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BOHL EXPONENTS OF LINEAR DIFFERENTIAL SYSTEMS

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1. Introduction. Upper Bohl exponents of both solutions and sets of solutions to differential systems were introduced as indices by Bohl in his basic work [1]. In that work he proved that any solution to a system of differential equations satisfying some conditions in a neighborhood of the given solution is stable under permanent perturbations if and only if the system has negative upper general exponent ($\equiv$ upper Bohl exponent of the whole totality of solutions) in variations of the system along the considered solution. Later but independently, studying the notions of uniform and uniform asymptotic stability, Persidskiĭ came to the notion of the upper general exponent. It appeared that the necessary and sufficient condition for uniform asymptotic stability of a linear differential system was negativeness of its upper general exponent [2], and the necessary and sufficient condition for uniform stability by the first linear approximation was negativeness of the upper general exponent of the linear approximation system [3].

Upper Bohl exponents play to some extent the same role with respect to the notion of uniform stability as Lyapunov exponents do with respect to the notion of stability. The necessity of building up an analogue of Lyapunov exponents theory for Bohl exponents is imposed not only by that reason, but also by the fact that Bohl exponents carry an important information about solutions' behavior which could be used in Lyapunov exponents theory as well. Thus Bohl exponents were applied by Bylov in his theory of almost reducibility [4, 5] and by Millionshchikov in his investigations of linear differential systems with almost periodic or uniformly continious coefficients [6–8], relations of almost reducibility [9], and properties of systems with integral separation [10].

The review includes results obtained in the Bohl exponents theory.

2. Definitions and primary properties. Consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

with the piecewise continious coefficient matrix $A(\cdot) : [0, +\infty) \to \ \mathrm{End}\,\mathbb{R}^n$. We denote the class of all such systems by $\mathcal{M}_n$, and by $\mathcal{D}_n$ we denote its subclass consisting of diagonal systems (we shall write $A \in \mathcal{M}_n$ identifying system (1) with its coefficient matrix). Let $X_A(\cdot; \cdot)$ be the Cauchy matrix (operator) of the system (1). The upper and lower Bohl exponents of of a nonzero solution to system (1) are defined, respectively, by the formulas [1; 11, 171–172]:

$$\overline{\beta}[x] \overset{\text{def}}{=} \varlimsup_{t-\tau\to+\infty} \frac{1}{t-\tau}\ln\frac{\|x(t)\|}{\|x(\tau)\|} \ \text{ and }\ \underline{\beta}[x] \overset{\text{def}}{=} \varliminf_{t-\tau\to+\infty} \frac{1}{t-\tau}\ln\frac{\|x(t)\|}{\|x(\tau)\|}, \tag{2}$$

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and the quantities

$$\Omega^0(A) \overset{\text{def}}{=} \overline{\lim_{t-\tau \to +\infty}} \frac{1}{t-\tau} \ln \|X_A(t;\tau)\|,$$

$$\omega_0(A) \overset{\text{def}}{=} -\overline{\lim_{t-\tau \to +\infty}} \frac{1}{t-\tau} \ln \|X_A^{-1}(t;\tau)\| \tag{3}$$

are called, respectively, the senior upper and the junior lower general exponents of the system (1). Replacing the upper limits in the formulas (3) by the lower ones, we obtain, respectively, the definition of the senior lower $\Omega_0(A)$ and the junior upper $\omega^0(A)$ general exponents. The relation between the Bohl exponents and the general ones is described in [4] by Theorem 10. It should be noted that in comparison with [11] the limits in definitions (2) and (3) are lacking an additional condition $\tau \to +\infty$. It is easy to show using the boundedness of the coefficient matrix that adding this condition doesn't change the values of the corresponding exponents. Since the coefficient matrix of the system (1) is bounded, the Bohl and the general exponents are finite.

The exponents (2) determine the mappings $\overline{\beta_A}$ and $\underline{\beta}_A$ from $\mathbb{R}^n \setminus \{\mathbf{0}\}$ to $\mathbb{R}$ acting by the rule: $\overline{\beta_A}(\alpha) = \overline{\beta}[x(\cdot;\alpha)]$ and $\underline{\beta}_A = \underline{\beta}[x(\cdot;\alpha)]$, with $x(\cdot;\alpha)$ being a solution to system (1) with initial vector $\alpha = x(0;\alpha)$. We will call the functions $\overline{\beta_A}$ and $\underline{\beta}_A$ the upper and lower Bohl functions of the system (1), respectively, and the vector-function $\beta_A \overset{\text{def}}{=} (\underline{\beta}_A, \overline{\beta}_A) : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^2$ – the Bohl vector-function of the system (1).

We will mention below some primary properties of the Bohl exponents (2) and (3) following from their definition.

$\mathbf{1^0}$. Definitions (2) could be otherwise rewritten using passages to the limits by one variable as:

$$\overline{\beta}[x] \overset{\text{def}}{=} \overline{\lim_{T \to +\infty}} \frac{1}{T} \sup_{\Delta \geq 0} \ln \frac{\|x(T+\Delta)\|}{\|x(\Delta)\|} \quad \text{and} \quad \underline{\beta}[x] \overset{\text{def}}{=} \underline{\lim_{T \to +\infty}} \frac{1}{T} \inf_{\Delta \geq 0} \ln \frac{\|x(T+\Delta)\|}{\|x(\Delta)\|}.$$

The same is true with respect to the general exponents.

$\mathbf{2^0}$. The Bohl exponents and the general exponents are invariant under Lyapunov transformations.

The Bohl exponents as Lyapunov transformations' invariants can be included into the common theory of asymptotic invariants for linear differential systems [12], the necessity of developement of which was emphasized by Bogdanov.

$\mathbf{3^0}$. As norms are equivalent norms in finite dimensional spaces, the values $\overline{\beta}[x]$ and $\underline{\beta}[x]$ don't depend on the choice of a norm. Regardless to the triviality of this statement it is very useful for evaluations of the Bohl exponents of concrete solutions as the successfull choice of a norm essentially simplifies the evaluation.

$\mathbf{4^0}$. The arguments t and τ in the formulas (2) and (3) may be assumed to be positive integers. The same is true respectively to the general exponents. The above statement is a simple implication of the boundedness of the coefficient matrix.

$\mathbf{5^0}$. Let us accept some notation in order to formulate the following property . Let $\{\delta_k\}_{k \in \mathbb{N}}$ be a sequence of nonnegative numbers monotone increasing to $+\infty$ such that $\delta_1 = 0$ and $\delta_{k+1} - \delta_k \to +\infty$ as $k \to +\infty$. Let $\Delta_k \overset{\text{def}}{=} [\delta_k, \delta_{k+1})$, $k \in \mathbb{N}$. We will write $t \sim \tau$ if t and τ belong to the same interval Δ_k for some k. Then

$$\overline{\beta}[x] = \overline{\lim_{\substack{t-\tau \to +\infty \\ t \sim \tau}}} \frac{1}{t-\tau} \ln \frac{\|x(t,\alpha)\|}{\|x(\tau,\alpha)\|} \quad \text{and} \quad \underline{\beta}[x] = \underline{\lim_{\substack{t-\tau \to +\infty \\ t \sim \tau}}} \frac{1}{t-\tau} \ln \frac{\|x(t,\alpha)\|}{\|x(\tau,\alpha)\|}.$$

An analogous statement concerning the general exponents is also valid.

3. Description of the functions $\overline{\beta}_A$ and $\underline{\beta}_A$ and their ranges.

3.1. What are the functions $\overline{\beta}_A$ and $\underline{\beta}_A$ and their ranges ?

The answer to that question being restricted to the class $\mathcal{D}_n$ of diagonal systems is given by Theorems 1 and 2; more precisely, those theorems give a complete description of the following classes of functions

$$\overline{\mathcal{D}}_n \overset{\text{def}}{=} \{\overline{\beta}_A : A \in \mathcal{D}_n\} \quad \text{and} \quad \underline{\mathcal{D}}_n \overset{\text{def}}{=} \{\underline{\beta}_A : A \in \mathcal{D}_n\}.$$

Let $I(\xi)$ be the set of numbers $i \in \{1, \dots, n\}$ of the nonzero components of a vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

Theorem 1 ([13]). *A function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ belongs to the class $\overline{\mathcal{D}}_n$ if and only if it satisfies the following conditions:*

1) $f(\xi) = f(\eta)$ *if* $I(\xi) = I(\eta)$;

2) *for any* $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *there exists a number* $i_\xi \in I(\xi)$ *such that for any* $\eta \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *with* $I(\eta) \subset I(\xi)$ *and* $i_\xi \in I(\eta)$, *the inequality* $f(\xi) \leq f(\eta)$ *holds.*

Theorem 2 ([13]). *A function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \in \mathbb{R}$ belongs to the class $\overline{\mathcal{D}}_n$ if and only if it satisfies the following conditions:*

1) $f(\xi) = f(\eta)$ *if* $I(\xi) = I(\eta)$;

2) *for any* $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *there exists a number* $i_\xi \in I(\xi)$ *such that for any* $\eta \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *with* $I(\eta) \subset I(\xi)$ *and* $i_\xi \in I(\eta)$, *the inequality* $f(\xi) \geq f(\eta)$ *holds.*

Theorems 1 and 2 imply an important corollary directly proved in [14].

Corollary 1 ([14]). *The range of the upper (respectively, lower) Bohl function of a diagonal system (1) has no more than $2^n - 1$ different elements.*

For any positive integers n and $m \leq 2^n - 1$ there exists a system from $\mathcal{D}_n$ such that the range of its function $\overline{\beta}_A$ (respectively, $\underline{\beta}_A$) contains exactly m different elements.

A complete description of the function classes

$$\overline{\mathcal{B}}_n \overset{\text{def}}{=} \{\overline{\beta}_A : A \in \mathcal{M}_n\} \quad \text{and} \quad \underline{\mathcal{B}}_n \overset{\text{def}}{=} \{\underline{\beta}_A : A \in \mathcal{M}_n\}$$

is given by the following two theorems.

Theorem 3 ([15]). *A function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ belongs to the class $\overline{\mathcal{B}}_n$ if and only if it satisfies the following three conditions:*

1) *the function f is bounded;*

2) $f(\alpha) = f(r\alpha)$ *for any nonzero* $r \in \mathbb{R}$ *and any* $\alpha \in \mathbb{R}^n \setminus \{\mathbf{0}\}$;

3) *for any* $q \in \mathbb{R}$ *the Lebesgue set* $[f \geq q]$ *of f is a G_δ–set.*

Theorem 4 ([15]). *A function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ belongs to the class $\underline{\mathcal{B}}_n$ if and only if it satisfies the following three conditions:*

1) *the function f is bounded;*

2) $f(\alpha) = f(r\alpha)$ *for any nonzero* $r \in \mathbb{R}$ *and any* $\alpha \in \mathbb{R}^n \setminus \{\mathbf{0}\}$;

3) *for any* $q \in \mathbb{R}$ *the Lebesgue set* $[f > q]$ *of f is a F_σ–set.*

Theorems 3 and 4 allow us to find easily the range structure for the Bohl exponents of the system (1).

Corollary 2 ([15]). *The set is the range of the upper (respectively, lower) exponent of some system from the class $\mathcal{M}_n$ if and only if it is a bounded Souslin set of the real axis.*

Note that although the upper Bohl exponent is an analogue of the characteristic Lyapunov exponent, their properties differ essentially, as it is shown in Theorem 3 and Corollary 2.

3.2. As well as $\|X_A(t,\tau)\| = \max \left\{ \dfrac{\|x_A(t;\alpha)\|}{\|x_A(\tau;\alpha)\|} : \alpha \in \mathbb{R}^n \setminus \{\mathbf{0}\} \right\}$, the following inequalities are obvious due to the definitions (3) of the upper and lower general exponents:

$$\Omega^0(A) \geq \sup\{\overline{\beta}_A(\alpha) : \alpha \in \mathbb{R}^n \setminus \{\mathbf{0}\}\} \quad \text{and} \quad \omega_0(A) \leq \inf\{\underline{\beta}_A(\alpha) : \alpha \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}.$$

The following theorems show that the first (respectively, second) of those inequlities and its corollaries completely describe the relation between the upper (respectively, lower) general and upper (respectively, lower) Bohl exponents for all the systems of the class $\mathcal{M}_n$.

Theorem 5 ([15]). *For any pair (b, β), where $\beta : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ is a function satisfying the conditions 1)–3) of Theorem 3 and b is a number satisfying the inequality $b \geq \sup\{\beta(\alpha) : \alpha \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}$, there exists a system (1) such that $\Omega^0(A) = b$ for its upper general exponent, and $\overline{\beta}_A \equiv \beta$ for its upper Bohl exponent.*

Theorem 6 ([15]). *For any pair (b, β), where $\beta : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ is a function satisfying the conditions 1)–3) of Theorem 4 and b is a number satisfying the inequality $b \leq \inf\{\beta(\alpha) : \alpha \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}$, there exists a system (1) such that $\omega_0(A) = b$ for its lower general exponent, and $\underline{\beta}_A \equiv \beta$ for its lower Bohl exponent.*

Statements similar to theorems 5 and 6 hold also for the senior lower $\Omega_0(A)$ and the junior upper $\omega^0(A)$ general exponents.

3.3. Theorems 3 and 4 give the complete description of the upper and lower Bohl functions classes, respectively, considered separately. A complete description of the class of the Bohl vector functions $\beta_A = (\underline{\beta}_A, \overline{\beta}_A)$ of both general ($A \in \mathcal{M}_n$) and diagonal ($A \in \mathcal{D}_n$) systems remains unknown, and only partial results on that problem are obtained. In particular, the following theorem completely describes the range of the Bohl vector function of the general system.

Theorem 7 ([16]). *A set $\mathcal{B} \subset \mathbb{R}^2$ is the range of the Bohl vector function of some system (1) if and only if for some $q \in \{1, \dots, n\}$ the set $\mathcal{B}$ can be represented as the union of q separated sets $\mathcal{B}_i$, $i = 1, \dots, q$, such that their projections $pr_k\mathcal{B}_i$ to each component ($k = 1, 2$) are bounded Souslin sets and $\inf pr_2\mathcal{B}_i < \sup pr_1\mathcal{B}_{i+1}$ for every $i \in \{1, \dots, q-1\}$; moreover, if $q > [n/2]$, then the above representation can be chosen so that there are at least $2q - n$ singletons among those q sets.*

It is interesting to note that the functions $\underline{\beta}_A$ and $\overline{\beta}_A$, considered separately, are, as follows from Theorems 3 and 4, just specific functions of the second Baire class and neither of them determines any linear structure on $\mathbb{R}^n$, but when considered together, they determine some flag on $\mathbb{R}^n$ as well as the Lyapunov exponent does. To be more precise, [16], every Bohl vector function β_A naturally defines some flag $\mathcal{F} : \{\mathbf{0}\} \equiv L_0 \subset L_1 \subset \dots \subset L_{q-1} \subset\subset L_q \equiv \mathbb{R}^n$ on $\mathbb{R}^n$ such that for every $i \in \{2, \dots, q\}$ there is a vector $\alpha_0^i \in L_i \setminus L_{i-1}$ and there exists a basis $\{\alpha_1^i, \dots, \alpha_{d(i)}^i\}$, where $d(i) = \dim L_{i-1}$, of the lineal L_{i-1} for which the inequalities $\overline{\beta}_A(\alpha_l^i) < \underline{\beta}_A(\alpha_0^i)$ ($l = 1, \dots, d(i)$) hold. Then the sets $\mathcal{B}_i = \beta_A(L_i \setminus L_{i-1})$ can be chosen as the sets $\mathcal{B}_i$ from Theorem 7. Let $L_i = L_{i-1} \bigoplus L_i^*$ for each of the lineals L_i, $i \in \{2, \dots, q\}$, of the flag $\mathcal{F}$. Then an equality $\beta_A(\alpha) = \beta_A(\alpha + \varepsilon)$ holds for any vector $\varepsilon \in L_{i-1}$ and for almost all (in the sense of $\ln^\nu |\ln|$- Hausdorf measure, where the constant ν can be taken of any value less than -1) vectors $\alpha \in L_i^*$ [16].

4. The general and Bohl exponents under the coefficient matrix perturbations.

4.1. We will consider the system (1) and the perturbed system

$$\dot{y} = (A(t) + Q(t))y, \quad y \in \mathbb{R}^n, \quad t \geq 0,$$

with piecewise continious matrix $Q(\cdot) : [0, +\infty) \to \operatorname{End}\mathbb{R}^n$ satisfying some conditions of smallness to be defined below. It is proved [11, c. 180] that the upper general exponent $\Omega^0(A)$ cannot strongly increase and the lower general exponent $\omega_0(A)$ cannot strongly decrease under a small (by norm) variation of the matrix A, i. e.:

Theorem 8 ([11]). *The equalities*

$$\lim_{\varepsilon \to 0} \sup_{\|Q\| \leq \varepsilon} \Omega^0(A + Q) = \Omega^0(A) \quad and \quad \lim_{\varepsilon \to 0} \inf_{\|Q\| \leq \varepsilon} \omega_0(A + Q) = \omega_0(A).$$

are valid.

Analogous relations are valid for the senior lower $\Omega_0(A)$ and junior upper $\omega^0(A)$ general exponents, respectively.

At the same time, the exponents $\Omega^0(A)$ and $\omega_0(A)$ may undergo a constant jump downwards and upwards, respectively, under an arbitrarily small variation of, the matrix A.

Theorem 9 ([9]). *There exists such system* (1) *that for every $\varepsilon > 0$ a matrix $Q_\varepsilon(t)$ can be found satisfying the estimate $\sup\limits_{t \geq 0} Q_\varepsilon(t) \leq \varepsilon$, so that the relations*

$$\Omega^0(A) - \Omega^0(A + Q_\varepsilon) > \delta \quad and \quad \omega_0(A + Q_\varepsilon) - \omega_0(A) > \delta$$

are valid for any (independent of ε) number $\delta > 0$.

This important fact was established by Millionshchikov [9] was applied by him for the proof of nonsymmetry of the relation of almost reducibility of linear differential systems.

Rachimberdiev [17, 18], using Cauchy matrix, carried out evaluations of the upper and lower general exponents under small perturbations of the coefficient matrix, i. e. of the values

$$\lim_{\varepsilon \to 0} \sup_{\|Q\| \leq \varepsilon} \Omega^0(A + Q) \quad and \quad \lim_{\varepsilon \to 0} \inf_{\|Q\| \leq \varepsilon} \omega_0(A + Q).$$

We will not state Rachimberdiev's theorem here as we would need a whole range of definitions and notation.

Nevertheless, the general exponents of the system (1) are stable under decreasing to zero (as $t \to +\infty$) perturbations of the coefficient matrix [11, p. 181], i. e.

$$\Omega^0(A + Q) = \Omega^0(A) \quad and \quad \omega_0(A + Q) = \omega_0(A),$$

if $\|Q(t)\| \to 0$ as $t \to +\infty$. An analogous statement for the senior lower $\Omega_0(A)$ and junior upper $\omega^0(A)$ general exponents is also valid.

4.2. Now consider the question on the behavior of the Bohl exponents of the system (1) under its coefficient matrix perturbations. Vinograd [19] proved a theorem establishing relations between the Bohl and general exponents.

Theorem 10 ([19]). *The relations*

$$\Omega^0(A) = \lim_{\varepsilon \to 0} \sup_{\|Q\| \leq \varepsilon} \overline{\beta}(A + Q), \quad \omega^0(A) = \lim_{\varepsilon \to 0} \inf_{\|Q\| \leq \varepsilon} \overline{\beta}(A + Q),$$

$$\Omega_0(A) = \lim_{\varepsilon \to 0} \sup_{\|Q\| \leq \varepsilon} \underline{\beta}(A + Q), \quad \omega_0(A) = \lim_{\varepsilon \to 0} \inf_{\|Q\| \leq \varepsilon} \underline{\beta}(A + Q)$$

are valid.

The first of the above relations is proved in [19], the others can be proved similarly.

We denote the extreme Bohl exponents of the system (1) – the senior: upper $\overline{\beta}_1(A) \overset{\text{def}}{=} \sup \overline{\beta}[x]$ and lower $\underline{\beta}_1(A) \overset{\text{def}}{=} \sup \underline{\beta}[x]$, and the junior: upper $\overline{\beta}_0(A) \overset{\text{def}}{=} \inf \overline{\beta}[x]$ and lower $\underline{\beta}_0(A) \overset{\text{def}}{=} \inf \underline{\beta}[x]$ (sup and inf are evaluated by all nonzero solutions to the system (1)). It is well known that upwards and downwards unstability of the (extreme) characteristic Lyapunov exponents under decreasing to zero (as $t \to +\infty$) perturbations of the coefficient matrix is achieved even on diagonal systems (1). The following simple statement shows that the above property is not valid for the extreme (upper and lower) Bohl exponents of a diagonal system (1).

Theorem 11 ([20]). *The senior upper and lower Bohl exponents of a diagonal system* (1) *are upwards stable and the junior ones are downwards stable under the decreasing to zero* (*as $t \to +\infty$*) *coefficient matrix perturbations, i. e.,*

$$\overline{\beta}_1(A+Q) \leq \overline{\beta}_1(A), \quad \underline{\beta}_1(A+Q) \leq \underline{\beta}_1(A),$$

and

$$\overline{\beta}_0(A+Q) \geq \overline{\beta}_0(A), \quad \underline{\beta}_0(A+Q) \geq \underline{\beta}_0(A)$$

for diagonal matrix $A(\cdot)$ and any matrix $Q(\cdot)$ with $\|Q(t)\| \to 0$ as $t \to +\infty$.

Nevertheless, it is easy to bring examples showing that the extreme Bohl exponents of the diagonal system (1) can be unstable under decreasing to zero (as $t \to +\infty$) perturbations of the coefficient matrix in the direction opposite to the corresponding direction from Theorem 11.

As a consequence of Theorem 11, the following question naturally arises. Can the extreme Bohl exponents at all vary in the corresponding directions determined in Theorem 11 under decreasing to zero (as $t \to +\infty$) perturbations of the coefficient matrix? Considering the whole class of decreasing to zero (as $t \to +\infty$) perturbations, the positive answer to the stated question trivially follows from Theorem 10 and Theorems 5 and 6. The answer stays positive even if we restrict the class of decreasing to zero (as $t \to +\infty$) perturbations to an important in asymptotic theory class of exponentially decreasing perturbations [20]. It means in particular that a quantity $\overline{B}_\sigma(A) \overset{\text{def}}{=} \sup \left\{ \overline{\beta}_1(A+Q) : \|Q(t)\| \leq \text{const} \cdot \exp(-\sigma t) \right\}$ being an analogue of the senior σ-exponent $\nabla_\sigma(A)$ [21] is in general a nonconstant function $\sigma > 0$. By analogy with the corresponding property of the σ-exponent the exponent $\overline{B}_\sigma(A) \equiv \overline{\beta}_1(A)$ for all big enough σ. That fact directly follows from a theorem [22, 23] on kinematic similarity of the systems (1) and (2) if $\|Q(t)\| \leq \text{const} \cdot \exp(-\sigma t)$ and σ is big enough. Nevertheless, the following theorem shows that in all other respects, the properties of the function $\overline{B}_\sigma(A)$ essentially differ from those [24] of the function $\nabla_\sigma(A)$.

Theorem 12 ([20]). *There exists a system* (1) *for which the function $\overline{B}_\sigma(A)$ is a different from $\overline{\beta}_1(A)$ constant with respect to all suffuciently small σ.*

To conclude the review, we note that we haven't stated unsolved problems as far as they could be easily found comparing the Bohl and Lyapunov exponents theory.

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