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ON EFFECTIVE SUFFICIENT CONDITIONS FOR STABILITY OF LINEAR SYSTEMS OF IMPULSIVE EQUATIONS

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Consider a linear system of impulsive equations of the form given in [1]

$$\frac{dx}{dt} = Q(t)x + q(t) \quad \text{for} \quad t \in \mathbb{R}_+, \tag{1}$$

$$x(t_j+) - x(t_j-) = G_j x(t_j-) + g_j \quad (j = 1, 2, \dots), \tag{2}$$

where $Q : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $q : \mathbb{R}_+ \to \mathbb{R}^n$ are, respectively, matrix and vector functions such that each of their components is measurable and integrable function in Lebesgue sense on every closed segment from $\mathbb{R}_+$; G_j and g_j $(j = 1, 2, \dots)$ are, respectively, constant matrices and vectors; $t_j \in \mathbb{R}_+$ $(j = 1, 2, \dots)$, $0 < t_1 < t_2 < \dots$, $\lim\limits_{j\to+\infty} t_j = +\infty$.

In this note we give some effective sufficient conditions guaranteeing stability of the system (1), (2) in the Liapunov sense. The analogous conditions for stability are given in [2] for linear systems of ordinary differential equations and in [3] for linear systems of generalized ordinary differential equations.

The following notation and definitions will be used in the paper.

$\mathbb{R} =]-\infty, +\infty[$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \sum_{j=1}^{m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

$\det(X)$ is the determinant of the matrix $X \in \mathbb{R}^{n\times n}$. I_n is the identity $n \times n$-matrix.

$r(H)$ is the spectral radius of the matrix $H \in \mathbb{R}^{n\times n}$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ at the point t.

$L_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ such that each of their components is measurable and integrable function in Lebesgue sense on every closed segment from $\mathbb{R}_+$.

If I is an arbitrary interval from $\mathbb{R}_+$, then $\widetilde{C}_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ such that each of their components is an absolutely continuous function on every closed segment from I. $\widetilde{C}_{\text{loc}}(\mathbb{R}_+ \setminus T; \mathbb{R}^{n\times m})$, where $T = \{t_1, t_2, \dots\}$, is the set of all matrix-functions $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ the restrictions of which on $]t_j, t_{j+1}[$ belong to $\widetilde{C}_{\text{loc}}(]t_j, t_{j+1}[; \mathbb{R}^{n\times m})$ for every $j \in \{1, 2, \dots\}$.

Under a solution of the system (1), (2) we understand a continuous from the left vector-function $x \in \widetilde{C}_{\text{loc}}(\mathbb{R}_+ \setminus T; \mathbb{R}^n)$ satisfying the system (1) almost everywhere on $]t_j, t_{j+1}[$ and the relation (2) in the point t_j for every $j \in \{1, 2, \dots\}$.

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We will assume that $Q = (q_{ik})_{i,k=1}^n \in L_{\rm loc}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, $G_j = (g_{jik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$ $(j = 1,2,\dots)$ and

$$\det(I_n + G_j) \neq 0 \quad (j = 1,2,\dots).$$

Definition 1. Let $\xi \in \widetilde{C}_{\rm loc}(\mathbb{R}_+\backslash T;\mathbb{R}_+)$ be a continuous from the left nondecreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty. \tag{3}$$

A solution x_0 of the system (1), (2) is called ξ-exponentially asymptotically stable if there exists a positive number η such that for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that an arbitrary solution x of the system (1), (2), satisfying the inequality

$$\|x(t_0) - x_0(t_0)\| < \delta$$

for some $t_0 \in \mathbb{R}_+$, admits the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp(-\eta(\xi(t) - \xi(t_0))) \quad \text{for} \quad t \geq t_0.$$

Note that the exponentially asymptotically stability is a particular case of the ξ-exponentially asymptotically stability if we assume $\xi(t) \equiv t$.

Stability, uniformly stability and asymptotically stability are defined just in the same way as for systems of ordinary differential equations (see, e. g., [2], [4]).

Definition 2. The system (1), (2) is called stable in one or other sense if every solution of this system is stable in the same sense.

As in the case of differential equations, the system (1), (2) is stable in one or other sense if and only if its corresponding homogeneous system

$$\frac{dx}{dt} = Q(t)x \quad \text{for} \quad t \in \mathbb{R}_+, \tag{1_0}$$

$$x(t_j+) - x(t_j-) = G_j x(t_j-) \quad (j = 1,2,\dots) \tag{2_0}$$

is stable in the same sense.

Definition 3. The pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is called stable in one or other sense if the system (1_0), (2_0) is stable in the same sense.

Theorem 1. *Let the components q_{ik} $(i,k = 1,\dots,n)$ and g_{jik} $(i,k = 1,\dots,n; j = 1,2,\dots)$ of the matrix-function Q and of the constant matrices G_j $(j = 1,2,\dots)$, respectively, satisfy the conditions*

$$1 + g_{jii} \neq 0 \quad (i = 1,\dots,n; \quad j = 1,2,\dots), \tag{4}$$

$$\begin{gathered}\int_{t^*}^{t} \exp\Big(\int_{\tau}^{t} q_{ii}(s)ds\Big)|q_{ik}(\tau)| \prod_{\tau\leq t_j<t} |1 + g_{jii}|d\tau + \\ + \sum_{t^*\leq t_l<t} \exp\Big(\int_{t_l}^{t} q_{ii}(s)ds\Big)|g_{lik}| \prod_{t_l<t_j<t} |1 + g_{jii}| \leq h_{ik} \\ \textit{for} \quad t \geq t^* \quad (i \neq k; \quad i,k = 1,\dots,n; \quad j = 1,2,\dots)\end{gathered} \tag{5}$$

and

$$\sup\bigg\{\int_0^t q_{ii}(s)ds + \sum_{0\leq t_j<t} \ln|1 + g_{jii}| : t \geq t^*\bigg\} < +\infty \quad (i = 1,\dots,n),$$

where t^* and $h_{ik} \in \mathbb{R}_+$ $(i \neq k;\ i,k = 1,\dots,n)$. Let, moreover, the constant matrix $H = (h_{ik})_{i,k=1}^n$, where $h_{ii} = 0$ $(i = 1,\dots,n)$, be such that

$$r(H) < 1. \tag{6}$$

Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ is stable.

Theorem 2. *Let the components* q_{ik} $(i,k=1,\dots,n)$ *and* g_{jik} $(i,k=1,\dots,n; j=1,2,\dots)$ *of the matrix-function* Q *and of the constant matrices* G_j $(j=1,2,\dots)$, *respectively, satisfy the conditions* (4), (5) *and*

$$\sup\left\{\int\limits_{\tau}^{t} q_{ii}(s)ds+\sum_{\tau\le t_j<t}\ln|1+g_{jii}|: t\ge\tau\ge 0\right\}<+\infty \quad (i=1,\dots,n),$$

where t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k; i,k=1,\dots,n)$. *Let, moreover, the constant matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfy the condition* (6). *Then the pair* $(Q,\{G_j\}_{j=1}^{+\infty})$ *is uniformly stable.*

Corollary 1. *Let the components* q_{ik} $(i,k=1,\dots,n)$ *and* g_{jik} $(i,k=1,\dots,n; j=1,2,\dots)$ *of the matrix-function* Q *and of the constant matrices* G_j $(j=1,2,\dots)$, *respectively, satisfy the conditions*

$$-1<g_{jii}\le 0 \quad (i=1,\dots,n;\quad j=1,2,\dots), \tag{7}$$

$$q_{ii}(t)\le 0 \quad (i=1,\dots,n), \tag{8}$$

$$|q_{ik}(t)|\le -h_{ik}q_{ii}(t) \quad (i\ne k;\quad i,k=1,\dots,n)$$

and

$$|g_{jik}|\le -h_{ik}g_{jii} \quad (i\ne k;\quad i,k=1,\dots,n;\quad j=1,2,\dots)$$

almost everywhere on $[t^*,+\infty[$ *for some* $t^*\in\mathbb{R}_+$, *where* $h_{ik}\in\mathbb{R}_+$ $(i\ne k; i,k=1,\dots,n)$. *Let, moreover, the constant matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfy the condition* (6). *Then the pair* $(Q,\{G_j\}_{j=1}^{+\infty})$ *is uniformly stable.*

Theorem 3. *Let the components* q_{ik} $(i,k=1,\dots,n)$ *and* g_{jik} $(i,k=1,\dots,n; j=1,2,\dots)$ *of the matrix-function* Q *and of the constant matrices* G_j $(j=1,2,\dots)$, *respectively, satisfy the condition* (4),

$$\int\limits_{t^*}^{t} q_{ii}(s)ds+\sum_{t^*\le t_j<t}\ln|1+g_{jii}|\le-\xi(t)+\xi(t^*) \quad \textit{for}\quad t\in[t^*,+\infty[\setminus T \quad (i=1,\dots,n)$$

and

$$\int\limits_{t^*}^{t}\exp\left(\xi(t)-\xi(\tau)+\int\limits_{\tau}^{t}q_{ii}(s)ds\right)|q_{ik}(\tau)|\prod_{\tau\le t_j<t}|1+g_{jii}|d\tau+$$

$$+\sum_{t^*\le t_l<t}\exp\left(\xi(t)-\xi(t_l)+\int\limits_{t_l}^{t}q_{ii}(s)ds\right)|g_{lik}|\prod_{t_l<t_j<t}|1+g_{jii}|\le h_{ik}$$

$$\textit{for}\quad t\in[t^*,+\infty[\setminus T \quad (i\ne k;\quad i,k=1,\dots,n),$$

where t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k;\ i,k=1,\dots,n)$. *Let, moreover, the constant matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfy the condition* (6), *and* $\xi\in\widetilde{C}_{\rm loc}(\mathbb{R}_+\setminus T;\mathbb{R}_+)$ *be the continuous from the left nondecreasing function satisfying the condition* (3). *Then the pair* $(Q,\{G_j\}_{j=1}^{+\infty})$ *is asymptotically stable.*

Corollary 2. *Let the components* q_{ik} $(i,k=1,\dots,n)$ *and* g_{jik} $(i,k=1,\dots,n; j=1,2,\dots)$ *of the matrix-function* Q *and of the constant matrices* G_j $(j=1,2,\dots)$, *respectively, satisfy the conditions* (7), (8),

$$|q_{ik}(t)|\le -h_{ik}q_{ii}(t) \quad (i\ne k;\quad i,k=1,\dots,n) \tag{9}$$

and

$$|g_{jik}|\le -h_{ik}g_{jii}(1+g_{jii}) \quad (i\ne k;\quad i,k=1,\dots,n;\quad j=1,2,\dots) \tag{10}$$

almost everywhere on $[t^*,+\infty[$, *where* $t^*\in\mathbb{R}_+$, *the numbers* $h_{ik}\in\mathbb{R}_+$ $(i\ne k;\ i,k=1,\dots,n)$ *are such that the constant matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfies the condition* (6). *Let,moreover, there exist a function* $a_0 \in L_{\rm loc}(\mathbb{R}_+;\mathbb{R})$ *such that*

$$a_0(t) - a_0(\tau) \le \min\left\{\left|\int\limits_\tau^t q_{ii}(s)ds + \sum_{\tau\le t_j<t} \ln(1+g_{jii})\right| : i=1,\dots,n\right\} \quad for \quad t \ge \tau \ge t^*$$

and

$$\lim_{t\to+\infty} a_0(t) = +\infty.$$

Then the pair $(Q, \{G_j\}_{j=1}^{+\infty})$ *is both asymptotically and uniformly stable.*

Corollary 3. *Let the components* q_{ik} $(i,k=1,\dots,n)$ *and* g_{jik} $(i,k=1,\dots,n;$ $j=1,2,\dots)$ *of the matrix-function* Q *and of the constant matrices* G_j $(j=1,2,\dots)$, *respectively, satisfy the conditions* $(7)-(10)$, *where* $t^* \in \mathbb{R}_+$, *and the numbers* $h_{ik} \in \mathbb{R}_+$ $(i \ne k;\ i,k=1,\dots,n)$ *are such that the constant matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfies the condition (6). Let, moreover,*

$$\int\limits_0^{+\infty} \eta(s)ds + \sum_{0\le t_j<+\infty} \ln(1+\eta_j) = -\infty,$$

where $\eta(t) = \max\{q_{ii}(t) : i=1,\dots,n\}$, $\eta_j(t) = \max\{g_{jii} : i=1,\dots,n\}$ $(j=1,2,\dots)$. *Then the conclusion of Corollary* 2 *is true.*

Theorem 4. *Let the components* q_{ik} $(i,k=1,\dots,n)$ *and* g_{jik} $(i,k=1,\dots,n; j=1,2,\dots)$ *of the matrix-function* Q *and of the constant matrices* G_j $(j=1,2,\dots)$, *respectively, satisfy the conditions* (4),

$$\sup\left\{(\xi(t)-\xi(\tau))^{-1}\left(\int\limits_\tau^t q_{ii}(s)ds + \sum_{\tau\le t_j<t} \ln|1+g_{jii}|\right) : t\ge\tau\ge t^*,\right.$$
$$\left.\xi(t)\ne\xi(\tau);\quad t,\tau\in\mathbb{R}_+\backslash T\right\} < -\gamma \quad (i=1,\dots,n) \tag{11}$$

and

$$\int\limits_{t^*}^t \exp\left(\gamma(\xi(t)-\xi(\tau)) + \int\limits_\tau^t q_{ii}(s)ds\right)|q_{ik}(\tau)| \prod_{\tau\le t_j<t} |1+g_{jii}|d\tau +$$
$$+\sum_{t^*\le t_l<t} \exp\left(\gamma(\xi(t)-\xi(t_l)) + \int\limits_{t_l}^t q_{ii}(s)ds\right)|g_{lik}| \prod_{t_l<t_j<t} |1+g_{jii}| \le h_{ik}$$
$$for \quad t\in[t^*,+\infty[\backslash T \quad (i\ne k;\quad i,k=1,\dots,n),$$

where $\gamma>0$, t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k; i,k=1,\dots,n)$, *the constant matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfies the condition* (6), *and* $\xi\in\widetilde{C}_{\rm loc}(\mathbb{R}_+\backslash T;\mathbb{R}_+)$ *is the continuous from the left nondecreasing function satisfying the condition* (3). *Then the pair* $(Q,\{G_j\}_{j=1}^{+\infty})$ *is* ξ*-exponentially asymptotically stable.*

Corollary 4. *Let the components* q_{ik} $(i,k=1,\dots,n)$ *and* g_{jik} $(i,k=1,\dots,n; j=1,2,\dots)$ *of the matrix-function* Q *and of the constant matrices* G_j $(j=1,2,\dots)$, *respectively, satisfy the conditions* $(7)-(11)$ *almost everywhere on* $[t^*,+\infty[$, *where* $\gamma>0$, t^* *and* $h_{ik}\in\mathbb{R}_+$ $(i\ne k;\ i,k=1,\dots,n)$. *Let, moreover, the constant matrix* $H=(h_{ik})_{i,k=1}^n$, *where* $h_{ii}=0$ $(i=1,\dots,n)$, *satisfy the condition* (6), *and* $\xi\in\widetilde{C}_{\rm loc}(\mathbb{R}_+\backslash T;\mathbb{R}_+)$ *be the continuous from the left nondecreasing function satisfying the condition* (3). *Then the pair* $(Q,\{G_j\}_{j=1}^{+\infty})$ *is* ξ*-exponentially asymptotically stable.*

These results immediately follow from the analogous results given in [3] for the system of so called generalized linear ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \quad \text{for} \quad t \in \mathbb{R}_+, \tag{12}$$

where $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}_+ \to \mathbb{R}^n$ are, respectively, matrix and vector-functions with bounded variation components on every closed interval from $\mathbb{R}_+$ (see [5]).

Under a solution of the system (12) we understand a vector-function $x : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ with bounded variation components on every closed interval from $\mathbb{R}_+$, such that

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \quad \text{for} \quad 0 \le t \le s,$$

where the integral is understood in Lebesgue–Stiltjes sense.

The system (1), (2) is a particular case of the system (12) if we assume

$$A(t) \equiv \int_0^t Q(\tau)d\tau + \sum_{0\le t_j<t} G_j, \quad f(t) \equiv \int_0^t q(\tau)d\tau + \sum_{0\le t_j<t} g_j.$$

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