $\omega(\zeta)$ and $w(\zeta)$, as $\zeta \to t$, $\zeta \in \mathrm{Im}(\zeta) > 0$ will be denoted by $z(t) = x(t) + iy(t)$, $\omega(t) = \varphi(t) + i\psi(t)$, $w(t) = u(t) - iv(t)$, while the complex conjugates to the functions $z(t)$, $\omega(t)$ and $w(t)$ will be denoted by $\overline{z(t)}$, $\overline{\omega(t)}$, and $\overline{w(t)}$.

Introduce the vectors $\Phi(\zeta) = [\omega(\zeta), z(\zeta)]$, $\overline{\Phi(\zeta)} = [\overline{\omega(t)}, \overline{z(t)}]$, $\Phi'(\zeta) = [\omega'(\zeta), z'(\zeta)]$, $\overline{\Phi'(\zeta)} = [\overline{\omega'(\zeta)}, \overline{z'(\zeta)}]$, $f(t) = [f_1(t), f_2(t)]$. Then the boundary conditions (2.2) can be written as follows

$$(a_{k2} + ia_{k1})\omega(t) + (a_{k4} + ia_{k3})z(t) = (a_{k2} - ia_{k1})\overline{\omega(t)} + \\ +(a_{k4} - ia_{k3})\overline{z(t)} + 2if_k(t), \quad -\infty < t < +\infty, \quad k = 1, 2. \tag{2.9}$$

The condition (2.9) by means of the vector $\Phi(z)$ can be rewritten as

$$\Phi(t) = g(t)\overline{\Phi(t)} + 2iG^{-1}f(t), \quad -\infty < t < +\infty, \tag{2.10}$$

where $g(t) = G^{-1}(t)\overline{G(t)}$ is a piecewise constant nonsingular second order matrix with the discontinuity points $t = e_k$, $k = \overline{1,n}$. $G^{-1}(t)$ is the inverse to $G(t)$ matrix and $\overline{G(t)}$ is the complex-conjugate to $G(t)$ matrix.

Below, instead of $a_{kj}(t)$, $k = 1, 2$, $j = \overline{1,4}$ we will write a_{kj}, $k = 1, 2$, $j = \overline{1,4}$.

Matrices $G(t)$ and $G^{-1}(t)$ are defined by the formulas

$$G(t) = \begin{pmatrix} a_{12} + ia_{11}, & a_{14} + ia_{13} \\ a_{22} + ia_{21}, & a_{24} + ia_{23} \end{pmatrix} \tag{2.11}$$

and

$$G^{-1}(t) = \frac{1}{\det G(t)} \begin{pmatrix} a_{24} + ia_{23}, & -(a_{14} + ia_{13}) \\ -(a_{22} + ia_{21}), & a_{12} + ia_{11} \end{pmatrix}. \tag{2.12}$$

The matrix $g(t)$ in the interval (a_j, a_{j+1}) is defined as

$$g_j(t) = G_j^{-1}\overline{G}_j = \frac{1}{\det G_j(t)} \begin{pmatrix} A_{11}^{*j}, & iA_{12}^{*j} \\ iA_{21}^{*j}, & \overline{A}_{11}^{*j} \end{pmatrix}, \quad a_j < t < a_{j+1}, \tag{2.13}$$

but for $j = n-1$ we have

$$\begin{aligned} A_{11}^{*(n-1)} &= (-1)(A_{13}^{n-1} + iA_{12}^{n-1}), \\ A_{12}^{*(n-1)} &= (-2)A_{14}^{(n-1)},\ A_{21}^{*(n-1)} = 2A_{11}^{(n-1)}, \\ A_{11}^{*(n-1)} &= a_{24}a_{12} + a_{23}a_{11} - a_{14}a_{22} - a_{13}a_{21} + \\ &\quad + i(a_{23}a_{12} - a_{24}a_{11} + a_{21}a_{14} - a_{13}a_{22}). \end{aligned} \tag{2.14}$$

The function $\overline{A}_{11}^{*(n-1)}$ is the complex-conjugate to $A_{11}^{*(n-1)}$.

Differentiation of (2.10) yields

$$\Phi'(t) = g(t)\overline{\Phi}'(t), \quad -\infty < t < +\infty. \tag{2.15}$$

It can be easily verified that the equality $\overline{g(t)} = [g(t)]^{-1} = \overline{G}^{-1}G$ holds, where $[g(t)]^{-1}$ is the matrix, inverse to $g(t)$, and $\overline{g}(t)$ is the matrix, complex-conjugate to $g(t)$.

For the point $t = e_j$ we compose the characteristic equation

$$\det(g_{j+1}^{-1}(e_i + 0)g_j(e_j - 0) - \lambda E) = 0, \tag{2.16}$$

where $g_{j+1}^{-1}(e_j + 0)g_j(e_j - 0)$ is a matrix, E is the unit matrix, λ is the parameter, and $g_j(e_j+0)$, $g_{j-1}(e_j-0)$ are the limiting values of matrices $g_j(t)$, $g_{j-1}(y)$ at the point $t = e_j$ from the right and from the left, respectively; $g_j^{-1}(e_j + 0)$ is the inverse to $g_j(e_j + 0)$ matrix.

If we denote by λ_{kn} the characteristic numbers of the matrix $g_{(n-1)}(t)$, then the equalities

$$\begin{gathered} \lambda_{1n} + \lambda_{2n} = [A_{11}^{*(n-1)} + \overline{A}_{11}^{*(n-1)}]/[2\det G_{n-1}], \\ \lambda_{1n} \cdot \lambda_{2n} = \det \overline{G}_{n-1}/\det G_{n-1}, \\ |\lambda_{1n}||\lambda_{2n}| = 1, \quad |\det g(t)| = 1, \quad \overline{\lambda}_{1n} \cdot \overline{\lambda}_{2n} = 1/[\lambda_{1n} \cdot \lambda_{2n}], \\ 1/\lambda_{1n} + 1/\lambda_{2n} = \overline{\lambda}_{1n} + \overline{\lambda}_{2n}, \quad \lambda_{1n} + \lambda_{2n} = \lambda_{1n}\lambda_{2n}(\overline{\lambda}_{1n} + \overline{\lambda}_{2n}) \end{gathered}$$

hold [1-31].

Let us introduce the characteristic numbers $\alpha_{kn} = \frac{1}{2\pi i}\ln\lambda_{kn}$, $k = 1, 2$. Then $\alpha_{1n} + \alpha_{2n} = \alpha_{0j}$, where $\alpha_{0j} = \frac{1}{2\pi i}\arg\det(\overline{G}_j/G_j)$,

$$\alpha_{1n} - \alpha_{2n} = \frac{1}{2\pi i}\ln(\lambda_{1n}/\lambda_{2n}) = \nu_n, \tag{2.17}$$

where $\pi\nu_n$ is the interior angle of the contour $l(w)$ of $s(w)$ at the point A_n.

The roots λ_{kn}, $k = 1, 2$, for the point $t = e_n$ are calculated by the formula [1–7]

$$\begin{gathered} \lambda_{kn} = [A_{11}^{*(n-1)} + \overline{A}_{11}^{*n} \pm \\ \pm i\sqrt{4\det G_n \det \overline{G}_n - (A_{11}^{*n} + \overline{A}_{11}^{*n})^2}]/[2\det G_n],\ k = 1, 2. \end{gathered} \tag{2.18}$$

For the points $t = e_j$, $j = 1, 2, \ldots, n-1$, we have

$$g_{j+1}^{-1}(a_j+0)g_j(e_j-0) = \overline{G}_{j+1}^{-1} G_{j+1} G_j^{-1} \overline{G}_j,$$

$$g_{j+1}^{-1}(a_j+0)g_j(e_j-0) =$$

$$= \frac{1}{\det \overline{G}_{j+1}} \cdot \frac{1}{\det G_j} \begin{pmatrix} \overline{A}_{11}^{*(j+1)}, & -iA_{12}^{*(j+1)} \\ -iA_{21}^{*(j+1)}, & A_{11}^{*(j+1)} \end{pmatrix} \begin{pmatrix} \overline{A}_{11}^{*j)}, & iA_{12}^{*j)} \\ iA_{21}^{*j}, & \overline{A}_{11}^{*j} \end{pmatrix}, \quad (2.19)$$

$$\lambda_{1j} + \lambda_{2j} = [\overline{A}_{11}^{*(j+1)} A_{11}^{*j} + A_{12}^{*(j+1)} A_{21}^{*j} +$$

$$+ A_{21}^{*(j+1)} A_{12}^{*j} + A_{11}^{*(j+1)} \overline{A}_{11}^{*j}]/[\det \overline{G}_{j+1} \det G_j], \quad (2.20)$$

$$\lambda_{1j}\lambda_{2j} = \det G_{j+1} \det \overline{G}_j / [\det \overline{G}_{j+1} \det G_j]. \quad (2.21)$$

Using (2.19), (2.20) and (2.21), we can calculate

$$\lambda_{1j}/\lambda_{2j}, \quad \alpha_{1j},\ \alpha_{2j}, \quad \alpha_{1j} + \alpha_{2j} = \alpha_{0j}^*, \quad \alpha_{1j} - \alpha_{2j} = \nu_j, \quad (2.22)$$

$$\alpha_{0j}^* = \frac{1}{\pi}[\alpha_{0(j+1)} - \alpha_{0j}], \quad \det G_j = R_0 \exp(i\alpha_{0j}).$$

The characteristic numbers α_{kj}, $k = 1, 2$, $j = \overline{1, (n+1)}$, must satisfy the Fuchs condition [1–31]

$$\sum_{j=1}^{n+1} [1 - (\alpha_{1j} + \alpha_{2j})] = 2,$$

$$\alpha_{1(n+1)} = 3, \quad \alpha_{2(n+1)} = 2, \quad t_\infty = a_{n+1} = \infty. \quad (2.23)$$

The equality $\alpha_{1j} + \alpha_{2j} = 1$, $j = \overline{1, n}$, under the condition (2.5) may fail to be fulfilled, and hence we are unable to apply the equation (1.4) for solving the equation (2.15). As it will be seen below, to solve (2.15) completely it suffices to use the linearly independent solutions (1.11).

Of all singular angular points of the boundaries $l(z)$ and $l(\omega)$, we select such angular points to which on the boundary $l(w)$ of $s(w)$ there correspond regular nonangular points. Such angular points on the boundaries $l(z)$ and $l(\omega)$ are usually called removable singular points [1–7]. For the sake of simplicity we assume that the number of removable singular points is equal to two. Denote these points by $t = e_k$ and $t = e_{k+j}$. The angles corresponding to such points on the contours $l(z)$ and $l(\omega)$ are equal to $\pi/2$. To remove those singular points from the boundary conditions (2.15), we introduce the new unknown vector $\Phi_1(\zeta)$ by the formula

$$\Phi'(\zeta) = \Phi_1(\zeta) \sqrt{\frac{(\zeta - e_{k-1})(\zeta - e_{k+j-1})}{(\zeta - e_k)(\zeta - e_{k+j})}},$$

$$\sqrt{\frac{(\zeta - e_{k-1})(\zeta - e_{k+i-1})}{(\zeta - e_k)(\zeta - e_{k+j})}} > 0, \quad \zeta > e_{k+j}. \quad (2.24)$$

When passing from the vector $\Phi(\zeta)$ to $\Phi_1(\zeta)$, the matrix $g(t)$ in the interval $(e_{k-1}, e_k), (e_{k+j-1}, e_{k+j})$ is multiplied by (-1).

We enumerate the remaining singular points along the t-axis as $t = a_k$, $k = \overline{1,m}$. To these points there correspond the points A_k, $k = \overline{1,m}$, on the contour $l(w)$. In what follows, the notation for the matrices $g(t) = G^{-1}\overline{G}$ will remain unchanged, but all the changes which occurred while introducing $\Phi_1(\zeta)$ will be taken into account.

If one or several elements in the matrix $g(t)$ are equal to zero, and moreover, $\det g(t) \neq 0$, then the problem (2.10) is solved completely by means of the Cauchy type integral [1–31]. Besides the above-mentioned one we come across the cases where all the elements in the matrix $g(t)$ are different from zero and then the problem (2.10) is solved by elementary means [16, 26].

The boundary condition with respect to $\Phi_1(\zeta)$ can be written as

$$\Phi_1(t) = g(t)\overline{\Phi}_1(t), \quad -\infty < t < +\infty. \tag{2.25}$$

To solve the problem (2.25), we first find all the roots λ_{kj}, $k = 1, 2$, $j = \overline{1, m+1}$, from (2.16) and then, taking into account (2.23), we find α_{ki}, $k = 1, 2$, $j = \overline{1, m+1}$ [1,7]. Having found the above-mentioned quantities, we substitute α_{kj}, $k = 1, 2$, $j = \overline{1,m}$ into (1.11).

All the equations and formulas (1.11)–(1.16) remain valid and will be used later on for solving of (2.10), (2.15) and (2.25).

3. The Fuchs Class Equation in the Form of a System

The equation (1.11) in the neighborhood of every singular point $t = a_k$, $k = \overline{1, m+1}$, and in the neighborhood of any regular point, where $p(\zeta)$ and $q(\zeta)$ are analytic, has two linearly independent local solutions which are constructed by means of infinite series whose coefficients are defined in the well-known manner. These series converge respectively in the circles with centers at the points for which these series have been constructed, and the convergence radii of the series are bounded by the distance from the centers of the given circles to the nearest to the centers singular points.

We denote the local linearly independent solutions of the equation (1.11) for singular points $\zeta = a_k$, $k = 1, 2, \dots, m+1$, by $v_{kj}(\zeta)$, $j = \overline{1, (m+1)}$, and for $t = a_j^* = (a_j + a_{j+1})/2$, $j = 1, 2, \dots, m-1$, by $\sigma_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, \dots, m-1$.

Suppose

$$u_1(\zeta) = pu_{1j}(\zeta) + qu_{2j}(\zeta), \quad u_2(\zeta) = ru_{1j}(\zeta) + su_{2j}(\zeta), \tag{3.1}$$

where p, q, r, s are the integration constants of (1.15).

The equation (1.11) can be written in the form of the system

$$\chi_1'(\zeta) = \chi_1(\zeta)\mathcal{P}(\zeta), \tag{3.2}$$

$$\chi_1(\zeta) = \begin{pmatrix} u_1(\zeta), & u_1'(\zeta) \\ u_2(\zeta), & u_2'(\zeta) \end{pmatrix}, \mathcal{P}(\zeta) = \begin{pmatrix} 0, & -q(\zeta) \\ 1, & -p(\zeta) \end{pmatrix}, \tag{3.3}$$

where $u_1(\zeta)$, $u_2(\zeta)$ are linearly independent solutions of (1.11).

A solution of the boundary value problem (2.25) will be sought by means of the matrix $\chi_1(\zeta)$. It is known that if the matrix $\chi_1(\zeta)$ is a solution of (3.2), then the matrix $T\chi_1(\zeta)$ is also the solution of (3.2), where

$$T = \begin{pmatrix} p, & q \\ r, & s \end{pmatrix}, \quad \det T \neq 0. \tag{3.4}$$

If we construct the local linearly independent solutions $u_{kj}(\zeta)$ and $\sigma_{kj}(\zeta)$ of (1.11), for the points $\zeta = a_j$, $j = \overline{1, m+1}$, $\zeta = a_j^* = (a_j + a_{j+1})/2$, $j = \overline{1, m-1}$, respectively, then the local fundamental matrices for (3.2) will have the form

$$\Theta_j(\zeta) = \begin{pmatrix} u_{1j}(\zeta), & u'_{1j}(\zeta) \\ u_{2j}(\zeta), & u'_{2j}(\zeta) \end{pmatrix}, \quad j = \overline{1, m+1}, \tag{3.5}$$

$$\sigma_j(\zeta) = \begin{pmatrix} \sigma_{1j}(\zeta), & \sigma'_{1j}(\zeta) \\ \sigma_{2j}(\zeta), & \sigma'_{2j}(\zeta) \end{pmatrix}, \quad j = \overline{1, m-1}. \tag{3.6}$$

Suppose that the inequality $|a_m| > |a_1|$ holds. Then at the point $a_m^* = -|a_m|$ we construct the local series $\sigma_{*k}(\zeta)$, $k = 1, 2$, and the corresponding local matrix $\sigma_{*j}(\zeta)$. The convergence radii of these series are bounded by the distance from the point $t = a_m$ to the singular point $t = a_1$, and if $|a_1| > |a_m|$, then we construct at the point $a_1^* = |a_1|$ the local series $\sigma_{*k}(\zeta)$, $k = 1, 2$, and the matrix $\sigma^*(\zeta)$. The convergence radius of these series will be bounded by the distance from the point a_1^* to the point $t = a_m$.

It becomes evident that there exists a finite number of circles with centers $\zeta = a_j$, $j = \overline{1, m+1}$, $\zeta = a_j^* = (a_j + a_{j+1})/2$, $j = \overline{1, m-1}$, $\zeta = a_m^*$ (or $\zeta = a_1^*$) which cover completely the x-axis, $-\infty < t < +\infty$. Note that the circle with the center $\zeta = \infty$ is assumed to be the exterior of the circle $|\zeta| < r_0$, where r_0 is equal to the largest (in absolute value) of the numbers a_1 and a_m .

The equation (1.11) in the neighborhood of $\zeta = a_j$ can be written as

$$(\zeta - a_j)^2 v''(\zeta) + (\zeta - a_j) p_j(\zeta) v'(\zeta) + q_j(\zeta) v(\zeta) = 0, \tag{3.7}$$

where

$$\begin{aligned} p_j(\zeta) &= p_{0j} + \sum_{n=1}^{\infty} p_{nj}(\zeta - a_j)^n, \\ p_{nj} &= (-1)^{n-1} \sum_{k=1, k\neq j}^{m} [1 - \alpha_{1k} - \alpha_{2k}](a_j - a_k)^n, \end{aligned} \tag{3.8}$$

$$\begin{aligned} p_{0j} &= 1 - \alpha_{1j} - \alpha_{2j}, \\ q_j(\zeta) &= \alpha_{1j}\alpha_{2j} + c_j^*(\zeta - a_j) + \sum_{n=2}^{\infty} q_{nj}(\zeta - a_j)^n, \end{aligned} \tag{3.9}$$

$$q_{nj} = (-1)^{n-2} \sum_{k=1, k\neq j}^{m} [\alpha_{1k}\alpha_{2k}(n-1) + c_k^*(a_j - a_k)](a_j - a_k)^{-n}, \tag{3.10}$$

$n = 2, 3, \ldots$

$$q_{0j} = \alpha_{1j}\alpha_{2j}, \quad q_{1j} = c_j^*, \quad j = \overline{1,m}. \tag{3.11}$$

The local solutions of (3.7) for the point $t = a_j$, $j = \overline{1,m}$, will be sought in the form

$$u_j(\zeta) = (\zeta - a_j)^{\alpha_j}\widetilde{u}_j(\zeta), \quad \widetilde{u}_j(\zeta) = 1 + \sum_{n=1}^{\infty}\gamma_{nj}(\zeta - a_j)^n. \tag{3.12}$$

For definition of the coefficients γ_{nj}, $n = \overline{1,\infty}$, $j = \overline{1,m}$, we have the following recursion formulas:

$$f_{0j}(\alpha_j) = \alpha_j(\alpha_j - 1) + p_{0j}\alpha_j + q_{0j} = 0, \tag{3.13}$$

$$\gamma_{1j}f_0(\alpha_j + 1) + f_1(\gamma_j) = 0, \tag{3.14}$$

$$\gamma_{2j}f_0(\alpha_{j+2}) + \gamma_{1j}f_1(\alpha_j + 1) + f_2(\alpha_j) = 0, \tag{3.15}$$

. . . .

$$\begin{gathered}\gamma_{nj}f_0(\alpha_j + n) + \gamma_{(n-1)j}f_1(\alpha_j + n - 1)+\\ + \gamma_{(n-2)j}f_2(\alpha_j + n - 2) + \cdots + \gamma_{1j}f_{(n-1)}(\alpha_j + 1) + f_n(\alpha_j) = 0,\end{gathered} \tag{3.16}$$

where

$$f_n(\alpha_j) = \alpha_j p_{nj} + q_{nj}. \tag{3.17}$$

The defining equation (3.13) for every point $t = a_j$, $j = \overline{1,m}$, has two roots, α_{1j} and α_{2j}. If the difference $\alpha_{1j} - \alpha_{2j}$ is not an integer, then using the formulas (3.14)–(3.16), we can construct for every point $t = a_j$ two linearly independent solutions

$$u_{kj}(\zeta) = (\zeta - a_j)^{\alpha_{kj}}\widetilde{u}_{kj}(\zeta), \quad \widetilde{u}_{kj} = 1 + \sum_{n=1}^{\infty}\gamma_{nj}^k(\zeta - a_j)^n, \quad k = 1, 2. \tag{3.18}$$

But if the difference $\alpha_{1j} - \alpha_{2j}$ is an integer, then $u_{1j}(\zeta)$, $j = \overline{1,m}$, can be constructed by the formulas (3.14)–(3.16), while if $u_{2j}(\zeta)$ involves a logarithmic term, $u_{2j}(\zeta)$ can be constructed with the help of the Frobenius method [15, 27-31].

Let us pass now to the construction of $u_{2j}(\zeta)$ when the difference $\alpha_{1j} - \alpha_{2j} = 2$ and $u_{2j}(\zeta)$ does not involve a logarithmic term. For such a point $t = a_j$, on the contour $l(w)$ there is a cut (circular or linear) with the angle 2π. P. Ya. Polubarinova-Kochina has proved [2] that $u_{2j}(\zeta)$ does not contain a logarithmic term. She also obtained the equation connecting the parameters a_j, c_j^* [1-7, 25-31]. To construct $u_{2j}(\zeta)$, we will act as follows [25-31].

For the point $t = a_k$, the equality (3.15) fails to be fulfilled because

$$f_{0j}(\alpha_j + 2) = 0 \tag{3.19}$$

as $\alpha_j \to \alpha_{2j}$.

In order for the equality (3.15) to take place as $\alpha_j \to \alpha_{2j}$, it is necessary and sufficient that the condition

$$\gamma_{1j}f_j(\alpha_j + 1) + f_2(\alpha_j) = 0, \quad \alpha_j \to \alpha_{2j} \tag{3.20}$$

be fulfilled.

After certain transformations, the equation (3.20) takes the form

$$q_{2j} + q_{1j}^2 + q_{1j}p_{1j} = 0. \tag{3.21}$$

Note that for the cut end $t = a_j$ with the angle 2π the equality $dw(\zeta)/d\zeta = 0$ holds for $t = a_j$, where $w(\zeta)$ is the general solution of (1.1) or (1.18).

To construct $u_{2j}(\zeta)$ for the cut end, it suffices to calculate $\gamma_{2j}^2(\alpha_{2j})$ uniquely; the remaining coefficients $\gamma_{nj}^2(\alpha_{2j})$, $n = 1, 3, 4, 5, \dots$, can be calculated by the formula (3.16). Under the conditions (3.19) and (3.20) the equation (3.15) is fulfilled.

To define $\gamma_{2j}^2(\alpha_{2j})$ and, consequently, $u_{2j}(\zeta)$ uniquely, we suppose that $\alpha_j \neq \alpha_{2j}$. Then (1.5) implies that

$$\gamma_{2j}(\alpha_j) = -[\gamma_{1j}(\alpha_j)f_{1j}(\alpha_j + 1) + f_{2j}(\alpha_j)]/f_{0j}(\alpha_j + 2). \tag{3.22}$$

The numerator and denominator on the right-hand side of (3.22) vanish as $\alpha_j \to \alpha_{2j}$, and hence there is an indeterminacy. Developing this indeterminacy by the L'Hospital rule, we obtain

$$\gamma_{2j}^2 = -0,5[p_{1j}(p_{1j} + 2q_{1j}) + p_{2j}]. \tag{3.23}$$

Thus γ_{2j}^2, and hence $u_{2j}(\zeta)$, are defined uniquely.

Let us proceed now to the determination of the local solutions in the neighborhood of the point $\zeta = a_{m+1} = \infty$.

Represent $p(\zeta)$ and $q(\zeta)$ in the neighborhood of $\zeta = \infty$ as follows

$$p(\zeta) = \zeta^{-1}\sum_{n=0}^{\infty} p_{n\infty}\zeta^{-n}, \quad q(\zeta) = \zeta^{-2}\sum_{n=0}^{\infty} q_{n\infty}\zeta^{-n}, \tag{3.24}$$

where

$$p_{n\infty} = \sum_{k=1}^{m}[1 - (\alpha_{1k} + \alpha_{2k})]a_k^n, \quad p_{0\infty} = 6, \tag{3.25}$$

$$q_{n\infty} = \sum_{k=1}^{m}[\alpha_{1k}\alpha_{2k}(n+1) + c_k^* a_k]a_k^n, \tag{3.26}$$

$$q_{0\infty} = \sum_{k=1}^{m}[\alpha_{1k}\alpha_{2k} + c_k^* a_k], \tag{3.27}$$

$$q_{1\infty} = \sum_{k=1}^{m}[2\alpha_{1k}\alpha_{2k}a_k + c_k^* a_k^2]. \tag{3.28}$$

The local solutions in the neighborhood of the point $t = \infty$ will be sought in the form

$$u_\infty(\zeta) = \zeta^{-\alpha_\infty} + \sum_{n=1}^{\infty}\gamma_{n\infty}\zeta^{-\alpha_\infty - n}. \tag{3.29}$$

For definition of $\gamma_{n\infty}$, $n = \overline{1, \infty}$, we have the formulas

$$f_{0\infty}(\alpha_\infty) = \alpha_\infty(\alpha_\infty + 1) - p_{0\infty}\alpha_\infty + q_{0\infty} = 0, \tag{3.30}$$

$$\gamma_{1\infty} f_{0\infty}(\alpha_\infty + 1) - p_{1\infty} + q_{1\infty} = 0, \tag{3.31}$$

$$\gamma_{2\infty} f_{0\infty}(\alpha_\infty + 2) + \gamma_{1\infty} f_{1\infty}(\alpha_\infty + 1) - p_{2\infty}\alpha_\infty + q_{2\infty} = 0, \tag{3.32}$$

. . . .

$$\begin{aligned}&\gamma_{n\infty} f_{0\infty}(\alpha_\infty + n) + \gamma_{(n-1)\infty} f_{1\infty}(\alpha_\infty + n - 1) + \\ &\quad + \gamma_{(n-2)\infty} f_{2\infty}(\alpha_\infty + n - 2) + \cdots + \gamma_{1\infty} f_{(n-1)}(\alpha_\infty + 1) - \\ &\quad - p_{n\infty}\alpha_\infty + q_{n\infty} = 0,\end{aligned} \tag{3.33}$$

where

$$f_{k\infty} = q_{k\infty} - (\alpha_\infty + k) p_{k\infty}. \tag{3.34}$$

Owing to the fact that $t = \infty$ is the image of the nonangular point, the equation (3.30) must have the roots $\alpha_{1\infty} = 3$ and $\alpha_{2\infty} = 2$, and hence the free term $q_{0\infty}$ must satisfy the condition

$$q_{0\infty} = \sum_{k=1}^{m} [\alpha_{1k}\alpha_{2k} + a_k c_k^*] = 6. \tag{3.35}$$

Since $\alpha_{1\infty} - \alpha_{2\infty} = 1$, the equality (3.31) fails to be fulfilled, therefore the formulas (3.31)–(3.33) allow one to determine only $\gamma'_{n\infty}$, $n = \overline{1, \infty}$, and hence the solution $u_{1\infty}(\zeta)$. For the equality (3.31) to take place for $\alpha_\infty = \alpha_{2\infty}$, it is necessary and sufficient that the condition

$$q_{1\infty} - p_{1\infty}\alpha_{2\infty} = 0 \tag{3.36}$$

be fulfilled.

To define $\gamma^2_{1\infty}$, we act as follows: from (3.31) for $\alpha_\infty \neq \alpha_{2\infty}$ we define $\gamma_{1\infty}$ and obtain

$$\gamma_{1\infty} = [p_{1\infty} - q_{1\infty}]/f_{0\infty}(\alpha_\infty + 1). \tag{3.37}$$

Since the numerator and the denominator in (3.37) vanish as $\alpha_\infty \to \alpha_{2\infty}$, we can develop the indeterminacy in the well-known manner and get

$$\gamma^2_{1\infty} = p_{1\infty}. \tag{3.38}$$

After that we define $\gamma^2_{n\infty}$, $n = \overline{2, \infty}$, by the formulas (3.32)–(3.33). Thus we have obtained the solution $u_{2\infty}(\zeta)$.

Finally, we have

$$u_{k\infty}(\zeta) = \zeta^{-\alpha_{k\infty}} + \sum_{n=1}^{\infty} \gamma^n_{n\infty} \zeta^{-\alpha_{2\infty} - n}, \quad k = 1, 2. \tag{3.39}$$

The equations (1.21) coincide respectively with the equations (1.13), (3.35) and (3.36).

4. Local Representations of the Matrices $\chi_j(\zeta)$, $j = \overline{1, m+1}$

Of each set of branches of the functions $\exp[\alpha_{kj} \ln(t - a_j)]$ appearing in the local solutions $u_{kj}(\zeta)$, we choose one as follows:

$$\exp[\alpha_{kj} \ln(t - a_j)] > 0, \quad t > a_j,$$

$$[\exp[\alpha_{kj} \ln(t - a_j)]]^{\pm} = \exp[\pm i\alpha_{kj}] \exp[\alpha_{kj} \ln(a_j - t)], \quad t < a_j,$$

$$[\exp[-\alpha_{k\infty}\ln(-t)]]^{\pm} > 0, \quad -\infty < t < a_j;$$
$$[\exp[-\alpha_{k\infty}\ln t]]^{\pm} = \exp[\pm i\pi(-\alpha_{k\infty})]\exp[-\alpha_{k\infty}\ln t], \quad a_m < t < \infty.$$

Along with (3.5) and (3.6), we introduce the matrices

$$\Theta_j^*(t) = \begin{pmatrix} u_{1j}^*(t), & u_{1j}'^*(t) \\ u_{2j}^*(t), & u_{2j}'^*(t) \end{pmatrix}, \quad a_{j-1} < t < a_j, \tag{4.1}$$

where

$$\begin{gathered} u_{kj}^*(t) = (a_j - t)^{\alpha_{kj}}\widetilde{u}_{kj}(t), \\ u_{kj}'(t) = -(a_j - t)^{\alpha_{kj}-1}\widetilde{u}_{kj}'^*(t), \quad u_{kj}'(t) = du_{kj(t)/dt}, \\ \widetilde{u}_{kj}'^*(t) \equiv \alpha_{kj} + \sum_{n=1}^{\infty}\gamma_{nj}^k(\alpha_{kj}+n)(t-a_j)^n. \end{gathered} \tag{4.2}$$

Between the matrices $\Theta_j(t)$ and $\Theta_j^*(t)$ there is the connection

$$\Theta_j^{\pm}(t) = \theta_j^{\pm}\Theta_j^*(t), \quad a_{j-1} < t < a_j, \tag{4.3}$$
$$\Theta_{\infty}^{\pm}(t) = \theta_{\infty}^{\pm}(t)\Theta_{\infty}^*(t), \quad a_m < t < +\infty, \tag{4.4}$$

where the matrices $\theta_j^{\pm}$ are defined by the formula

$$\theta^{\pm} = \begin{pmatrix} \exp(\pm i\pi\alpha_{1j}), & 0 \\ 0, & \exp(\pm i\pi\alpha_{2j}) \end{pmatrix} \tag{4.5}$$

for $\alpha_{1j} - \alpha_{2j} \neq n$, while for $n = 0, 1, 2$ they are defined by the equality

$$\theta_j^{\pm} = e^{\pm\pi\alpha_{2j}}\begin{pmatrix} 1, & 0 \\ \mp\pi i, & 1 \end{pmatrix}, \ n=0,2; \ \ \theta_j^{\pm} = e^{\pm i\pi\alpha_{2i}}\begin{pmatrix} -1, & 0 \\ \pm\pi i, & -1 \end{pmatrix}, \ n=1. \tag{4.6}$$

For the cut end $w = A_j$, the matrices $\theta_j^{\pm}$ are defined in the following manner. If the eigenvalues are of the type $\alpha_{1j} = 3/2$, $\alpha_{2j} = -1/2$, then $\theta_j^{\pm} = \mp iE$, where E is the unit matrix, but if $\alpha_{1i} = 2$, $\alpha_{2j} = 0$, then $\theta_j^{\pm} = E$.

The elements of the matrix $\Theta_j^*(t)$ involving logarithmic terms are defined by the formulas

$$u_{2j}^*(t) = (a_j - t)^{\alpha_{2i}}[(t-a_j)^n\widetilde{u}_{1j}(t)\ln(a_j - t) + \widetilde{u}_{2i}^2(t)], \tag{4.7}$$
$$\begin{aligned} u_{2i}'^*(t) = -(a_j - t)^{\alpha_{2j}-1}\{[(a_j - t)^n e^{i\pi n}\widetilde{u}_{2j}^1\ln(a_j - t) + \widetilde{u}_{1j}(t)] + \\ +\widetilde{u}_{2j}^2(t)\}, \quad n = 0, 1, 2. \end{aligned} \tag{4.8}$$

5. The Fundamental Matrix

Let us construct the matrix $\chi(\zeta)$. The domains of convergence of the matrices $\Theta_i(t)$ and $\sigma_j(t)$ have always a part in common in which one can write the equalities

$$\Theta_j^*(t) = T_j^*\sigma_j(t), \ \ \sigma_j(t) = T_{0j}\Theta_{j-1}(t), \ \ T_{j-1} = T_j^*T_{0j}, \ \ a_{j-1} < t < a_j, \tag{5.1}$$
$$\Theta_1^*(t) = T_{-m}\sigma_{-m}(t), \quad \sigma_{-m}(t) = T_{-\infty}\Theta_{\infty}(t), \quad -\infty < t < a_1 \tag{5.2}$$
$$\Theta_{\infty}^*(t) = T_{\infty}\Theta_m(t), \quad a_m < t < +\infty. \tag{5.3}$$

Here T_j^*, T_{0j}, T_{-m}, T_{j-1}, $T_{-\infty}$, T_∞ are constant real matrices defined by the equalities (5.1)–(5.3). In these equalities we can fix t arbitrarily in the domain where the two local matrices, appearing in the above-mentioned equalities, converge.

Define the matrix $\chi_1(\zeta)$ along the t-axis of the plane ζ as

$$\chi_1^\pm(t) = T\Theta_m^\pm(t), \quad \Theta_m^+(t) = \Theta_m^-(t), \quad a_m < t < +\infty, \tag{5.4}$$

$$\chi_1^\pm(t) = T\theta_m^\pm\Theta_m^*(t), \quad a_{m-1} < t < a_m; \tag{5.5}$$

$$\chi_1^\pm(t) = T\theta_m^\pm T_{m-1}\Theta_{m-1}^\pm(t), \quad T_{m-1} = T_m^* T_{0m}, \quad a_{m-1} < t < a_m; \tag{5.6}$$

$$\chi_1^\pm(t) = T\theta_m^\pm T_{m-1}\theta_{m-1}^\pm\Theta_{m-1}^*(t), \quad a_{m-2} < t < a_{m-1}; \tag{5.7}$$

. . . .

$$\chi_1^\pm(t) = T\theta_m^\pm T_{m-1}\dots T_1\theta_1^\pm\Theta_1^*(t), \quad -\infty < t < a_1, \tag{5.8}$$

$$\chi_1^\pm(t) = T\theta_m^\pm T_{m-1}\dots \theta_1^\pm T_{-\infty}\Theta_\infty(t), \quad -\infty < t < a_1, \tag{5.9}$$

$$\chi_1^\pm(t) = TT_\infty\Theta_\infty^\pm(t), \quad a_m < t < +\infty, \tag{5.10}$$

where the matrix T is defined by the formula (3.4).

The upper signs ($\pm$) in the matrices (5.4)–(5.10) denote the limiting values of the matrix $\chi(\zeta)$ respectively from the upper (when $\zeta \in \operatorname{Im}(\zeta) > 0$, $\zeta \to t$) and in the lower (when $\zeta \in \operatorname{Im}(\zeta) < 0$, $\zeta \to t$) half-planes. The limiting values of $\chi^+(t)$ and of $\chi^-(t)$ are connected follows: $\chi^-(t) = \overline{\chi^+(t)}$, where $\overline{\chi^+(t)}$ is the complex conjugate of the matrix $\chi^+(t)$.

6. Solution of the Boundary Value Problem (2.25)

A straightforward checking shows that the matrices (5.4)–(5.10) satisfy the equation (3.2). Therefore, by appropriate choice of the parameters a_j, c_j, $j = \overline{1,m}$, p, q, r, s, the same matrices must satisfy the condition (2.25). Indeed, we start our proof from the interval $(a_m, +\infty)$. We have

$$\begin{gathered} T\Theta_m^+(t) = g_m(t)\overline{T}\Theta_m^-(t), \quad g_m(t) = E, \\ \Theta_m^+(t) = \Theta_m^-(t), \quad T = \overline{T}, \quad a_m < t < +\infty. \end{gathered} \tag{6.1}$$

For the interval (a_{m-1}, a_m) in the neighborhood of $A = a_m$ we obtain the equality

$$T\theta_m^+\Theta_m^*(t) = g_{m-1}T\theta_m^-\Theta_m^*(t), \quad a_{m-1} < t < a_m. \tag{6.2}$$

The expressions (6.1) and (6.2) result in the matrix equations

$$(\theta_m^+)^2 = T^{-1}G_{m-1}^{-1}\overline{G}_{m-1}T, \tag{6.3}$$

from which one can see that the matrices $(\theta_m^+)^2$ and $G_{m-1}^{-1}\overline{G}_{m-1}$ are similar.

The matrix equation (6.2) can be rewritten in the form

$$T\begin{pmatrix} \widetilde{\lambda}_{1(m)}, & 0 \\ 0, & \widetilde{\lambda}_{2(m)} \end{pmatrix} = \begin{pmatrix} A_{11}^{*(m-1)}, & iA_{12}^{*(m-1)} \\ iA_{21}^{*(m-1)}, & \overline{A}_{11}^{*(m-1)} \end{pmatrix} T, \tag{6.4}$$

$$\lambda_{km} = \widetilde{\lambda}_{km}/\det G_{m-1}, \quad k = 1, 2, \tag{6.5}$$

which in its turn results in the system consisting of two equations

$$r/p=\left\{\sqrt{\det G_{m-1}\det\overline{G}_{m-1}-(\operatorname{Re}A_{11}^{*(m-1)})^2}-\operatorname{Im}A_{11}^{*(m-1)}\right\}/A_{12}^{*(m-1)}, \quad (6.6)$$

and

$$s/q=\left\{\operatorname{Im}A_{11}^{*(m-1)}-\sqrt{\det G_{m-1}\det\overline{G}_{m-1}-(\operatorname{Re}A_{11}^{*(m-1)})^2}\right\}/A_{21}^{*(m-1)}. \quad (6.7)$$

Analogously to the matrix equation (6.3), we find the matrix equations successively for the points $\zeta = a_j$, $j = m-1, m-2, \dots, 2, 1$. We have

$$T\theta_m^+T_{m-1}\theta_{m-1}^+ = g_{m-2}(t)T\theta_m^-T_{m-1}\theta_{m-1}^-, \quad (6.8)$$

$$T\theta_m^+T_{m-1}\theta_{m-1}^+T_{m-2}\theta_{m-2}^+ = g_{m-3}T\theta_m^-T_{m-1}\theta_{m-1}^-T_{m-2}\theta_{m-2}^-, \quad (6.9)$$

. . . .

$$T\theta_m^+T_{m-1}\theta_{m-1}^+T_{m-2}\theta_{m-2}^+\dots T_1\theta_1^+ =$$
$$= T\theta_m^-T_{m-1}\theta_{m-1}^-T_{m-2}\theta_{m-2}^-\dots T_1\theta_1^-. \quad (6.10)$$

Similarly to the system of equations (6.6) and (6.7), from the matrix equations (6.8)–(6.10) we get two equations for every singular point.

The matrix equation (6.3) can be written as

$$p\cdot\exp(i\pi\alpha_{1m}) =$$
$$= A_{11}^{*(m-1)}p\cdot\exp(-i\pi\alpha_{1m}) + iA_{12}^{*(m-1)}r\cdot\exp(-i\pi\alpha_{1m}), \quad (6.11)$$

$$r\cdot\exp(i\pi\alpha_{1m}) =$$
$$= iA_{21}^{*(m-1)}p\cdot\exp(-i\pi\alpha_{1m}) + \overline{A}_{11}^{*(m-1)}r\cdot\exp(-i\pi\alpha_{1m}), \quad (6.12)$$

$$q\cdot\exp(i\pi\alpha_{2m}) =$$
$$= A_{11}^{*(m-1)}q\cdot\exp(-i\pi\alpha_{2m})+iA12^{*(m-1)}s\cdot\exp(-i\pi\alpha_{2m}), \quad (6.13)$$

$$s\cdot\exp(i\pi\alpha_{2m}) =$$
$$= iA_{21}^{*(m-1)}q\cdot\exp(-i\pi\alpha_{2m}) + \overline{A}_{11}^{*(m-1)}s\cdot\exp(-i\pi\alpha_{2m}). \quad (6.14)$$

Dividing the corresponding parts of the equations (6.11) and (6.12), (6.13) and (6.14), one can see that the ratios p/r, q/s in the interval (a_{m-1}, a_m) satisfy the boundary condition (2.25),

$$\frac{p}{r} = \frac{iA_{11}^{*(m-1)}p/r + A_{12}^{*(m-1)}}{iA_{21}^{*(m-1)}p/r + i\overline{A}_{11}^{*(m-1)}}, \quad \frac{q}{s} = \frac{A_{11}^{*(m-1)}q/s + iA_{12}^{*(m-1)}}{iA_{21}^{*(m-1)}q/s + \overline{A}_{11}^{*(m-1)}}. \quad (6.15)$$

The coordinates of the points $w = A_m$, $w = A'_m$ also satisfy the same condition and, consequently,

$$p/r = A_m, \quad q/s = A'_m, \quad (6.16)$$

where A'_m is the second point of intersection of the two neighboring circumferences.

Remind that by A_k, A'_k, $k = 1, 2, \dots, m$, we have denoted the complex coordinates of the angular points of the circular polygon $s(w)$ at which two neighboring circumferences may intersect; note that the point A'_k lies more often outside of the contour $l(w)$.

On the plane w, if the origin coincides with the point $w = A_m$, then $A_m = 0$ and $A'_m = \infty$. Consequently, $p = 0$ and $s = 0$. It should be noted that for the interval (a_{m-1}, a_m), if $\nu_m \neq 0$, one can always suppose that

$$G_{m-1} = \begin{pmatrix} A_{11}^{*(m-1)}, & 0 \\ 0, & \overline{A}_{11}^{*(m-1)} \end{pmatrix}. \tag{6.17}$$

Consider the matrix equation (6.8),

$$T_{*(m-1)}\theta^+_{m-1} = g_{m-2}\overline{T}_{*(m-1)}\theta^-_{m-1}, \quad T_{*(m-1)} = T\theta^+_m T_{m-1}. \tag{6.18}$$

From (6.18) we get the system of equations

$$p_{*(m-1)}/r_{*(m-1)} = A_{m-1}, \quad q_{*(m-1)}/s_{*(m-1)} = A'_{m-1}, \tag{6.19}$$

where $p_{*(m-1)}$, $q_{*(m-1)}$, $r_{*(m-1)}$ and $s_{*(m-1)}$ are the elements of the matrix $T_{*(m-1)}$. Taking into account (6.18), the equalities (6.19) can be rewritten as follows:

$$\frac{p_*p_{m-1} + q_*r_{m-1}}{r_*p_{m-1} + s_*r_{m-1}} = A_{m-1}, \quad \frac{p_*q_{m-1} + q_*s_{m-1}}{r_*p_{m-1} + s_*s_{m-1}} = A'_{m-1}, \tag{6.20}$$

where p_*, q_*, r_* and s_* are the elements of the matrix $T_* = T\theta^+_m$.

The equalities (6.20) with regard for (6.19) can in their turn be rewritten as

$$\begin{aligned} &\frac{r_*p_{m-1}A_m + s_*r_{m-1}A'_m}{r_*p_{m-1} + s_*r_{m-1}} = A_{m-1}, \\ &\frac{r_*q_{m-1}A_m + s_*s_{m-1}A'_m}{r_*q_{m-1} + s_*s_{m-1}} = A'_{m-1}. \end{aligned} \tag{6.21}$$

After simplification, the equations (6.21) take the form

$$r_*p_{m-1}(A_m - A_{m-1}) + s_*r_{m-1}(A'_m - A_{m-1}) = 0, \tag{6.22}$$

$$r_*q_{m-1}(A_m - A'_{m-1}) + s_*s_{m-1}(A'_m - A'_{m-1}) = 0. \tag{6.23}$$

The condition of compatibility of (6.22) and (6.23) with respect to r_* and s_* has the form

$$\frac{p_{m-1}s_{m-1}}{r_{m-1}q_{m-1}} = \frac{A'_m - A_{m-1}}{A_m - A_{m-1}} \cdot \frac{A_m - A'_{m-1}}{A'_m - A'_{m-1}}. \tag{6.24}$$

From the matrix equation (6.9) we obtain the system of equations

$$\begin{aligned} &\frac{p_{*(m-1)}p_{m-2} + q_{*(m-1)}r_{m-2}}{r_{*(m-1)}p_{m-2} + s_{*(m-1)}r_{m-2}} = A_{m-2}, \\ &\frac{p_{*(m-1)}q_{m-2} + q^*_{*(m-1)}s_{m-2}}{r_{*(m-1)}q_{m-2} + s_{*(m-1)}s_{m-2}} = A'_{m-2}, \end{aligned} \tag{6.25}$$

where $p_{*(m-1)}$, $q_{*(m-1)}$, $r_{*(m-1)}$, $S_{*(m-1)}$ are the elements of the matrix

$$T_{*(m-1)} = T\theta^+_m T_{m-1}\theta^+_{m-1}. \tag{6.26}$$

After certain transformations the above system takes the form

$$r_{*(m-1)}p_{m-2}(A_{m-1}-A_{m-2})+s_{*(m-1)}r_{m-2}(A'_{m-1}-A_{m-2})=0, \quad (6.27)$$

$$r_{*(m-1)}q_{m-2}(A_{m-1}-A'_{m-2})+s_{*(m-1)}s_{m-2}(A'_{m-1}-A'_{m-2})=0. \quad (6.28)$$

The equations (6.27) and (6.28) imply

$$\frac{p_{m-2}s_{m-2}}{r_{m-2}q_{m-2}}=\frac{A'_{m-1}-A_{m-2}}{A_{m-1}-A_{m-2}}\cdot\frac{A_{m-1}-A'_{m-2}}{A'_{m-1}-A'_{m-2}}, \quad (6.29)$$

The remaining matrix equations can be investigated analogously [25-31].

The equations (6.24) and (6.29) are nothing but the invariant cross-ratios of four points of the same circumference at which the given circumference intersects with the two neighboring ones.

From (6.3)–(6.10) we can get all the needed equations with respect to a_k, c_k, $k=\overline{1,m}$, and to the integrations constants p, q, r and s.

For every point $t=a_j$ we have obtained a system of two equations which are homogeneous with respect to the elements of the matrices T_k, $k=\overline{1,m}$; their conditions of compatibility for, e.g., the points $t=a_m$ and a_{m-1} have the form (6.24) and (6.29). The above-mentioned systems of equations have been obtained under the assumption that $\alpha_{1j}-\alpha_{2j}\neq n$, $n=0,1,2$.

Consider briefly the case where $\alpha_{1j}-\alpha_{2j}=n$, $n=0,1,2$. According to the representations (5.4)–(5.10), the unknown matrices $\chi^+(t)$ and $\chi^-(t)$ in the interval (a_{j-1},a_j) must satisfy the boundary condition

$$\begin{aligned} &T\theta_m^+T_{m-1}\theta_{m-1}^+T_{m-2}\theta_{m-2}^+\dots T_j\theta_j^+ = \\ &= g_{j-1}T\theta_m^-T_{m1}\theta_{m-1}^-T_{m-2}\theta_{m-2}^-\dots T_j\theta_j^-, \end{aligned} \quad (6.30)$$

where

$$\theta_j^+=e^{i\pi\alpha_{2j}}\begin{pmatrix}1, & 0\\ \pm\pi i, & 1\end{pmatrix}, \quad \theta_j^-=\overline{\theta}_j^+, \quad n=0,2;$$

$$\theta_j^+=e^{i\pi\alpha_{2j}}\begin{pmatrix}-1, & 0\\ -\pi i, & 1\end{pmatrix}, \quad n=1, \quad \theta_j^-=\overline{\theta}_j^+.$$

It can immediately be verified that (6.30) leads to a usual system of two equations with respect to p_j, q_j, r_j, s_j, but the condition of their compatibility does not provide now the relations analogous to (6.24) and (6.29).

As is mentioned above, matrix equations similar to (6.1)–(6.10) can be obtained for all the points $\zeta=a_k$, with the exclusion of the points $\zeta=a_j$ to which there correspond the cut ends of the boundary $l(w)$ of the circular polygon $w=A_j$ for which $\nu_j=2$. For such points we have either the condition (3.20) or (3.21). This allows one to obtain one equation for each point, the second equation being obtained after determination of $l(z)$, $l(\omega)$ and $l(w)$.

From the matrix representations we first define $u_1^+(t)$ and $u_2^+(t)$ and then compose the ration $w^+(t)=u_1^+(t)/u_2^+(t)$. According to (5.4)–(5.10), the function $w^+(t)$ for the interval (a_j, a_{j+1}) can be represented as

$$w^+(t) = [A_j^* u_{1j}^+(t) + B_j^* u_{2j}^*(t)]/[C_j^* u_{1j}^+ + D_j^* u_{2j}^+(t)], \tag{6.31}$$

where A_j^*, B_j^*, C_j^*, D_j^* are defined by (5.4)–(5.10).

Calculating the limit as $\zeta \to a_j$ by means of (6.31), we obtain the equation

$$A_j = B_j^*/D_j^*. \tag{6.32}$$

The corresponding equations for other points $t = a_k$, $k = \overline{1, m+1}$, can be obtained analogously.

Finally, for every point $t = a_j$ we obtain two real, homogeneous with respect to p_j, q_j, r_j and s_j equations, for example, (6.6) and (6.7). From the condition of compatibility of homogeneous equations for $\nu_j \neq 0, 1, 2$, we obtain invariant cross-ratios for four points of one circumference, for example, (6.24) and (6.29). In the case where $\nu_j = 0, 1, 2$, the conditions of compatibility of two equations provide certain, equations which, however are not anharmonic.

From each system of two equations we can take one equation and, in addition, one more equation of compatibility, i.e. we take two equations for each point $\zeta = a_j$. The number of equations is equal to $2m$ and the number of unknown parameters a_k, c_k^* $k = \overline{1, m}$, p, q, r, s with $ps - rq \neq 0$ is $2m - 3$. Consequently, the number of equations will be greater by three than the number of unknown parameters. It should be noted here that from the very beginning we have supposed that the linear fractional transformation over the domain $s(w)$ was performed with a view to have the equation $G_m = E$ (E is the unit matrix) on one part of the boundary $l(w)$. Thus the parameters p, q, r, and s turned out to be real and their number equals to three, since $ps - rq \neq 0$. The above-described method of constructing the functions $w(\zeta)$, $\omega(\zeta)$ and $z(\zeta)$ and the system of equations with respect to a_j, c_j^*, $j = \overline{1, m}$, is assumed to be much more convenient than some other methods. One can give up transformation of the domain $s(w)$. In this case the parameters p, q, r and s will be complex and the number of the unknown parameters will equal to $2m$. The appearance of three additional equations can be explained just as in the case of linear polygons.

Having constructed the system of equations for determining a_k, c_k^*, p, q, r and s, $k = \overline{1, m}$, we have first to establish the intervals of variation of the parameters c_k^*, $k = \overline{1, m}$, then to solve the system with respect to a_k, c_k^*, $k = \overline{1, m}$ and finally, to define p, q, r and s.

Remind that p_j, q_j, r_j and s_j, $j = \overline{1, m}$, depend implicitly, through the coefficients of the generalized hypergeometric series, on the parameters a_k, c_k^*, $k = \overline{1, m}$. The intervals of variation of the parameters can be established according to [27].

As is known, the series $u_{kj}(\zeta)$, $j = \overline{1, m+1}$, $k = 1, 2$, converge, respectively, in the neighborhood of the points $\zeta = a_j$, $j = \overline{1, m+1}$, $t = a_{m+1} = \infty$ and the series $\sigma_{kj}(\zeta)$ in the neighborhood of the points $a_j^* =$

$(a_j + a_{j+1})/2$. The convergence radii of these series are bounded by the distance from the given point $t = a_j$ (or from the point a_j^*) to the nearest points $\zeta = a_{j-1}, a_{j+1}$.

The series $u_{kj}(\zeta)$, $k = 1, 2$, $j = \overline{1, m}$, are whole functions of the parameters c_j^*, $j = \overline{1, m}$, but with respect to ζ these series converge slowly, which makes numerical calculations difficult. As n increases, the coefficients γ_{nj}^k sometimes rapidly increase, although their multipliers $(\zeta - a_j)^n$ on the contrary rapidly decrease. Electronic computers fail to multiply γ_{nj}^k by $(t-a_j)^n$ despite the fact that the series converge. To eliminate this drawback, we have suggested to write the same series in the form of rapidly and uniformly convergent functional series [28-31].

Let us consider the structure of the recurrence formulas (3.15)–(3.16), (3.31)–(3.33). The sum of the first lower indices in the expressions $\gamma_{(k-n)j} \cdot f_{nj}(\alpha_j + k - n)$ is always equal to k, i.e. to the exponent of $(t-a_j)^k$. Instead of the series (3.18) let us consider the function series of the type

$$u_{kj}(t) = (t-a_j)^{\alpha_{kj}} \widetilde{u}_{kj}(t-a_j), \quad \widetilde{u}_{kj}(t) = \sum_{n=0}^{\infty} \gamma_{nj}^k (t-a_j), \quad \gamma_{0j}^k = 1, \tag{6.33}$$

where γ_{nj}^k is defined, according to (3.15)–(3.16), in terms of $\gamma_{1j}, \gamma_{2j}, \ldots, \gamma_{(n-1)j}$, and the latter in their turn are defined in terms of $f_{kj}(\alpha_j)$, where

$$f_{kj}(t - a_j, \alpha_j) = \alpha_j p_{kj}(t - a_j) + q_{kj}(t - a_j) \tag{6.34}$$

$$p_{nj}(t - a_j) = (-1)^{n-1} \sum_{k=1, j\neq k}^{k} [1 - \alpha_{1k} - \alpha_{2k}] \Big(\frac{t - a_j}{a_j - a_k}\Big)^n, \tag{6.35}$$

$n = 1, 2, \ldots,$

$$q_{nj}(t-a_j) = (-1)^{n-2} \sum_{k=1, k\neq j} [\alpha_{1k}\alpha_{2k}(n-1) + c_k^*(a_j - a_k)] \Big(\frac{t - a_j}{a_j - a_k}\Big)^n, \tag{6.36}$$

$n = 2, 3, \ldots,$

$$\Big|\frac{t - a_j}{a_j - a_k}\Big| < 1, \quad k \neq j \tag{6.37}$$

$$p_{n\infty}(t) = \sum_{k=1}^{m} [1 - \alpha_{1k}\alpha_{2k}](a_k/t)^n, \tag{6.38}$$

$$q_{n\infty}(t) = \sum_{k=1}^{\infty} [\alpha_{1k}\alpha_{2k}(n+1) + c_k^* a_k](a_k/t)^n, \tag{6.39}$$

$n = 0, 1, 2, \ldots,$

$$|a_k/t| < 1. \tag{6.40}$$

The formulas (6.34)–(6.40) show that the fundamental series (6.33) and the series

$$u_{k\infty}(t) = \zeta^{-\alpha k\infty} \Big[1 + \sum_{n=1}^{\infty} \gamma_{n\infty}^k(t)\Big] \tag{6.41}$$

converge in the domain $|\zeta - a_j|$ more rapidly than the series (3.18).

The matrices $\chi^{\pm}(t)$ defined by the formulas (5.1)–(5.10) satisfy the boundary condition (2.25).

7. Definition of the Functions $\omega(\zeta)$ and $z(\zeta)$

Along the real t-axis, the function $w^+(t)$ is defined by the equality

$$w^+(t) = u_1^+(t)/u_2^+(t), \quad -\infty < t < +\infty, \tag{7.1}$$

where $u_1^+(t)$ and $u_2^+(t)$, being the linear independent solutions of (1.11), are defined by the formulas (5.1)–(5.10).

Knowing $w(\zeta)$ along the entire real t-axis of the plane, we can find $w(\zeta)$ for $\mathrm{Im}(\zeta) > 0$ for all $t = e_k$, $k = \overline{1, n+1}$, with the help of the well-known formula given in [16].

Note that using the matrix $\chi(\zeta)$ defined by the formulas (5.4)–(5.10), we can construct a canonical matrix for the corresponding homogeneous problem (2.10) with regard for all singular points $t = e_k$, $k = \overline{1, n+1}$, after which it becomes possible to solve the nonhomogeneous boundary value problem (2.10) by means of the Cauchy type integral. This has been done by us in [27]. In the present paper we find the solution of (2.10) in a more simple way than that described in [27] and [29]. We rely here on the linear independent solutions (1.11) and on the general solution of (1.18).

Let us multiply the functions $u_1^+(t)$ and $u_2^+(t)$ by

$$\chi_0(\zeta) = \sqrt{\frac{(\zeta - e_{k-1})(\zeta - e_{k+j-1})}{(\zeta - e_k)(\zeta - e_{k+j})}}.$$

The matrix $\chi_1(\zeta)$ defined by the formulas (5.4)–(5.10) satisfies the boundary condition (2.25), as far as we take for granted that the equalities (6.1)–(6.32) are fulfilled. This means that the columns of the matrix $\chi_1(\zeta)$ defined by the formulas (5.4)–(5.10) satisfy the boundary condition (2.25). To obtain the solution $\Phi_1(\zeta)$, we have to take the elements of the first column of the matrix $\chi_1(\zeta)$ which are defined by the formulas (6.1)–(6.32) and then to compose the vector $\Phi_1(\zeta) = [u_1(\zeta), u_2(\zeta)]$, $\mathrm{Im}(\zeta) \geq 0$.

We have taken the elements of the first column of the matrix $\chi_1(\zeta)$ because the relation $w(\zeta) = u_1(\zeta)/u_2(\zeta)$ gives the general solution of the Schwarz equation with the right-hand side (1.18), while the ratio $u_1'(\zeta)/u_2'(\zeta)$ does not satisfy the equation (1.18).

The vector $\Phi'(\zeta) = \Phi_1(\zeta)\chi_0(\zeta)$, where $\chi_0(\zeta) = \sqrt{\frac{(\zeta - e_{k-1})(\zeta - e_{k+j-1})}{(\zeta - e_k)(\zeta - e_{k+j})}}$, will be a solution of the problem (2.15). Consequently, the elements of the vector $\Phi'(\zeta)$, $\omega'(\zeta) = u_1^+(\zeta)\chi_0(\zeta)$ and $z'(\zeta) = u_2^+(\zeta)\chi_0(\zeta)$, satisfy both the boundary conditions (2.15) and the conditions at the singular points $t = e_k$, $k = \overline{1, n+1}$.

Now we can write the following equalities:

$$d\omega(t) = u_1^+(t)\chi_0^+(t)dt, \quad -\infty < t < +\infty, \tag{7.2}$$

$$dz(t) = u_2^+(t)\chi_0^+(t)dt, \quad -\infty < t < +\infty. \tag{7.3}$$

Integrating the equalities (7.2) and (7.3) in the intervals $(-\infty, t)$, (e_j, t), $j = 1, 2, \ldots, n$, we obtain

$$\omega^+(t) = \int\limits_{-\infty}^{t} u_1^+(t)\chi_0^+(t)dt + \omega^+(-\infty), \quad -\infty < t < e_1, \tag{7.4}$$

$$z^+(t) = \int\limits_{-\infty}^{t} u_2^+(t)\chi_0^+(t)dt + z^+(-\infty), \quad -\infty < t < e_1, \tag{7.5}$$

$$\omega^+(t) = \int\limits_{e_j}^{t} u_1^+(t)\chi_0^+(t)dt + \omega_j^+(e_j), \quad j = \overline{1, n+1}, \quad e_j < t < e_{j+1}, \tag{7.6}$$

$$z^+(t) = \int\limits_{e_j}^{t} u_2^+(t)\chi_0^+(t)dt + z_j^+(e_j), \quad j = \overline{1, n+1}, \quad e_j < t < e_{j+1}, \tag{7.7}$$

where $\omega^+(-\infty)$, $z^+(-\infty)$, $\omega^+(e_j)$, $z^+(e_j)$ are the limiting values of the corresponding functions $\omega^+(t)$, $z^+(t)$ from the right at the points $-\infty$, e_j, $j = \overline{1, n+1}$.

Obviously, the functions $\omega^+(t)$, $z^+(t)$ defined by the formulas (7.4)–(7.7) satisfy the boundary conditions (2.10).

In the formulas (7.4)–(7.7) we can separate the real and imaginary parts and get expressions for the functions $\varphi(t)$, $\psi(t)$, $\chi(t)$ and $y(t)$.

Passing in the formulas (7.4) and (7.5) to the limit as $t \to e_j$ from the left, we arrive at

$$\omega^+(e_1) = \int\limits_{-\infty}^{e_1} u_1^+(t)\chi_0^+(t)dt + \omega^+(-\infty), \tag{7.8}$$

$$z^+(e_1) = \int\limits_{-\infty}^{e_1} u_2^+(t)\chi_0^+(t)dt + z^+(-\infty), \tag{7.9}$$

$$\omega^+(e_{j+1}) = \int\limits_{e_j}^{e_{j+1}} u_1^+(t)\chi_0^+(t)dt + \omega^+(e_j), \quad j = \overline{1, n+1}, \tag{7.10}$$

$$z + (e_{j+1}) = \int\limits_{e_j}^{e_{j+1}} u_2^+(t)\chi_0^+(t)dt + z^+(e_j), \quad j = \overline{1, n+1}, \tag{7.11}$$

where $\omega^+(e_{j+1})$, $z^+(e_{j+1})$, are the limiting values of the functions $\omega^+(t)$, $z^+(t)$ from the left at the point $t = e_{j+1}$.

In the formulas (7.4)–(7.11) it is assumed that the integrands at the points $t = -\infty$, $t = e_j$, $j = \overline{1, n}$ are integrable. In case the integrands are nonintegrable at some point $t = e_j$ of e_1, $e_2, e_2, \dots, e_{n+1}$, we take the integrals from the other end of the interval, where they are integrable. But if the above-mentioned functions are nonintegrable at both ends of the interval, then we take any interior point of the interval and from that point (as the lower limit) the integral is taken.

For determination of the parameters a_j and c_j, $j = \overline{1,m}$, we have obtained a system of higher transcendent equations, e.g., the equations (6.6)–(6.32); as for the parameters $t = e_j$, $j = \overline{1,n}$, which do not coincide with the parameters $t = a_j$ and the function $\chi_0(\zeta)$ depends on, and also as for the parameter Q which is connected with the liquid discharge, for their determination we have obtained the system of equations (7.8)–(7.11).

Having found all the unknown parameters on which the functions $\omega(\zeta)$, $z(\zeta)$, and $w(\zeta)$ depend, by the formulas (7.6)–(7.7) we can find the equations for the unknown parts of the boundaries of the domains $s(z)$, $s(\omega)$ and $s(w)$, as well as for other geometric and mechanical parameters of the liquid flow [30, 31].

8. Another Method of Solving the System (6.3)–(6.10) with Respect to p_j/r_j, s_j/q_j.

Of the system (6.3)–(6.10), we consider the matrix equations for two neighboring points $t = a_j$ and $t = a_{j-1}$. We have

$$A^+_{j+1} = g_j A^-_{j+1}, \quad A^+_{j+1} T_j \theta^+_j = g_{j-1} A^-_{j+1} T_j \theta^-_j, \tag{8.1}$$

where

$$A^\pm_{j+1} = T\theta^\pm_m T_{m-1}\theta^\pm_{m-1} \dots T_{j+1}\theta^\pm_{j+1}. \tag{8.2}$$

Excluding A^+_{j+1} from the system (8.1), we obtain the equation with respect to T_j:

$$T_j(\theta^+_j)^2 = B^j T_j, \quad B^j = \begin{pmatrix} B^j_{11}, & B^j_{12} \\ B^j_{21}, & B^J_{22} \end{pmatrix} = (A^-_{j+1})^{-1} g_j^{-1} g_{j-1} A^-_{j+1}. \tag{8.3}$$

When solving (8.3), consider the following cases: (1) the difference $\alpha_{1j} - \alpha_{2j}$ is not an integer; (2) the difference $\alpha_{1j} - \alpha_{2j}$ is an integer.

1. The solution of (8.3) with respect to the elements of the matrix T_j has the form

$$p_j/r_j = B^j_{12}[\lambda_{1j} - B^j_{11}]^{-1}, \quad p_j/r_j = (B^j_{21})^{-1}[\lambda_{1j} - B^j_{22}] \tag{8.4}$$

$$s_j/q_j = (B^j_{12})^{-1}[\lambda_{2j} - B^j_{11}]^{-1}, \quad s_j/q_j = B^j_{21}[\lambda_{2j} - B^j_{22}]^{-1}. \tag{8.5}$$

We take one equation from each of (8.4) and (8.5) because the second equations coincide with the first ones owing to the fact that

$$\det B^j = \lambda_{1j}\lambda_{2j}, \quad B^j_{11} + B^j_{22} = \lambda_{1j} + \lambda_{2j}. \tag{8.6}$$

Consequently, the solution of (8.3) for one point is given in the form of two scalar equations with respect to the parameters a_j, c_j, $j = \overline{1,m}$. Recall (5.1) and (5.2) in which it is seen that the parameters p_j, q_j, r_j, s_j depend implicitly on a_j, c_j, $j = \overline{1,m}$.

The solution of the matrix equation

$$T\theta_m^+ = g_{m-1}T\theta_m^-, \quad g_{m-1} = \begin{pmatrix} g_{11}^{m-1}, & g_{12}^{m-1} \\ g_{21}^{m-1}, & g_{22}^{m-1} \end{pmatrix} \tag{8.7}$$

have the form

$$p/r = g_{12}^{m-1}[\lambda_{1m} - g_{11}^{m-1}]^{-1}, \quad s/q = B_{12}^{m-1}[\lambda_{2m} - g_{11}^{m-1}], \tag{8.8}$$

where p/r and s/q are the integration constants of the Schwarz differential equation (1.15).

We can immediately verify that the solutions (8.4) and (8.5) are real, hence the equation

$$p_j s_j/(r_j q_j) = [\lambda_{2j} - B_{11}^j][\lambda_{1j} - B_{11}^j]^{-1}, \tag{8.9}$$

is real as well. This equation is connected with the invariant cross-ratio of four intersection points of one circumference with two neighboring circumferences (see, e.g., (6.29) or (6.24)).

2(a). The difference $\lambda_{1j} - \lambda_{2j} = n$, $n = 0, 2$. In this case the equation (8.3) takes the form

$$\lambda_{2j}T_j \begin{pmatrix} 1, & 0 \\ 2\pi i, & 1 \end{pmatrix} = B^j T_j. \tag{8.10}$$

The solution (8.10) has the form

$$\lambda_{2j}(p_j + 2\pi i q_j) = B_{11}^j p_j + B_{12}^j r_j, \quad \lambda_{2j}[r_j + 2\pi i s_j) = B_{21}^j p_j + B_{22}^j r_j, \tag{8.11}$$

$$s_j/q_j = [\lambda_{2j} - B_{11}^j](B_{12}^j)^{-1}, \quad s_j/q_j = B_{2j}^j[\lambda_{2j} - B_{22}^j]^{-1}. \tag{8.12}$$

We take one equation from each of (8.11) and (8.12) because the second equations coincide with the first ones. Indeed, this is obvious for (8.12), while for (8.11) it is necessary to indicate the way of proving. First we define q_j/s_j from (8.12) and substitute the obtained value into the first of the equations (8.11), then we divide by s_j the left and right sides of both equations (8.11) and obtain

$$\frac{p_j}{s_j}(\lambda_{2j} - B_{11}^j) - \frac{r_j}{s_j}B_{12}^j = -2\pi i \lambda_{1j} B_{12}^j[\lambda_{1j} - B_{11}^j], \tag{8.13}$$

$$\frac{p_j}{s_j}(-B_{21}^j) + (\lambda_{2j} - B_{22}^j)\frac{r_j}{s_j} = 2\pi i \lambda_{2j}. \tag{8.14}$$

These equations coincide because the coefficients (including the free terms) are proportional.

2(b). If $\alpha_{1j} - \alpha_{2j} = 1$, then the equation (8.3) takes the form

$$\lambda_{2j}T_j \begin{pmatrix} 1, & 0 \\ -2\pi i, & 1 \end{pmatrix} = B^j T_j. \tag{8.15}$$

In this case (8.12) remains invariable, and proportionality of the coefficients (including free terms) is not violated if in the systems (8.13) and (8.14) we replace $2\pi i$ by $-2\pi i$.

Defining the elements p_j, q_j, r_j and s_j from (5.1) as depending on a_j, c_j, $j=\overline{1,m}$, and substituting them in (8.4), (8.5), (8.9), (8.11) and (8.12), we obtain equations with respect to a_j, c_j, $j=\overline{1,m}$.

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Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia