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A MAXIMUM PRINCIPLE AND THE IMPLICIT SIGNORINI PROBLEM

Abstract. In the present paper is given the analogue of the maximum principle for a scalar, linear eliptic equation, in coercive case (Lemma 1.4). The result is applied to locate the set of coincidence in the classical problem of Signorini for some concrete cases (Corollaries 1.6–1.8) and also, for the formulation of the maximum principle for the same problem (Theorem 1.5). An implicit Signorini problem was studied earlier by Bensoussan and Lions. They investigated the mentioned problem, proved existence, but the uniqueness result was still open. From the above mentioned results are derived uniqueness of a solution under asserted conditions. If some of asserted conditions is missing, the existence might fail; in particular, there are found a system of data, under which the problem has no solution at all. Next we state more general Siniorini's Implicit problem. In some cases, there is proved uniqueness of solution and is given a sufficient condition of solvability of the problem (Theorem 3.1). Further, is consider the implicit Signorini problem in elasticity with the Diriclet and the Neumann boundary conditions (Problem (4.20)–(4.21)). Existence of solution and, in some cases, also uniqueness is proved (Theorem 4.4). In general, uniqueness of solution, can equivalently be reduced to some assumption, similar to "maximum principle" (Lemma 1.4), of the theory of elasticity.

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1. A Maximum Principle in the Classical Signorini Problem

Let Ω be a bounded domain in $\mathbb{R}^n$, Γ be the boundary of $\Omega : \Gamma = \partial\Omega$, $\Gamma \in C^2, \nu$ be the outward normal to Ω; and $H^s(\Omega)$ and $H^s(\Gamma)$ be real Sobolev spaces. The norm in these spaces will be denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\Gamma}$, respectively. Define the bilinear form on the space $H^1(\Omega)\times H^1(\Omega)$ as follows:

$$a(u,v) = \sum_{i,j=1}^{n} \int\limits_{\Omega} a_{ij}(u,v)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx + \sum_{i=1}^{n}\int\limits_{\Omega} a_i \frac{\partial u}{\partial x_i}dx + \int\limits_{\Omega} a_0 uvdx,$$

$$a_{ij}, a_i, a_0 \in L^\infty(\mathbb{R}^n), \quad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha_0|\xi|^2, \tag{1.1}$$

$$a_0(x) \geq a^0, \quad a^0 = \text{const} > 0.$$

Suppose that the form $a(u,v)$ is coercive:

$$a(u,u) \geq \alpha||u||^2_{1,\Omega}, \quad u \in H^1(\Omega). \tag{1.2}$$

Define the following operators:

$$A(x,\partial) = -\sum_{ij=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{n} a_i\frac{\partial}{\partial x_i} + a_0,$$

$$\frac{\partial}{\partial \nu_A} = \sum_{ij=1}^{n} a_{ij}\nu_j\frac{\partial}{\partial x_i}. \tag{1.3}$$

As it is known, if $u \in H^1(\Omega),\ Au \in L_2(\Omega)$, then $\dfrac{\partial u}{\partial \nu_A} \in H^{-\frac{1}{2}}(\Gamma)$ and the following Green formula is true:

$$a(u,v) = \left\langle \frac{\partial u}{\partial \nu_A}, v\right\rangle_\Gamma + \int\limits_\Omega Auvdx. \tag{1.4}$$

Here $\langle\cdot\,,\cdot\rangle_\Gamma$ is the relation of duality between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$.

Let us now formulate some definitions.

Let $F \in H^{-\frac{1}{2}}(\Gamma)$ and $\Gamma_0 \subset \Gamma$ be any measurable subset of the boundary, $\operatorname{mes}\Gamma_0 > 0$.

Definition. (i) We say that $F|_{\Gamma_0} \geq 0$ if

$$\langle F,\varphi\rangle_\Gamma \geq 0, \qquad \varphi \in H^{\frac{1}{2}}(\Gamma),\ \varphi \geq 0,\ \varphi|_{\Gamma\setminus\Gamma_0} = 0.$$

In the sequel we will write $F \geq 0$ when $\Gamma = \Gamma_0$.

(ii) We say $F|_{\Gamma_0} = 0$ if

$$\langle F,\varphi\rangle_\Gamma = 0, \qquad \varphi \in H^{\frac{1}{2}}(\Gamma),\ \varphi|_{\Gamma\setminus\Gamma_0} = 0.$$

The classical Signorini problem is defined as follows: find $u \in H^1(\Omega)$ such that

$$\begin{aligned} &Au = f, \qquad f \in L_2(\Omega);\\ &u|_\Gamma \geq h, \qquad \left.\frac{\partial u}{\partial \nu_A}\right|_\Gamma \geq 0, \quad h \in H^{\frac{1}{2}}(\Gamma);\\ &\left\langle \frac{\partial u}{\partial \nu_A}, u - h \right\rangle_\Gamma = 0. \end{aligned} \tag{1.5}$$

This problem is known to be uniquely solvable in the space $H^1(\Omega)$ (since $a(u,v)$ is coercive).

It is intersting to consider the problem: where the identity $u = h$ holds?

To answer this and other questions, we give some lemmas.

Lemma 1.1. *If* $u \in H^1(\Omega)$, *then*

(i) $\left.\dfrac{\partial u}{\partial x_i}\right|_D = 0$, $1 \leq i \leq n$, *where* $D = \{x \in \Omega, u(x) = 0\}$.

(ii) $\max(u,0) \in H^1(\Omega)$ *and*

$$[\max(u,0)]_{x_i} = \begin{cases} u_{x_i}, & \{x \in \Omega, \ u(x) > 0\}, \\ 0, & \{x \in \Omega, \ u(x) \leq 0\}. \end{cases}$$

Remark. All equalities and inequalities of this lemma are assumed to be fulfilled almost everywhere.

For the proof of the lemma see [3].

Lemma 1.2. *If* $u \in H^1(\Omega)$, *then* $\max(u,0)|_\Gamma = \max(u|_\Gamma, 0)$.

Proof. First we will show that if

$$u^k \in H^1(\Omega), \quad u^k \xrightarrow{H^1(\Omega)} u, \quad \text{then} \quad \max(u^k,0) \xrightarrow{H^1(\Omega)} \max(u,0), \quad \text{i.e.,}$$

$$\max(u^k,0) \xrightarrow{L_2(\Omega)} \max(u,0) \quad \text{and} \quad [\max(u^k,0)]_{x_i} \xrightarrow{L_2(\Omega)} [\max(u,0)]_{x_i},$$

$1 \leq i \leq n$. We begin by proving the latter claim. Take

$$E = \{x \in \Omega, u^k(x) \to u(x), u^k_{x_i}(x) \to u_{x_i}(x), 1 \leq i \leq n\} \setminus \{x \in \Omega, \exists i \in \overline{1,n}, u(x) = 0, u_{x_i}(x) \neq 0\}.$$

By virtue of Lemma 1.1. (i) it is obvious that $\operatorname{mes} E = \operatorname{mes} \Omega$. Show that if $x \in E$, then

$$[\max(u^k,0)]_{x_i}(x) \to [\max(u,0)]_{x_i}(x).$$

Indeed, assume that $u(x) > 0$. Since $x \in E$, there exists $k \in N$ such that if $k \geq K$, then $u^k(x) > 0$. Now, due to (ii) of Lemma 1.1, for $k \geq K$ we have

$$[\max(u^k,0)]_{x_i}(x) = u^k_{x_i}(x) \to u_{x_i}(x) = [\max(u,0)]_{x_i}(x), \quad k \to \infty.$$

Similarly, for $u(x) < 0$ there exists $k \in N$ such that if $k \geq K$, then $u^k(x) \leq 0$ and therefore by (ii) of Lemma 1.1 we obtain

$$0 = [\max(u^k, 0)]_{x_i}(x) \to [\max(u, 0)]_{x_i}(x) = 0, \quad k \geq K, \quad k \to \infty.$$

When $u(x) = 0$, one can split the sequence $u^k(x)$ into two parts: $u^p(x) > 0$ and $u^m(x) \leq 0$. Consider the case where these parts are infinite. Clearly, $[\max(u^m, 0)]_{x_i}(x) \to [\max(u, 0)]_{x_i}(x)]$. Show that the above reasoning holds for the second sequence, too. Since $x \in E$, then, by Lemma 1.1,

$$[\max(u^p, 0)]_{x_i}(x) = u^p_{x_i}(x) \to u_{x_i}(x) = 0 = [\max(u, 0)]_{x_i}(x),$$

i.e., $[\max(u^k, 0)]_{x_i} \to [\max(u, 0)]_{x_i}$ almost everywhere in the domain Ω. On the other hand,

$$\|[\max(u^k, 0)]_{x_i}\|_{L_2(\Omega)} \leq \|u^k_{x_i}\|_{L_2(\Omega)} \to \|u_{x_i}\|_{L_2(\Omega)} < \infty.$$

By Lebesgue's theorem

$$[\max(u^k, 0)]_{x_i} \xrightarrow{L_2(\Omega)} [\max(u, 0)]_{x_i}.$$

Similarly, we can prove that

$$\max(u^k, 0) \xrightarrow{L_2(\Omega)} \max(u, 0).$$

Hence

$$\max(u^k, 0) \xrightarrow{H^1(\Omega)} \max(u, 0).$$

Returning to Lemma 1.2, if we take $u^k \in C^1(\overline{\Omega})$, $u^k \xrightarrow{H^1(\Omega)} u$, then $\max(u^k, 0) \xrightarrow{H^1(\Omega)} \max(u, 0)$ and, obviously,

$$\max(u^k|_\Gamma, 0) = \max(u^k, 0)|_\Gamma \xrightarrow{H^{\frac{1}{2}}(\Gamma)} \max(u, 0)|_\Gamma.$$

Since $u^k|_\Gamma \xrightarrow{H^{\frac{1}{2}}(\Gamma)} u|_\Gamma$, we obtain $\max(u^k|_\Gamma, 0) \xrightarrow{L_2(\Gamma)} \max(u|_\Gamma, 0)$, i.e.,

$$\max(u^k, 0)|_\Gamma \xrightarrow{L_2(\Gamma)} \max(u|_\Gamma, 0) = \max(u, 0)|_\Gamma. \quad \square$$

Lemma 1.3. *If* $F \in H^{-\frac{1}{2}}(\Gamma)$, $\varphi \in H^{\frac{1}{2}}(\Gamma)$, $\varphi \geq 0$, $F \geq 0$, $\langle F, \varphi \rangle_\Gamma = 0$ *and*

$$\Gamma_d = \{x \in \Gamma, \ \varphi(x) \geq d\}, \quad d \in R, \quad d > 0,$$

then $F|_{\Gamma_d} = 0$.

Proof. Let $\psi \in H^{\frac{1}{2}}(\Gamma)$, $\psi|_{\Gamma\setminus\Gamma_d} = 0$. Then, if $|\psi| \leq M$, $M \in \mathbb{R}^+$, we have $\varphi \pm \dfrac{d}{M}\psi \geq 0$ and

$$\Big\langle F, \varphi \pm \frac{d}{M}\psi \Big\rangle_\Gamma = \pm \frac{d}{M}\langle F, \psi\rangle_\Gamma \geq 0.$$

Let $\operatorname{essup} \psi = \infty$. Set

$$\psi^\pm = \max(\pm\psi, 0), \quad \Psi \in H^1(\Omega), \quad \Psi|_\Gamma = \psi^+, \quad \Psi_k = \min(\Psi, k).$$

Then it is clear that

$$\psi = \psi^+ - \psi^-, \quad \Psi_k = (\Psi - \max(\Psi - k, 0)) \in H^1(\Omega).$$

Let us show that $\Psi_k \xrightarrow{H^1(\Omega)} \Psi$, $k \to \infty$, i.e., $\max(\Psi - k, 0) \xrightarrow{H^1(\Omega)} 0$. Since $\operatorname{mes} E_k \to 0$, $k \to \infty$, where $E_k = \{x \in \Omega, \Psi(x) \geq k\}$, then, by (ii) of Lemma 1.1 and the property that the Lebesgue integral is absolutely continuous, we obtain

$$\|\max(\Psi - k, 0)\|^2_{L_2(\Omega)} = \int\limits_{E_k} |\Psi - k|^2 dx \leq \int\limits_{E_k} |\Psi|^2 dx \to 0,$$

$$\|[\max(\Psi - k, 0)]_{x_i}\|^2_{L_2(\Omega)} = \int\limits_{E_k} |[\Psi - k]_{x_i}|^2 dx = \int\limits_{E_k} |\Psi_{x_i}|^2 dx \to 0,$$

i.e., $\Psi_k \xrightarrow{H^1(\Omega)} \Psi$, $k \to \infty$. By virtue of Lemma 1.2 we have

$$\min(\psi^+, k) = \psi^+ - \max(\psi^+ - k, 0) = \Psi_k|_\Gamma \xrightarrow{H^{\frac{1}{2}}(\Gamma)} \psi^+.$$

Since $|\min(\psi^+, k)| \leq k$, from what has been proved above, it follows that

$$\langle F, \psi^+\rangle_\Gamma = \lim_{k\to\infty} \langle F, \min(\psi^+, k)\rangle_\Gamma = 0.$$

Similarly, we obtain $\langle F, \psi^-\rangle_\Gamma = 0$. □

Define a constant M_f of the form (1.1) as follows:

$$M_f = \inf_{x\in\Omega} \frac{|f(x)|}{a_0(x)}, \quad f \in L_2(\Omega).$$

Lemma 1.4 (An analogue of the week maximum principle). *Let $u \in H^1(\Omega)$, $Au \in L_2(\Omega)$,*

(i) *if $Au \leq 0$, $\left.\dfrac{\partial u}{\partial \nu_A}\right|_{\Gamma_0} \leq 0$, where $\Gamma_0 \subset \Gamma$ is any measurable subset $\operatorname{mes} \Gamma_0 > 0$, and $\operatorname{essup}_{\Gamma\setminus\Gamma_0} u = M$. Then $u \leq \max(M, -M_{Au})$, $x \in \overline{\Omega}$.*

(ii) *if $Au \geq 0$, $\left.\dfrac{\partial u}{\partial \nu_A}\right|_{\Gamma} \geq 0$, then $u \geq M_{Au}$, $x \in \overline{\Omega}$.*

Proof. Let

$$M' = \max(M, -M_{Au}), \quad w = \max(u - M', 0).$$

We have to show that $w = 0$. Due to (ii) of Lemma 1.1, we have the following properties of the function w:

$$w \in H^1(\Omega), \quad \frac{\partial w}{\partial x_i}\frac{\partial u}{\partial x_j} = \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}, \quad \frac{\partial u}{\partial x_i} w = \frac{\partial w}{\partial x_i} w, \qquad u \cdot w = w^2 + M'w. \tag{1.6}$$

Now, if we insert the function w into the form (1.1) and take into acount (1.6), we obtain

$$a(u, w) = a(w, w) + M' \int_\Omega a_0 w dx. \tag{1.7}$$

Since $M' \geq M$, by Lemma 1.2 we have $w|_{\Gamma \setminus \Gamma_0} = 0$, whereas $M' \geq -M_{Au}$ and $Au \leq 0$ yield $M' a_0 - Au \geq 0$, which, when combined with (1.7), (1.4) and $w \geq 0$, gives

$$0 \geq \left\langle \frac{\partial u}{\partial \nu_A}, w \right\rangle_\Gamma = -\int_\Omega Auwdx + a(w, w) + M' \int_\Omega a_0 w dx \geq$$
$$\geq \int_\Omega (M' a_0 - Au) w dx + \alpha \|w\|^2_{1,\Omega}.$$

Thus $w = 0$ and claim (i) of Lemma 1.4 is proved.

(ii) Let $w = \max(M_{Au} - u, 0)$. Then, similarly to the above, we can show that

$$a(u, w) = -a(w, w) + M_{Au} \int_\Omega a_0 w dx,$$

and from (1.4) we get

$$0 \leq \left\langle \frac{\partial u}{\partial \nu_A}, w \right\rangle_\Gamma = -\int_\Omega Auwdx - a(w, w) + M_{Au} \int_\Omega a_0 w dx \leq$$
$$\leq \int_\Omega (M_{Au} a_0 - Au) w dx - \alpha \|w\|^2_{1,\Omega}.$$

Since $\int_\Omega (M_{Au} a_0 - Au) w dx \leq 0$, then $w = 0$. This means that $u(x) \geq M_{Au}$, $x \in \overline{\Omega}$. □

Theorem 1.5. *If u is a solution of the Signorini problem* (1.5)*, $f \leq 0$ and* $\operatorname{essup}_\Gamma h \geq -M_f$*, then*

$$\operatorname{essup}_{\overline{\Omega}} u = \operatorname{essup}_\Gamma h. \tag{1.8}$$

Proof. Let us first assume that $\frac{\partial u}{\partial \nu_A} = 0$. Then by virtue of (ii) of Lemma 1.4 we obtain $u \le -M_f$, $x \in \overline{\Omega}$. Since $u|_\Gamma \ge h$, we have $-M_f \ge \operatorname*{essup}_\Gamma u \ge \operatorname*{essup}_\Gamma h \ge -M_f$, i.e.,

$$\operatorname*{essup}_{\overline{\Omega}} u = \operatorname*{essup}_\Gamma h = -M_f. \tag{1.9}$$

Now let us consider the case $\frac{\partial u}{\partial \nu_A} \ne 0$. We have $\inf_\Gamma (u-h) = 0$. Clearly, if $u - h|_\Gamma \ge d > 0$, then because $\frac{\partial u}{\partial \nu_A} \ge 0$, by Lemma 1.3 we obtain $\frac{\partial u}{\partial \nu_A} = 0$, i.e., $\inf_\Gamma (u-h) = 0$. Let

$$\Gamma_n = \Big\{ x \in \Gamma, u(x) - h(x) \ge \frac{1}{n} \Big\}, \quad n \in N.$$

Then, provided that $\operatorname{mes} \Gamma \setminus \Gamma_n > 0$, by Lemma 1.3 we find that $\left.\frac{\partial u}{\partial \nu_A}\right|_{\Gamma_n} = 0$ holds for all $n \in N$. If there exists $n \in \mathbb{N}$ such that $\operatorname*{essup}_{\Gamma\setminus\Gamma_n} u < -M_f$, then due to (i) of Lemma 1.4 we have $u \le -M_f$, $x \in \overline{\Omega}$, and, as it has been shown above, we obtain (1.9). Let $\operatorname*{essup}_{\Gamma\setminus\Gamma_n} u \ge -M_f$ for all $n \in N$. Then, again using (i) of Lemma 1.4, we get

$$\operatorname*{essup}_{\overline{\Omega}} u = \operatorname*{essup}_{\Gamma\setminus\Gamma_n} u \le \operatorname*{essup}_\Gamma h + \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

which implies (1.8) when $n \to \infty$. □

Corollary 1.6. *If u is a solution of the problem* (1.5) *and $f \ge 0$, then the following inclusion takes the place:*

$$\{x \in \Gamma, u(x) = h(x)\} \subset \{x \in \Gamma, h(x) \ge M_f\}. \tag{1.10}$$

Proof. Since $\frac{\partial u}{\partial \nu_A} \ge 0$ and $Au = f \ge 0$, by (ii) of Lemma 1.4 we have $u \ge M_f$, $x \in \overline{\Omega}$, i.e., (1.10) holds. Moreover, because $M_f \ge 0$, we have

$$\{x \in \Gamma, u(x) = h(x)\} \subset \{x \in \Gamma, h(x) \ge 0\}. \quad \square$$

Corollary 1.7. *Let u be a solution of* (1.5) *and Γ_0 be any measurable set on the boundary Γ, $\operatorname{mes}\Gamma_0 > 0$.*

(i) *If*

$$h \le M, \quad h|_{\Gamma_0} = M, \quad M \ge -M_f,$$

then $u|_{\Gamma_0} = h|_{\Gamma_0}$.

(ii) *If there exists a function* $v \in H^1(\Omega)$, $Av \in L_2(\Omega)$ *and a number* d, $d \in R$, $d \geq 0$, *such that*

$$Av \geq f, \quad 0 \leq \frac{\partial v}{\partial \nu_A} \leq \frac{\partial u}{\partial \nu_A}, \quad v|_\Gamma \geq h - d, \quad v|_{\Gamma_0} = h - d|_{\Gamma_0},$$

then $u = h|_{\Gamma_0}$.

Proof. (i) is trivial since Theorem 1.5 immediately implies $\operatorname{essup}_{\overline{\Omega}} u = M$, i.e., $M \geq u|_{\Gamma_0} \geq h|_{\Gamma_0} = M$.

(ii) Let $w = u - v$. It is easy to show that

$$Aw = f - Av, \quad \frac{\partial w}{\partial \nu_A} \geq 0, \quad w|_\Gamma \geq h - v|_\Gamma,$$

$$0 \leq \Big\langle \frac{\partial w}{\partial \nu_A}, w - h + v \Big\rangle_\Gamma = \Big\langle \frac{\partial (u-v)}{\partial \nu_A}, u - h \Big\rangle_\Gamma = -\Big\langle \frac{\partial v}{\partial \nu_A}, u - h \Big\rangle_\Gamma \leq 0,$$

i.e., $\Big\langle \frac{\partial w}{\partial \nu_A}, w - h + v \Big\rangle_\Gamma = 0$ and w is a solution of the classical Signorini problem with the data $(f - Av, h - v)$.

Since

$$f - Av \leq 0, \quad h - v|_\Gamma \leq d, \quad (h - v)|_{\Gamma_0} = d, \quad d \geq 0 \geq -M_{f-Av},$$

by (i) we get $w = (h - v)|_{\Gamma_0}$, i.e., $(u = h)|_{\Gamma_0}$. □

Corollary 1.8. *If* $h \in H^{\frac{3}{2}}(\Gamma)$ *and the coefficients of the form* (1.1) *are elements of the space* $C^1(\overline{\Omega})$, *then, there exists* $f_h \in L_2(\Omega)$ *such that, for any function* $f \in L_2(\Omega)$ *with* $f \leq f_h$, *the solution of the classical Signorini problem with the data* (f, h) *satisfies the condition* $(u = h)|_\Gamma$.

Proof. Since $h \in H^{\frac{3}{2}}(\Gamma)$ and the coeficients of the form (1.1) are in the space $C^1(\overline{\Omega})$, there exists a function $v \in H^2(\Omega)$ such that

$$v|_\Gamma = h, \quad \frac{\partial v}{\partial \nu_A} = 0,$$

cf. [4], [5]. Denote

$$f_h = Av.$$

To prove Corollary 1.8, it is sufficient to take in Corollary 1.7

$$\Gamma_0 = \Gamma, \quad d = 0. \quad \square$$

2. The Implicit Signorini Problem

Let us state the implicit Signorini problem for the operators (1.3). Find such $u \in H^1(\Omega)$, that

$$
\begin{aligned}
&Au = f,\\
&u|_\Gamma \geq h - \Big\langle \frac{\partial u}{\partial \nu_A}, \varphi \Big\rangle_\Gamma, \quad \frac{\partial u}{\partial \nu_A}\Big|_\Gamma \geq 0,\\
&\Big\langle \frac{\partial u}{\partial \nu_A}, u - h + \Big\langle \frac{\partial u}{\partial \nu_A}, \varphi \Big\rangle_\Gamma \Big\rangle_\Gamma = 0,
\end{aligned} \tag{2.1}
$$

where

$$
f \in L_2(\Omega), \quad h, \varphi \in H^{\frac{1}{2}}(\Gamma), \quad \varphi \geq 0. \tag{2.2}
$$

This problem is considered in the monograph [1]. The existence of solutions was proved, but the question as to the number of solutions remained open. Using the results of §1, we will show that the problem (2.1) has a unique solution. Let u_λ for $\lambda \geq 0$ be a solution of the following classical Signorini problem:

$$
\begin{aligned}
&Au_\lambda = f, \qquad u_\lambda \in H^1(\Omega),\\
&u_\lambda|_\Gamma \geq h - \lambda, \quad \frac{\partial u_\lambda}{\partial \nu_A}\Big|_\Gamma \geq 0,\\
&\Big\langle \frac{\partial u_\lambda}{\partial \nu_A}, u_\lambda - h + \lambda \Big\rangle_\Gamma = 0,
\end{aligned} \tag{2.3}
$$

where f and h are the data of the problem (2.1).

Define the mapping $F : R^+ \to R^+$ as follows:

$$
F(\lambda) = \Big\langle \frac{\partial u_\lambda}{\partial \nu_A}, \varphi \Big\rangle_\Gamma, \quad \lambda \geq 0. \tag{2.4}
$$

Clearly, if $F(\lambda) = \lambda$, then the corresponding u_λ is a solution of the problem (2.1). Conversely, if u is a solution of the problem (2.1), then for $\lambda = \Big\langle \frac{\partial u}{\partial \nu_A}, \varphi \Big\rangle_\Gamma$ we have $u_\lambda = u$, i.e., the problem of solvability and the number of the solutions of the problem (2.1) are reduced to defining the number of statical points of the mapping (2.4). The continuity of the function $F(\lambda)$ for a more general case will be proved in §3 below. To prove the uniqueness of a solution of the problem (2.1) with the conditions (2.2), it is sufficient to show that $F(\lambda)$ is a nonincreasing function, i.e.,

$$
F(\lambda_1) - F(\lambda_2) = \Big\langle \frac{\partial (u_{\lambda_1} - u_{\lambda_2})}{\partial \nu_A}, \varphi \Big\rangle_\Gamma \geq 0, \quad 0 \leq \lambda_1 < \lambda_2.
$$

Since $\varphi \geq 0$ is an arbitrary function of the space $H^{\frac{1}{2}}(\Gamma)$ and $0 \leq \lambda_1 < \lambda_2$ are arbitrary numbers, we have to show that

$$\frac{\partial(u_0 - u_\lambda)}{\partial \nu_A} \geq 0, \quad \lambda > 0, \tag{2.5}$$

where u_0 and u_λ are the solutions of the problem (2.3) with data h and $h - \lambda$, respectively.

Let

$$h_0 = h - u_\lambda|_\Gamma. \tag{2.6}$$

Clearly,

$$h_0 \in H^{\frac{1}{2}}(\Gamma), \quad h_0 \leq \lambda. \tag{2.7}$$

Let us consider the Signorini problem

$$\begin{aligned} &Aw = 0, \quad w \in H^1(\Omega). \\ &w|_\Gamma \geq h_0, \quad \left.\frac{\partial w}{\partial \nu_A}\right|_\Gamma \geq 0, \\ &\left\langle \frac{\partial w}{\partial \nu_A}, w - h_0 \right\rangle_\Gamma = 0. \end{aligned} \tag{2.8}$$

If $\operatorname{essup}_\Gamma h_0 < 0$, then $w = 0$. If $\operatorname{essup}_\Gamma h_0 \geq 0$, then, due to Theorem 1.5, we have

$$\operatorname{essup}_{\overline{\Omega}} w = \operatorname{essup}_\Gamma h_0,$$

i.e. (see (2.7)),

$$w \leq \lambda, \quad x \in \overline{\Omega}. \tag{2.9}$$

Let

$$v = u_\lambda + w.$$

Let us show that v is a solution of the problem (2.3) when $\lambda = 0$, i.e., $v = u_0$. Indeed, we have

$$\begin{gathered} Av = f, \quad \left.\frac{\partial v}{\partial \nu_A}\right|_\Gamma \geq 0, \\ v|_\Gamma = (u_\lambda + w)|_\Gamma \geq u_\lambda|_\Gamma + h_0|_\Gamma = h. \end{gathered} \tag{2.10}$$

Now, by virtue of (2.6), (2.9) and (2.10), the problem (2.8) implies

$$0 \leq \left\langle \frac{\partial v}{\partial \nu_A}, v - h \right\rangle_\Gamma = \left\langle \frac{\partial u_\lambda}{\partial \nu_A} + \frac{\partial w}{\partial \nu_A}, u_\lambda + w - h \right\rangle_\Gamma =$$

$$= \left\langle \frac{\partial u_\lambda}{\partial \nu_A} + \frac{\partial w}{\partial \nu_A}, w - h_0 \right\rangle_\Gamma = \left\langle \frac{\partial u_\lambda}{\partial \nu_A}, w - h_0 \right\rangle_\Gamma \leq \left\langle \frac{\partial u_\lambda}{\partial \nu_A}, \lambda + u_\lambda - h \right\rangle_\Gamma = 0,$$

i.e., $\left\langle \frac{\partial v}{\partial \nu_A}, v - h \right\rangle_\Gamma = 0$ and $v = u_0$ because of the uniqueness of a solution of the problem (2.3). Therefore

$$\frac{\partial(u_0 - u_\lambda)}{\partial \nu_A} = \frac{\partial w}{\partial \nu_A} \geq 0.$$

Thus (2.5) is proved. Hence the function (2.4) is nonincreasing and continuous and therefore has one statical point, which means that the problem (2.1) under the conditions (2.2) has a unique solution.

In the monograph [1], the problem (2.1) is also considered when no restriction $\varphi \geq 0$ is imposed on the data, and the following fact is proved:

$$\sup_{\lambda \geq 0} F(\lambda) < +\infty, \quad \frac{F(\lambda)}{\lambda} \to \left\langle \frac{\partial v}{\partial \nu_A}, \varphi \right\rangle_\Gamma, \quad \lambda \to -\infty, \tag{2.11}$$

where $F(\lambda)$ is the function defined in (2.4) for all $\lambda \in \mathbb{R}$ and $v \in H^1(\Omega)$ is a solution of the problem

$$\begin{gathered} Av = 0; \\ v|_\Gamma \leq -1, \quad \left.\frac{\partial v}{\partial \nu_A}\right|_\Gamma \leq 0, \\ \left\langle \frac{\partial v}{\partial \nu_A}, v + 1 \right\rangle_\Gamma = 0. \end{gathered} \tag{2.12}$$

Remark. Since $-v$ is a solution of the Signorini problem with the data $f = 0$, $h = 1$, by virtue of Corollary 1.7 v satisfies the problem

$$Av = 0, \quad v|_\Gamma = -1.$$

Due to (2.11) it is clear that when we have no restriction $\varphi \geq 0$, for the problem (2.1) to be solvable it is sufficient that

$$\left\langle \frac{\partial v}{\partial \nu_A}, \varphi \right\rangle_\Gamma < 1. \tag{2.13}$$

Show that if no restriction $\varphi \geq 0$ is imposed, then the problem (2.1) may have no solution. One can easily see that, for all $f \in L_2(\Omega)$, it is possible to choose functions h and φ from $H^{\frac{1}{2}}(\Gamma)$ such that for the data f, h, φ of the implicit Signorini problem, the following conditions hold:

$$\varphi \leq 0, \quad u_0|_\Gamma = h, \quad \left\langle \frac{\partial u_0}{\partial \nu_A}, \varphi \right\rangle_\Gamma \neq 0, \tag{2.14}$$

$$\left\langle \frac{\partial v}{\partial \nu_A}, \varphi \right\rangle_\Gamma \geq 1. \tag{2.15}$$

For $f \le 0$, by virtue of Theorem 1.5 and Lemma 1.3 we can write the data h and φ in the conditions (2.14), (2.15), for example, as follows:

$$h = 1, \quad \varphi \le d, \quad d \in R^-, \quad \Big\langle \frac{\partial v}{\partial \nu_A}, \varphi \Big\rangle_\Gamma \ge 1.$$

Let us show that, if in (2.2) the conditions (2.14), (2.15) hold instead of $\varphi \ge 0$ then the problem (2.1) has no solution, i.e., that the function $F(\lambda)$ defined by (2.4) for all $\lambda \in \mathbb{R}$ has no stationary point. Since $\varphi \le 0$, $F(\lambda) \le 0$, so, the mapping $F : R \to R^-$ may have a stationary point only on R^-.

Let

$$w = \frac{u_\lambda - u_0}{\lambda}, \quad \lambda < 0.$$

We will show that w satisfies the problem (2.12), i.e., $w = v$. Since $u_0|_\Gamma = h$, the first three conditions are obvious by virtue of the problem (2.3) and (2.5). Prove the fourth condition

$$0 \le \Big\langle \frac{\partial w}{\partial \nu_A}, w + 1 \Big\rangle_\Gamma = \frac{1}{\lambda^2} \Big\langle \frac{\partial (u_\lambda - u_0)}{\partial \nu_A}, u_\lambda - h + \lambda \Big\rangle_\Gamma = \\ = \frac{1}{\lambda^2} \Big\langle - \frac{\partial u_0}{\partial \nu_A}, u_\lambda - h + \lambda \Big\rangle_\Gamma \le 0.$$

Hence, $w = v$ and thus

$$u_\lambda = \lambda v + u_0, \quad \lambda \le 0,$$

$$F(\lambda) = \Big\langle \frac{\partial v}{\partial \nu_A}, \varphi \Big\rangle_\Gamma \lambda + \Big\langle \frac{\partial u_0}{\partial \nu_A}, \varphi \Big\rangle_\Gamma, \quad \lambda \le 0,$$

i.e., $F(\lambda) < \lambda$, when $\lambda \le 0$. This implies that the mapping F has no stationary point on R^- (Fig. 1) and, as we have noted, does not have one on R^+ either. Thus, if $f \in L_2(\Omega)$, $\varphi, h \in H^{\frac{1}{2}}(\Gamma)$ satisfy the conditions (2.14), (2.15), then the problem (2.1) has no solution. Note that if the conditions (2.14) hold for the problem (2.1) to be solvable, the condition (2.13) is necessary and sufficent.

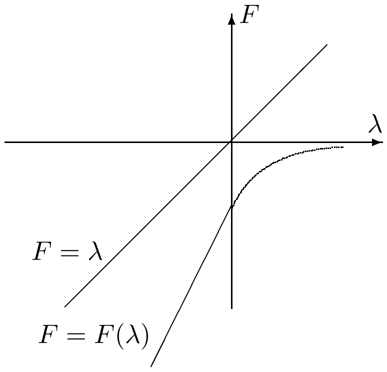

Fig.1

Thus, we have proved the following

Theorem 2.1. *The problem* (2.1)*, has a unique solution under the conditions* (2.2)*, and it has no solution, when instead of $\varphi \geq 0$ in the data, the conditions* (2.14)*,* (2.15) *hold.*

3. Statement of the Implicit Signorini Problem in More General Terms

The implicit Signorini problem can be stated for the operators (1.3) in more general terms as follows: find such $u \in H^1(\Omega)$, that

$$\begin{aligned} &Au = f, \\ &u|_\Gamma \geq h - \Big\langle \frac{\partial u}{\partial \nu_A}, \varphi \Big\rangle_\Gamma \psi, \quad \frac{\partial u}{\partial \nu_A}\Big|_\Gamma \geq 0, \\ &\Big\langle \frac{\partial u}{\partial \nu_A}, u - h + \Big\langle \frac{\partial u}{\partial \nu_A}, \varphi \Big\rangle_\Gamma \psi \Big\rangle_\Gamma = 0, \end{aligned} \tag{3.1}$$

where

$$f \in L_2(\Omega), \quad h, \varphi, \psi \in H^{\frac{1}{2}}(\Gamma). \tag{3.2}$$

Let u_λ be a solution of the following classical Signorini problem for all $\lambda \in R$:

$$\begin{aligned} &Au_\lambda = f, \quad u_\lambda \in H^1(\Omega), \\ &u_\lambda|_\Gamma \geq h - \lambda\psi, \quad \frac{\partial u_\lambda}{\partial \nu_A}\Big|_\Gamma \geq 0, \\ &\Big\langle \frac{\partial u_\lambda}{\partial \nu_A}, u_\lambda - h + \lambda\psi \Big\rangle_\Gamma = 0, \end{aligned} \tag{3.3}$$

where the functions f, h, ψ are the data of the problem (3.2). Define the mapping $F : R \to R$ as

$$F(\lambda) = \Big\langle \frac{\partial u_\lambda}{\partial \nu_A}, \varphi \Big\rangle_\Gamma, \quad \lambda \in \mathbb{R}. \tag{3.4}$$

As in the case of the problem (2.1), the question of the number of solutions of problem (3.1) is reduced to determining the number of stationary points of the function $F(\lambda)$. Let us consider this question. On the space $H^{\frac{1}{2}}(\Gamma)$, define the norm as follows:

$$\|g\|_{\frac{1}{2},\Gamma} = \inf_{\substack{G \in H^1(\Omega) \\ G|_\Gamma = g}} \|G\|_{1,\Omega}. \tag{3.5}$$

This is equivalent to the standard definition of the norm on this space. Further, we use such a defined norm on the space $H^{\frac{1}{2}}(\Gamma)$. Define the constant c for the bilinear form (1.1):

$$c = \inf\{d \in R,\ a(u,v) \le d\|u\|_{1,\Omega}\|v\|_{1,\Omega},\ u,v \in H^1(\Omega)\}. \qquad (3.6)$$

Obviously $c < +\infty$, because

$$c \le \max_{1\le i,j\le n}(\|a_{ij}\|_{L^\infty}, \|a_i\|_{L^\infty}, \|a_0\|_{L^\infty}).$$

Show the continuity of the function $F(\lambda)$. For all $\lambda_1,\ \lambda_2 \in R$, from the definitions of the functions u_{λ_1}, u_{λ_2}, we get:

$$\Big\langle \frac{\partial u_{\lambda_1}}{\partial \nu_A} - \frac{\partial u_{\lambda_2}}{\partial \nu_A}, u_{\lambda_1} - u_{\lambda_2} + (\lambda_1 - \lambda_2)\psi \Big\rangle_\Gamma =$$
$$= \Big\langle \frac{\partial u_{\lambda_1}}{\partial \nu_A} - \frac{\partial u_{\lambda_2}}{\partial \nu_A}, u_{\lambda_1} - h + \lambda_1\psi - (u_{\lambda_2} - h + \lambda_2\psi) \Big\rangle_\Gamma \le 0. \qquad (3.7)$$

Using the Green formula (1.4) and the formulas (1.2), (3.6), (3.7), we obtain

$$\alpha\|u_{\lambda_1} - u_{\lambda_2}\|^2_{1,\Omega} \le a(u_{\lambda_1} - u_{\lambda_2}, u_{\lambda_1} - u_{\lambda_2}) =$$
$$= \Big\langle \frac{\partial(u_{\lambda_1} - u_{\lambda_2})}{\partial \nu_A}, u_{\lambda_1} - u_{\lambda_2} \Big\rangle_\Gamma \le (\lambda_2 - \lambda_1)\Big\langle \frac{\partial(u_{\lambda_1} - u_{\lambda_2})}{\partial \nu_A}, \psi \Big\rangle_\Gamma =$$
$$= (\lambda_2 - \lambda_1)a(u_{\lambda_1} - u_{\lambda_2}, \Psi) \le c|\lambda_1 - \lambda_2|\|u_{\lambda_1} - u_{\lambda_2}\|_{1,\Omega}\|\Psi\|_{1,\Omega}, \qquad (3.8)$$

where c is the constant defined in (3.6) and $\Psi \in H^1(\Omega)$ is an arbitrary function with $\Psi|_\Gamma = \psi$. Hence, by virtue of (3.5), from (3.8) we have

$$\|u_{\lambda_1} - u_{\lambda_2}\|_{1,\Omega} \le \alpha^{-1}c|\lambda_1 - \lambda_2|\,\|\psi\|_{\frac{1}{2},\Gamma}. \qquad (3.9)$$

By a reasoning similar to that we have used for the mates (3.8), from (3.9) we have

$$|F(\lambda_1) - F(\lambda_2)| = \Big|\Big\langle \frac{\partial(u_{\lambda_1} - u_{\lambda_2})}{\partial \nu_A}, \varphi \Big\rangle_\Gamma\Big| \le$$
$$\le c\|u_{\lambda_1} - u_{\lambda_2}\|_{1,\Omega}\|\varphi\|_{\frac{1}{2},\Gamma} \le c^2\alpha^{-1}|\lambda_1 - \lambda_2|\|\psi\|_{\frac{1}{2},\Gamma}\|\varphi\|_{\frac{1}{2},\Gamma}. \qquad (3.10)$$

Thus, $F(\lambda)$ belongs to the Lipschitz class and if

$$\|\psi\|_{\frac{1}{2},\Gamma}\|\varphi\|_{\frac{1}{2},\Gamma} < \frac{\alpha}{c^2},$$

then $F(\lambda)$ is a contractive mapping and has a unique stationary point. Therefore, the problem (3.1) has a unique solution.

Let us consider the situation

$$\phi = \gamma\psi, \quad \gamma = \text{const} > 0.$$

When $\lambda_1 < \lambda_2$, for some estimate given in (3.8), we get

$$F(\lambda_1) - F(\lambda_2) = \left\langle \frac{\partial(u_{\lambda_1} - u_{\lambda_2})}{\partial \nu_A}, \gamma\psi \right\rangle_\Gamma \geq 0.$$

Thus the function F is nonincreasing and, as shown in (3.10), it is also continuous and therefore has a unique stationary point. Thus the problem (3.1) has a unique solution. Let us give sufficient conditions for the problem (3.1) to be solvable. As it has been mentioned above, (2.11) is established for the problem (2.1) in [1]. Repeating an analogous reasoning for any $\psi \in H^{\frac{1}{2}}(\Gamma)$, we can show for the problem (3.1) that

$$\frac{F(\lambda)}{\lambda} \to \left\langle \frac{\partial v}{\partial \nu_A}, \varphi \right\rangle_\Gamma, \quad \lambda \to -\infty \tag{3.11}$$

$$\frac{F(\lambda)}{\lambda} \to \left\langle \frac{\partial w}{\partial \nu_A}, \varphi \right\rangle_\Gamma, \quad \lambda \to +\infty, \tag{3.12}$$

where $v \in H^1(\Omega)$ and $w \in H^1(\Omega)$ are respectively solutions of the following problems:

$$\begin{gathered} Av = 0, \\ v|_\Gamma \leq -\psi, \quad \left.\frac{\partial v}{\partial \nu_A}\right|_\Gamma \leq 0, \\ \left\langle \frac{\partial v}{\partial \nu_A}, v + \psi \right\rangle_\Gamma = 0; \end{gathered} \tag{3.13}$$

$$\begin{gathered} Aw = 0, \\ w|_\Gamma \geq -\psi, \quad \left.\frac{\partial w}{\partial \nu_A}\right|_\Gamma \geq 0, \\ \left\langle \frac{\partial w}{\partial \nu_A}, w + \psi \right\rangle_\Gamma = 0. \end{gathered} \tag{3.14}$$

Since the function $F(\lambda)$ is continuous, (3.11) and (3.12) clearly imply that each of the conditions

$$\left\langle \frac{\partial v}{\partial \nu_A}, \varphi \right\rangle_\Gamma > 1, \quad \left\langle \frac{\partial w}{\partial \nu_A}, \varphi \right\rangle_\Gamma > 1 \tag{3.15}$$

or

$$\left\langle \frac{\partial v}{\partial \nu_A}, \varphi \right\rangle_\Gamma < 1, \quad \left\langle \frac{\partial w}{\partial \nu_A}, \varphi \right\rangle_\Gamma < 1. \tag{3.16}$$

is sufficient for the problem (3.1) to be solvable

Obviously, if $\psi \geq 0$, then $w = 0$, and if $\psi \leq 0$, then $v = 0$. In these cases, is each of the conditions is sufficient conditions of the solvability of the problem (3.1) are

$$\psi \leq 0, \quad \left\langle \frac{\partial w}{\partial \nu_A}, \varphi \right\rangle_\Gamma < 1, \tag{3.17}$$

$$\psi \geq 0, \quad \Big\langle \frac{\partial v}{\partial \nu_{_A}}, \varphi \Big\rangle_{_\Gamma} < 1. \tag{3.18}$$

(3.17), (3.18) hold in the following situations, each of which also represent sufficient conditions for the problem (3.1) to be solvable:

$$\psi \leq 0, \quad \varphi \leq 0.$$
$$\psi \geq 0, \quad \varphi \geq 0.$$

Let us clarify the uniqueness problem. We will show that, when correspoding $F(\lambda) \neq \text{const}$, for any data from (3.2) there exists α_1, $\alpha_2 \in R$ such that the problem (3.1) with

$$\breve{f} = f, \quad \breve{h} = h + \alpha_1 \psi, \quad \breve{\varphi} = \alpha_2 \varphi, \quad \breve{\psi} = \psi, \tag{3.19}$$

has at least two solutions. Indeed, for any fixed α_1, $\alpha_2 \in R$, we denote by $\breve{u}_{_\lambda}$ a solution of the problem (3.3) stated in the data (3.19), and by $\breve{F}(\lambda)$ the corresponding mapping (3.4). Now it easily follows that

$$\breve{u}_{_\lambda} = u_{_{\lambda - \alpha_1}}, \quad \breve{F}(\lambda) = \alpha_2 F(\lambda - \alpha_1), \quad \forall \alpha_1, \alpha_2, \lambda \in R,$$

where $u_{_\lambda}$ and $F(\lambda)$ are defined by the initial data. Let us choose λ_1, $\lambda_2 \in R$ such that $F(\lambda_1) \neq F(\lambda_2)$. If now we take

$$\alpha_2 = \frac{\lambda_2 - \lambda_1}{F(\lambda_2) - F(\lambda_1)}, \quad \alpha_1 = \alpha_2 F(\lambda_1) - \lambda_1, \tag{3.20}$$

then, obviously, the stationary points of the function $\breve{F}(\lambda)$ are $\alpha_2 F(\lambda_1)$, $\alpha_2 F(\lambda_2)$, which means that, under data (3.19), the problem (3.1) has at least two solutions.

Thus, we have proved the following theorem.

Theorem 3.1. *The problem* (3.1) *with the conditions* (3.2) *has a unique solution if*

1) $\|\psi\|_{\frac{1}{2},\Gamma} \|\varphi\|_{\frac{1}{2},\Gamma} < \dfrac{\alpha}{c^2}$,
where, α *and* c *are the constants from* (1.2) *and* (3.6), *and* $\|\cdot\|_{\frac{1}{2},\Gamma}$ *is defined in* (3.5).

2) $\phi = \gamma\psi$, $\gamma = \text{const} > 0$.

3) *The problem* (3.1) *with the conditions* (3.2) *has a solution if one of the pair of the following conditions*

$$\Big\langle \frac{\partial v}{\partial \nu_{_A}}, \varphi \Big\rangle_{_\Gamma} > 1, \quad \Big\langle \frac{\partial w}{\partial \nu_{_A}}, \varphi \Big\rangle_{_\Gamma} > 1.$$
$$\Big\langle \frac{\partial v}{\partial \nu_{_A}}, \varphi \Big\rangle_{_\Gamma} < 1, \quad \Big\langle \frac{\partial w}{\partial \nu_{_A}}, \varphi \Big\rangle_{_\Gamma} < 1.$$

holds, where v *and* w *are solutions of the problems* (3.13) *and* (3.14)*, respectively. Therefore the following four pairs of conditions are also sufficient for the problem* (3.1) *to be solvable:*

$$\psi \le 0, \quad \Big\langle \frac{\partial w}{\partial \nu_{A}}, \varphi \Big\rangle_{\Gamma} < 1,$$
$$\psi \ge 0, \quad \Big\langle \frac{\partial v}{\partial \nu_{A}}, \varphi \Big\rangle_{\Gamma} < 1,$$
$$\psi \le 0, \quad \varphi \le 0,$$
$$\psi \ge 0, \quad \varphi \ge 0.$$

4) *For any data* (3.2)*, where the corresponding function* $F(\lambda) \neq \text{const}$*, there exist* α_1*,* $\alpha_2 \in R$ *such that the problem* (3.1) *under conditions*

$$\breve{f} = f, \quad \breve{h} = h + \alpha_1 \psi, \quad \breve{\varphi} = \alpha_2 \varphi, \quad \breve{\psi} = \psi,$$

has at least two solutions.

4. An implicit Signorini problem in the elasticity theory

Let Ω, Γ and ν be as in §1. Assume, that $\mathcal{L}_2(\Omega) = (L_2(\Omega))^n$, $\mathcal{H}^s(\Omega) = (H^s(\Omega))^n$, $\mathcal{H}^s(\Gamma) = (H^s(\Gamma))^n, s \in \mathbb{R}$, where $H^s(\Omega)$ and $H^s(\Gamma)$ are real Sobolev spaces. The norms in the spaces $\mathcal{H}^s(\Omega)$ and $\mathcal{H}^s(\Gamma)$ will be denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\Gamma}$. The norm $\|\cdot\|_{\frac{1}{2},\Gamma}$ we understand to be defined by means of (3.5). Let us define the matrix-diferential operators $A(x,\partial)$ and $T(x,\partial,\nu)$ and the bilinear form on the space $\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ as follows:

$$\begin{aligned} &A(x,\partial) = \|A_{jk}(x,\partial)\|_{n\times n}, \\ &A_{jk}(x,\partial) = \frac{\partial}{\partial x_i}\Big(a_{ijlk}(x)\,\frac{\partial}{\partial x_l}\Big), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &T(x,\partial,\nu) = \|T_{jk}(x,\partial,\nu)\|_{n\times n}, \\ &T_{jk}(x,\partial,\nu) = a_{ijlk}(x)\nu_i(x)\,\frac{\partial}{\partial x_l}, \end{aligned} \tag{4.2}$$

$$a_{ijlk} \in C^1(\overline{\Omega}), \quad a_{ijlk} = a_{jilk} = a_{lkij};$$
$$\forall x \in \overline{\Omega},\ \forall \xi_{ij} \in \mathbb{R}\ (\xi_{ij} = \xi_{ji}):\ \ a_{ijlk}(x)\xi_{ij}\xi_{lk} \ge \beta\xi_{ij}\xi_{ij}, \ \ \beta > 0;$$
$$B(u,v) = \int\limits_{\Omega} a_{ijlk}(x)\,\frac{\partial u_i}{\partial x_j}\,\frac{\partial v_l}{\partial x_k}\,dx, \quad \forall u,v \in \mathcal{H}^1(\Omega). \tag{4.3}$$

The operator (4.1) corresponds to the system of equilibrium equations of the elasticity theory for a nonhomogenous body, while (4.2) is a stress operator. It is clear that the form B has the following boundness property:

$$\begin{aligned} B(u,v) \le C\|u\|_{1,\Omega}\|v\|_{1,\Omega}, \quad \forall u,v \in \mathcal{H}^1(\Omega), \\ C = \sup_{x\in\overline{\Omega}}\big\{a_{ijlk}(x),\ \ i,j,l,k = \overline{1,n}\big\}. \end{aligned} \tag{4.4}$$

As it is known, if $u \in \mathcal{H}^1(\Omega)$, $Au \in \mathcal{L}_2(\Omega)$, then $Tu \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$ and the following Green formula holds:

$$\int_\Omega Au \cdot v dx = -B(u,v) + \langle Tu, v\rangle_\Gamma, \quad \forall v \in \mathcal{H}^1(\Omega), \tag{4.5}$$

where $\langle \cdot\,\cdot\rangle_\Gamma$ is the duality relation between the spaces $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ and $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$. Let us define v_ν and v_s, as the normal and tangential components for the vector-function $v \in \mathcal{H}^{\frac{1}{2}}(\Gamma)$, respectively, while $(Tu)_\nu$ and $(Tu)_s$ be, respectively normal and tangential components of stress. Then the following formula is true

$$\langle Tu, v\rangle_\Gamma = \langle (Tu)_\nu, v_\nu\rangle_\Gamma + \langle (Tu)_s, v_s\rangle_\Gamma, \ \ v \in \mathcal{H}^{\frac{1}{2}}(\Gamma). \tag{4.6}$$

Let $\Gamma_0, \Gamma_1, \Gamma_2 \neq \emptyset$, be open sets on the boundary Γ such that

$$\overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma, \quad \partial\Gamma_0 \cap \partial\Gamma_2 = \emptyset. \tag{4.7}$$

Recall that, for $s \geq 0$,

$$\widetilde{H}^s(\Gamma_0) = \{\varphi \in H^s(\Gamma), \ \varphi|_{\Gamma\setminus\Gamma_0} = 0\},$$

while $H^s(\Gamma_0)$ is the space of restrictions on Γ_0 of $H^s(\Gamma)$-functions, endowed with the norm

$$\|\varphi\|_{H^s(\Gamma_0)} = \inf_{\psi|_{\Gamma_0}=\varphi} \|\psi\|_{H^s(\Gamma)}.$$

Define

$$H^{-s}(\Gamma_0) = (\widetilde{H}^s(\Gamma_0))', \quad \widetilde{H}^{-s}(\Gamma_0) = (H^s(\Gamma_0))',$$

where $(\,\cdot\,)'$ means the self-adjointness of spaces. Analogously we define the spaces $\mathcal{H}^{\pm s}(\Gamma_0)$ and $\widetilde{\mathcal{H}}^{\pm s}(\Gamma_0)$.

We will use the notation

$$\mathcal{H}^1_*(\Omega) = \{v \in \mathcal{H}^1(\Omega), \ v|_{\Gamma_2} = 0\}. \tag{4.8}$$

Let us state several lemmas:

Lemma 4.1. (i) *If the functions F and φ are nonnegative on Ω and $f \in H^{-\frac{1}{2}}(\Omega) \cap L^2_{\text{loc}}(\Omega)$ and $\varphi \in H^{\frac{1}{2}}(\Omega)$, then $f\varphi \in L^1(\Omega)$.*

(ii) *If $u \in \mathcal{H}^{-\frac{1}{2}}(\Omega) \cap \mathcal{L}^2_{\text{loc}}(\Omega)$, $\ v \in \mathcal{H}^{\frac{1}{2}}(\Omega)$, and $uv \in L^1(\Omega)$, then*

$$\langle u, v\rangle_\Omega = \int_\Omega uv dx.$$

The proof see in [3].

Lemma 4.2. *Let $f \in \mathcal{H}^{-\frac{1}{2}}(\Omega) \cap \mathcal{L}^2_{\text{loc}}(\Omega)$, and $f \geq 0$ as the functional on the space $\mathcal{H}^{-\frac{1}{2}}(\Omega)$, in the sense of §1. Then $f \geq 0$ almost everywhere on Ω.*

Proof. Assume, there exists the set $\Omega' \subset \Omega$ with positive measure where $f < 0$. Then it is possible to find such a closed set $\Omega'' \subset \Omega'$ and a closed core B, that if we take $\Omega_0 = B \cap \Omega''$ $d = \operatorname{dist}(\partial\Omega, B)$, then

$$\operatorname{mes}\Omega_0 > 0, \quad d > 0.$$

Let $\Omega_k := \Omega^0_{\varepsilon_k}$, where $\Omega^0_{\varepsilon_k}$ is the ε_k-neighbourhood of the set Ω^0. Let us show that we can choose such sequence ε_k, that

$$\operatorname{mes}\Omega_k \longrightarrow \operatorname{mes}\Omega_0. \tag{4.9}$$

In fact, since Ω_0 is a closed, and therefore measurable set, there exist open sets Ω^k, such that $\bigcap_k \Omega^k = \Omega_0$, and $\operatorname{mes}\Omega^k \longrightarrow \operatorname{mes}\Omega^0$. There can be found such ε_k that $\Omega_k \subset \Omega^k$ for each $k \in \mathbb{N}$. Indeed, if $\operatorname{dist}(\partial\Omega^k, \Omega^0) = 0$ for some k, then there exist such sequences $\{x_m\} \subset \Omega^0$ and $\{y_m\} \subset \partial\Omega^k$, that $x_m - y_m \to 0$. Then it can be found such x, that for some sabsequences x'_m and y'_m of $\{x_m\}$ and $\{y_m\}$, there hold $\lim\limits_{m\to\infty} x'_m = \lim\limits_{m\to\infty} y'_m = x$. So $x \in \Omega_0 \cap \partial\Omega^k$, which is the contradiction. Hence $\operatorname{dist}(\Omega_0 \cap \partial\Omega^k) > 0$, which implies $\Omega_k \subset \Omega^k$ for each $k \in \mathbb{N}$. thus, there exists such sequence ε_k that (4.9) holds.

Let us choose the sequence ε_k in terms (4.9) and χ_k, such that

$$0 < \varepsilon_k < d/2, \quad \varepsilon_k > \varepsilon_{k+1}, \quad \chi_k \in C_0^\infty(\Omega_k), \quad 0 \le \chi_k \le 1, \quad \chi_k|_{\Omega_{k+1}} = 1.$$

So

$$\langle f, \chi_k\rangle_\Omega = \int\limits_{\Omega_k} f\chi_k dx = \int\limits_{\Omega_0} f dx + \int\limits_{\Omega_k\setminus\Omega_0} f\chi_k dx.$$

From (4.9) it follows that

$$\int\limits_{\Omega_k\setminus\Omega_0} f\chi_k dx \le \int\limits_{\Omega_k\setminus\Omega_0} |f| dx \longrightarrow 0.$$

Since $f|_{\Omega_0} < 0$, we can find $\chi_k \ge 0$ such that $\langle f, \chi_k\rangle_\Omega < 0$, which contradicts to our assumption. □

Lemma 4.3. *Let $F \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$, $\Gamma_0, \Gamma_1, \Gamma_2$ satisfy the conditions (4.7) and there exist a functional F_1 such that*

$$F_1|_{\Gamma_1} = F|_{\Gamma_1}, \quad F_1 \in \widetilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_1 \cup \overline{\Gamma}_2).$$

Then there exists a functional F_0 such that

$$F_0|_{\Gamma_0} = F|_{\Gamma_0}, \quad F_0 \in \widetilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_0 \cup \overline{\Gamma}_2),$$

and

$$\langle F, \varphi\rangle_\Gamma = \langle F_1, \varphi\rangle_{\Gamma_1} + \langle F_0, \varphi\rangle_{\Gamma_0}, \quad \varphi \in \mathcal{H}^{\frac{1}{2}}(\Gamma),\ \varphi|_{\Gamma_2} = 0.$$

Proof. Define the functional F_0 on the space $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ as follows:

$$\langle F_0, \varphi\rangle_\Gamma = \langle F, \varphi\rangle_\Gamma - \langle F_1, \varphi\rangle_{\Gamma_1 \cup \overline{\Gamma}_2}, \quad \varphi \in \mathcal{H}^{\frac{1}{2}}(\Gamma).$$

It is clear that the functional F_0 possesses the following properties:

$$\begin{aligned}
&\langle F_0, \varphi\rangle_\Gamma \le c_0 \|\varphi\|_{\frac{1}{2},\Gamma}, \\
&\langle F_0, \varphi\rangle_\Gamma = 0, \qquad\quad \varphi|_{\Gamma_0} = 0, \\
&\langle F_0, \varphi\rangle_\Gamma = \langle F, \varphi\rangle_\Gamma, \quad \varphi|_{\Gamma\setminus\Gamma_0} = 0,
\end{aligned}$$

i.e.,

$$F|_{\Gamma_0} = F_0|_{\Gamma_0}, \quad F_0 \in \widetilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_0).$$

If we consider the functional F_0 on $\{\varphi \in \mathcal{H}^{\frac{1}{2}}(\Gamma);\ \varphi|_{\Gamma_0} = 0\}$, the second claim of Lemma will be fulfilled. □

Now let us state the classical Signorini problem for the operator (4.1), with Dirichlet and Neumann boundary conditions: find a displacement $u \in H^1(\Omega)$, such that

$$\begin{aligned}
&Au = f, \\
&u|_{\Gamma_2} = \psi, \quad Tu|_{\Gamma_1} = P, \\
&(Tu)_s|_{\Gamma_0} = 0; \quad (Tu)_\nu|_{\Gamma_0} \ge 0; \\
&u_\nu|_{\Gamma_0} \ge h, \\
&\langle (Tu)_\nu, u_\nu - h\rangle_{\Gamma_0} = 0;
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
f \in \mathcal{L}_2(\Omega), \quad P \in \widetilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_1 \cup \overline{\Gamma}_2), \\
\psi \in \mathcal{H}^{\frac{1}{2}}(\Gamma_2), \quad h \in H^{\frac{1}{2}}(\Gamma_0).
\end{aligned} \tag{4.11}$$

Since $(Tu)_\nu \in H^{-\frac{1}{2}}(\Gamma)$, we understand the inequality $(Tu)_\nu|_{\Gamma_0} \ge 0$ in the sense of §1, while the inequality $u_\nu|_{\Gamma_0} \ge h$ is meant to be fulfilled almost everywhere.

This problem expresses the following mechanical process: the volum force f is acting on a body, on the part Γ_1 the displacement is fixed, while on the part Γ_2 the force P acts. The mechanical sense of the last four conditions can be interpreted in such a way: the body is placed into a rigid frame by its part Γ_0. Stresses on the part Γ_0 are caused only by its interaction with the frame, without friction. The problem (4.10) has a unique solution under the conditions (4.11). Show that this problem is stable with respect to the data ψ, p, h. State the same problem for the data ψ_k, P_k, h_k, from the conditions (4.11):

$$\begin{aligned}
&Au^k = f, \qquad u^k \in \mathcal{H}^1(\Omega);\\
&u^k|_{\Gamma_2} = \psi_k, \quad Tu^k|_{\Gamma_1} = P_k,\\
&(Tu^k)_s|_{\Gamma_0} = 0; \quad (Tu^k)_\nu|_{\Gamma_0} \geq 0;\\
&u^k_\nu|_{\Gamma_0} \geq h_k,\\
&\langle (Tu^k)_\nu, u^k_\nu - h_k\rangle_{\Gamma_0} = 0.
\end{aligned} \tag{4.12}$$

Show that when $P_k \xrightarrow{\mathcal{H}^{-\frac{1}{2}}(\Gamma_1)} P$, $\psi_k \xrightarrow{\mathcal{H}^{\frac{1}{2}}(\Gamma_2)} \psi$, $h_k \xrightarrow{H^{\frac{1}{2}}(\Gamma_0)} h$, then $u^k \xrightarrow{\mathcal{H}^1(\Omega)} u$, where u^k is the solution of the problem (4.11).

Let χ, ϕ_k and $\widetilde{\psi^k}$ be functions with the properties:

$$\begin{gathered}
\chi \in C^1(\overline{\Omega}), \quad \chi|_\Gamma = \nu,\\
A\widetilde{\psi_k} = 0, \qquad \widetilde{\psi_k} \in \mathcal{H}^1(\Omega).\\
\widetilde{\psi_k}|_{\Gamma_2} = \psi_k - \psi, \quad T\widetilde{\psi_k}|_{\Gamma_1} = 0, \quad \widetilde{\psi_k}|_{\Gamma_0} = 0.\\
\phi_k \in H^1(\Omega), \quad \phi_k|_{\Gamma_0} = h_k - h, \quad \phi_k|_{\Gamma_2} = 0,\\
\|\phi_k\|_{1,\Omega} \leq d\|h_k - h\|_{\frac{1}{2},\Gamma_0} \ \text{ for some constant } d.
\end{gathered} \tag{4.13}$$

Remark. Such ϕ_k exist. To prove this, first of all we construct the trace of ϕ_k on Γ. Extend $h_k - h$ from Γ_0 to the whole Γ by its minimal norm and multiply on χ_0 with

$$\chi_0 \in C^\infty(\Gamma), \quad \chi_0|_{\Gamma_0} = 1, \quad \chi_0|_{\Gamma_2} = 0.$$

Since the multiplication on a $C^\infty(\Gamma)$-function is a bounded operator in $H^{\frac{1}{2}}(\Gamma)$, due to the definition of the norm in $H^{\frac{1}{2}}(\Gamma_0)$, we obtain

$$\|\phi_k\|_{\frac{1}{2},\Gamma} \leq d\|h_k - h\|_{\frac{1}{2},\Gamma_0}.$$

To complete the proof, we have to recall that $\|\cdot\|_{\frac{1}{2},\Gamma}$ is understood as (3.5).

Passing to the properties of u and u^k, from the formulas (4.5) and (4.6), due to Lemma 4.3 we get

$$\begin{gathered}
B(u^k - u - \widetilde{\psi_k}, u^k - u - \widetilde{\psi_k}) =\\
= \langle (Tu^k)_\nu - (Tu)_\nu, u^k_\nu - h_k - (u_\nu - h) + h_k - h\rangle_{\Gamma_0} -\\
-\langle T\widetilde{\psi_k}, u^k_\nu - u_\nu)\rangle_{\Gamma_0} + \langle P_k - P, u^k - u - \widetilde{\psi_k}\rangle_{\Gamma_1} \leq\\
\leq \langle (Tu^k)_\nu - (Tu)_\nu, h_k - h\rangle_{\Gamma_0} - \langle T\widetilde{\psi_k}, u^k - u\rangle_{\Gamma_0} +\\
+\langle P_k - P, u^k - u - \widetilde{\psi_k}\rangle_{\Gamma_1} = B(u^k - u - \widetilde{\psi_k}, \phi_k\chi) -\\
-\langle T\widetilde{\psi_k}, u^k - u - \phi_k\chi\rangle_{\Gamma_0} + \langle P_k - P, u^k - u - \widetilde{\psi_k} - \phi_k\chi\rangle_{\Gamma_1}.
\end{gathered} \tag{4.14}$$

The form (4.3) is coercive on the space $\mathcal{H}^1_*(\Omega)$ defined in (4.8) (see[2]),

$$B(u,u) \geq \alpha \|u\|^2_{1,\Omega}. \tag{4.15}$$

Evidently, $(u^k - u - \widetilde{\psi_k}) \in \mathcal{H}^1_*(\Omega)$. From the estimates (4.14) and (4.15) we have:

$$\begin{aligned}
\alpha \|u^k - u - \widetilde{\psi_k}\|^2_{1,\Omega} &\leq B(u^k - u - \widetilde{\psi_k}, \phi_k \chi) - \\
-\langle T\widetilde{\psi_k}, u^k - u - \phi_k\chi)\rangle_{\Gamma_0} &+ \langle P_k - P, u^k - u - \widetilde{\psi_k} - \phi_k\chi\rangle_{\Gamma_1} \leq \\
\leq C\|\phi_k\chi\|_{1,\Omega}\|u^k - u - \widetilde{\psi_k}\|_{1,\Omega} &+ \|T\widetilde{\psi_k}\|_{-\frac{1}{2},\Gamma_0}\|u^k - u - \phi_k\chi\|_{\frac{1}{2},\Gamma_0} + \\
+\|P_k - P\|_{-\frac{1}{2},\Gamma_1}&\|u^k - u - \widetilde{\psi_k} - \phi_k\chi\|_{\frac{1}{2},\Gamma_1},
\end{aligned} \tag{4.16}$$

where C is defined from (4.4).

Define the norm $\|\cdot\|_{(C^1(\Gamma))^n}$ and the constant C_1 as follows:

$$\|g\|_{(C^1(\Gamma))^n} = \inf_{\substack{G \in (C^1(\overline{\Omega}))^n \\ G|_\Gamma = g}} \|G\|_{(C^1(\overline{\Omega}))^n}, \quad C_1 = \|\nu\|_{(C^1(\Gamma))^n}. \tag{4.17}$$

Since χ is any function from the conditions (4.13), then, we can choose it such

$$\|\phi_k\chi\|_{1,\Omega} \leq \|\chi\|_{(C^1(\overline{\Omega}))^n}\|\phi_k\|_{1,\Omega} \leq C_1 d\|h_k - h\|_{\frac{1}{2},\Gamma_0}.$$

From the estimate (4.16) we get

$$\begin{aligned}
\alpha\|u^k - u - \widetilde{\psi_k}\|^2_{1,\Omega} &\leq \big(CC_1 d\|h_k - h\|_{\frac{1}{2},\Gamma_0} + \|T\widetilde{\psi_k}\|_{-\frac{1}{2},\Gamma_0} + \\
+\|P_k - P\|_{-\frac{1}{2},\Gamma_1}\big)\|u^k - u - \widetilde{\psi_k}\|_{1,\Omega} &+ \|T\widetilde{\psi_k}\|_{-\frac{1}{2},\Gamma_0}\big(\|\widetilde{\psi_k}\|_{\frac{1}{2},\Gamma_0} + \\
+C_1 d\|h_k - h\|_{\frac{1}{2},\Gamma_0}\big)&C_1 d\|P_k - P\|_{-\frac{1}{2},\Gamma_1}\|h_k - h\|_{\frac{1}{2},\Gamma_0}.
\end{aligned} \tag{4.18}$$

Show that $\widetilde{\psi_k} \xrightarrow{\mathcal{H}^1(\Omega)} 0$, $\quad T\widetilde{\psi_k} \xrightarrow{\mathcal{H}^{-\frac{1}{2}}(\Gamma_0)} 0$. Since the form (4.3) is coercive on the space $\{v \in \mathcal{H}^1(\Omega), v|_{\Gamma_0} = 0\}$ (see[2]) and $\widetilde{\psi_k}$, which is defined in (4.13), assignes minimal value to the functional $B(v,v)$ on the closed convex set $V_k = \{v \in \mathcal{H}^1(\Omega), v|_{\Gamma_2} = \psi_k - \psi\}$, so

$$\alpha_1\|\widetilde{\psi_k}\|^2_{\mathcal{H}^1(\Omega)} \leq B(\widetilde{\psi_k}, \widetilde{\psi_k}) = \inf_{v \in V_k} B(v,v) \leq C\|\psi_k - \psi\|^2_{\frac{1}{2},\Gamma_2},$$

which implies that $\widetilde{\psi_k} \xrightarrow{\mathcal{H}^1(\Omega)} 0$.

Let $\varphi \in \mathcal{H}^{\frac{1}{2}}(\Gamma)$, $v \in \mathcal{H}^1(\Omega)$ and $v|_\Gamma = \chi_0\varphi$, where χ_0 is taken from the remark of (4.13). Then, by the Green formula (4.5),

$$\langle T\widetilde{\psi_k}, \varphi\rangle_{\Gamma_0} = \langle T\widetilde{\psi_k}, \chi_0\varphi\rangle_\Gamma = B(\widetilde{\psi_k}, v) \leq C\|\widetilde{\psi_k}\|_{1,\Omega}\|v\|_{1,\Omega}.$$

Due to the definition of norm $\|\cdot\|_{\frac{1}{2},\Gamma_0}$ and to the continuity of the operator of multiplication on a $C^\infty(\Gamma)$- function in $H^{\frac{1}{2}}(\Gamma),$ the last inequality implies

$$\langle T\widetilde{\psi_k},\varphi\rangle_{\Gamma_0} \le Cd\|\widetilde{\psi_k}\|_{1,\Omega}\|\varphi\|_{\frac{1}{2},\Gamma_0},$$

so that

$$\|T\widetilde{\psi_k}\|_{-\frac{1}{2},\Gamma_0} \le Cd\|\widetilde{\psi_k}\|_{1,\Omega}$$

and $T\widetilde{\psi_k} \xrightarrow{\mathcal{H}^{-\frac{1}{2}}(\Gamma_0)} 0.$

Based on these facts, we conclude from (4.18):

$$\text{if}\quad P_k \xrightarrow{\mathcal{H}^{-\frac{1}{2}}\Gamma_1)} P,\quad \psi_k \xrightarrow{\mathcal{H}^{\frac{1}{2}}(\Gamma_2)} \psi,\quad h_k \xrightarrow{H^{\frac{1}{2}}(\Gamma_0)} h,$$

$$\text{then}\quad u^k \xrightarrow{\mathcal{H}^1(\Omega)} u,\quad Tu^k \xrightarrow{\mathcal{H}^{-\frac{1}{2}}(\Gamma)} Tu. \tag{4.19}$$

Analogously to the problem (2.1), let us formulate the implicit Signorini problem for the operator (4.1), with Dirichlet and Neumann boundary conditions: find a displacement $u \in H^1(\Omega)$ such that

$$\begin{aligned} &Au = f,\\ &u|_{\Gamma_2} = \psi,\quad Tu|_{\Gamma_1} = P,\\ &(Tu)_s|_{\Gamma_0} = 0;\quad (Tu)_\nu|_{\Gamma_0} \ge 0;\\ &u_\nu|_{\Gamma_0} \ge h - \langle (Tu)_\nu,\varphi\rangle_{\Gamma_0},\\ &\langle (Tu)_\nu, u_\nu - h + \langle (Tu)_\nu,\varphi\rangle_{\Gamma_0}\rangle_{\Gamma_0} = 0; \end{aligned} \tag{4.20}$$

$$\begin{gathered} f \in \mathcal{L}_2(\Omega),\ P \in \widetilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_1 \cup \overline{\Gamma}_2),\ \psi \in \mathcal{H}^{\frac{1}{2}}(\Gamma_2),\\ h,\varphi \in H^{\frac{1}{2}}(\Gamma_0),\ \varphi \ge 0. \end{gathered} \tag{4.21}$$

In the mechanical process expressed by this problem, contrary to the classical problem, the frame has some elastic properties and expands uniformly with constant $\langle (Tu)_\nu,\varphi\rangle_{\Gamma_0}$. Show the solvability of the problem (4.20) under the conditions (4.21) and examine the unequeness problem.

Let, for any number $\lambda \ge 0$, u^λ be a solution of the following classical Signorini problem:

$$Au^\lambda = f,\qquad u^\lambda \in \mathcal{H}^1(\Omega), \tag{4.22}$$

$$u^\lambda|_{\Gamma_2} = \psi,\quad Tu^\lambda|_{\Gamma_1} = P, \tag{4.23}$$

$$(Tu^\lambda)_\nu|_{\Gamma_0} \ge 0;\quad (Tu^\lambda)_s|_{\Gamma_0} = 0, \tag{4.24}$$

$$u^\lambda_\nu|_{\Gamma_0} \ge h - \lambda, \tag{4.25}$$

$$\langle (Tu^\lambda)_\nu, u^\lambda_\nu - h + \lambda\rangle_{\Gamma_0} = 0. \tag{4.26}$$

Here f, P, ψ, h are the data of the problem (4.20). Define the mapping $F : \mathbb{R}^+ \to \mathbb{R}^+$ for the problem (4.20):

$$F(\lambda) = \langle (Tu^\lambda)_\nu, \varphi \rangle_{\Gamma_0}, \quad \lambda \geq 0. \tag{4.27}$$

Due to Lemma 4.3, such definition is correct. It is clear that $F(\lambda) \geq 0$ and if $F(\lambda) = \lambda$, then the corresponding u^λ is the solution of the problem (4.20), and vice versa, if u is a solution of (4.20), then $\lambda = \langle (Tu)_\nu, \varphi \rangle_{\Gamma_0}$ satisfies the condition $F(\lambda) = \lambda$, i.e., the number of solutions of the problem (4.20) coincides with the number of stationary points of the function $F(\lambda)$. Show that the problem (4.20) has a solution under the conditions (4.21), i.e., the function $F(\lambda)$ has at last one stationary point. First of all, we show the boundedness of the function $F(\lambda)$. From (4.24) and (4.26) we get

$$\langle (Tu^\lambda)_\nu, h - u^\lambda_\nu \rangle_{\Gamma_0} = \lambda \langle (Tu^\lambda)_\nu, 1 \rangle_{\Gamma_0} \geq 0, \quad \lambda \geq 0.$$

Let the functions $\widetilde{\phi}, \widetilde{\psi} \in \mathcal{H}^1(\Omega)$ satisfy the following conditions:

$$\begin{gathered} A\widetilde{\psi} = f; \\ \widetilde{\psi}|_{\Gamma_2} = \psi, \quad T\widetilde{\psi}|_{\Gamma_1} = P; \quad (T\widetilde{\psi})_\nu|_{\Gamma_0} = 0; \\ \widetilde{\phi}_\nu|_{\Gamma_0} = h; \quad \widetilde{\phi}|_{\Gamma_2} = \psi. \end{gathered}$$

Then from the formula (4.5) and the conditions (4.22), (4.23)

$$B(u^\lambda - \widetilde{\psi}, \widetilde{\phi} - u^\lambda) = \langle (Tu^\lambda)_\nu, h - u^\lambda_\nu \rangle_{\Gamma_0} \geq 0, \quad \lambda \geq 0. \tag{4.28}$$

Evidently $(u^\lambda - \widetilde{\psi}) \in \mathcal{H}^1_*(\Omega)$ and, as it was mentioned, for the form $B(u, v)$ (4.15) holds on the space $\mathcal{H}^1_*(\Omega)$. Therefore, from the (4.28) we get:

$$\begin{aligned} \alpha \|u^\lambda - \widetilde{\psi}\|^2_{1,\Omega} \leq B(u^\lambda - \widetilde{\psi}, u^\lambda - \widetilde{\psi}) \leq B(u^\lambda - \widetilde{\psi}, \widetilde{\phi} - \widetilde{\psi}) \leq \\ \leq C \|u^\lambda - \widetilde{\psi}\|_{1,\Omega} \|\widetilde{\phi} - \widetilde{\psi}\|_{1,\Omega}, \end{aligned}$$

i.e.,

$$\|u^\lambda - \widetilde{\psi}\|_{1,\Omega} < \infty.$$

Taking

$$\begin{aligned} \Psi \in H^1(\Omega), \quad \Psi|_{\Gamma_0} = 1, \quad \Psi|_{\Gamma_2} = 0, \\ \Phi \in H^1(\Omega), \quad \Phi|_{\Gamma_0} = \varphi \quad \Phi|_{\Gamma_2} = 0, \end{aligned} \tag{4.29}$$

from the formula (4.5), due to Lemma 4.3, we get

$$|F(\lambda)| = |\langle (Tu^\lambda)_\nu, \varphi \rangle_{\Gamma_0}| = |B(u^\lambda - \widetilde{\psi}, \Phi\chi)| \leq C \|u^\lambda - \widetilde{\psi}\|_{1,\Omega} \|\Phi\chi\|_{1,\Omega} < \infty.$$

Show that the mapping (4.32) is continuous. Make the following replacements in the problems (4.10) and (4.12):

$$h := h - \lambda_1, \quad h_k := h - \lambda_2, \quad \psi_k := \psi, \quad P_k := P. \tag{4.30}$$

Since the form (4.3), along with the property (4.4) and the symmetricity, has the coercivity property (4.15) on the space $\mathcal{H}^1_*(\Omega)$, therefore, it can be considered as a scalar product on the space $\mathcal{H}^1_*(\Omega)$ and, for this form the Schwarz inequality is true. Thus, from (4.13),(4.14), in view of (4.29),(4.30), we get:

$$B(u^{\lambda_1}-u^{\lambda_2},u^{\lambda_1}-u^{\lambda_2}) \leq (\lambda_2-\lambda_1)B(u^{\lambda_1}-u^{\lambda_2},\Psi\chi) \leq$$
$$\leq \lambda_1-\lambda_2\sqrt{B(u^{\lambda_1}-u^{\lambda_2},u^{\lambda_1}-u^{\lambda_2})}\,\sqrt{B(\Psi\chi,\Psi\chi)}\,,$$
$$|F(\lambda_1)-F(\lambda_2)| = |B(u^{\lambda_1}-u^{\lambda_2},\Phi\chi)| \leq$$
$$\leq \sqrt{B(u^{\lambda_1}-u^{\lambda_2},u^{\lambda_1}-u^{\lambda_2})}\,\sqrt{B(\Phi\chi,\Phi\chi)} \leq$$
$$\leq |\lambda_1-\lambda_2|\sqrt{B(\Psi\chi,\Psi\chi)}\,\sqrt{B(\Phi\chi,\Phi\chi)}\,.$$

Since Φ and Ψ are arbitrary functions, from the conditions (4.30), analogously to the above reasoning, we conclude:

$$|F(\lambda_1)-F(\lambda_2)| \leq CC_1^2|\lambda_1-\lambda_2|\,\|1\|_{H^{\frac{1}{2}}(\Gamma_0)}\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_0)}. \tag{4.31}$$

Here, the constants C and C_1 are defined in (4.4) and (4.17), respectively. Thus, the function $F(\lambda)$ is continuous. Since it is also bounded and $F(0)\geq 0$, it has at least one stationary point. Therefore, the corresponding problem (4.20) under the conditions (4.21) has a solution.

It is easy to see that

$$\|1\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq \|1\|_{H^1(\Omega)} = \operatorname{mes}\Omega.$$

Hence and from the estimate (4.31) we have

$$|F(\lambda_1)-F(\lambda_2)| \leq CC_1^2\operatorname{mes}\Omega|\lambda_1-\lambda_2|\,\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_0)}.$$

It is clear, that if

$$\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_0)} < \frac{1}{CC_1^2\operatorname{mes}\Omega},$$

then the function (4.27) is a contractive mapping and it has unique stationary point. Correspondingly, the problem (4.20) has a unique solution. From the estimates in (4.14), taking into account (4.30), we have

$$(\lambda_1-\lambda_2)\langle (Tu^{\lambda_2})_\nu-(Tu^{\lambda_1})_\nu,1\rangle_{\Gamma_0} = \langle (Tu^k)_\nu-(Tu)_\nu,h_k-h\rangle_{\Gamma_0} \geq 0.$$

Hence, if $\varphi=\ell$, $\ell=\text{const}>0$, and $\lambda_1<\lambda_2$, then

$$F(\lambda_1)-F(\lambda_2) = \ell\langle (Tu^{\lambda_1})_\nu-(Tu^{\lambda_2})_\nu,1\rangle_{\Gamma_0} \geq 0.$$

Therefore, the function $F(\lambda)$ is nonincreasing and we see that the problem (4.20) has a unique solution in the case $\varphi=\text{const}>0$ as well. Thus, we have proved following

Theorem 4.4. *The implicit Signorini problem* (4.20)*, under the conditions* (4.21)*, has a solution. This solution is unique if one of the following conditions holds:*

(i) $\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_0)} < \dfrac{1}{CC_1^2 \operatorname{mes}\Omega}$,

where C and C_1 are constants defined by (4.4) *and* (4.17)*, respectively.*

(ii) $\varphi = \ell, \quad \ell = \text{const} > 0.$

Consider the uniqueness question in general. Show that for the uniqueness of the solution of the problem (4.20), under the conditions (4.21) it is necessary and sufficient that the function defined by the equality (4.27) be nonincreasing. The sufficiency is clear, show the necessity. Suppose that for some data f, P, ψ, h, φ from the conditions (4.21) the nonincreaseness of the corresponding function $F(\lambda)$ is violated, i.e., let there exist nonnegative numbers λ_1, λ_2, that

$$\lambda_2 > \lambda_1, \quad F(\lambda_2) > F(\lambda_1).$$

Then, show that there exist constants α_1,α_2, $\alpha_2 > 0$ such that the problem (4.20) for

$$\overline{f} = f,\ \overline{P} = P,\ \overline{\psi} = \psi,\ \overline{h} = h + \alpha_1,\ \overline{\varphi} = \alpha_2 \varphi \tag{4.32}$$

has at last two solutions. Indeed, denote by $\overline{u}^\lambda$ the solution of the corresponding problem (4.22)–(4.26) with the data (4.32), and, by $\overline{F}(\lambda)$, the corresponding mapping (4.27) for the data. Then it is easy to check the validity of the following equalities:

$$\overline{u}^\lambda = u^{\lambda-\alpha_1}, \quad \overline{F}(\lambda) = \alpha_2 F(\lambda - \alpha_1).$$

Here u^λ and $F(\lambda)$ are defined by the data f, P, ψ, h, φ. If we take

$$\alpha_2 = \frac{\lambda_2 - \lambda_1}{F(\lambda_2) - F(\lambda_1)}, \quad \alpha_1 = \alpha_2 F(\lambda_1) - \lambda_1,$$

then it is evident that $\alpha_2 > 0$, and the stationary points of the function $\overline{F}(\lambda)$ are

$$\alpha_2 F(\lambda_1), \quad \alpha_2 F(\lambda_2),$$

which means that the problem (4.20), under the conditions (4.32), has at least two solutions. Hence, in order that this problem, in general, to have the unique solution, it is necessary and sufficient that for any functions f, P, ψ, h, φ, from the conditions (4.21) and for any numbers $\lambda_2 > \lambda_1 \geq 0$, the following condition holds:

$$\langle (Tu^{\lambda_1})_\nu - (Tu^{\lambda_2})_\nu, \varphi \rangle_{\Gamma_0} \geq 0, \quad \varphi \in H^{\frac{1}{2}}(\Gamma_0), \quad \varphi \geq 0.$$

Since the data φ have not participated in obtaining u^λ, this condition is equivalent to the following fact:

For any data from the conditions (4.11),

$$((Tu^0)_\nu - (Tu^\lambda)_\nu)|_{\Gamma_0} \geq 0. \tag{4.33}$$

Here, u^λ is the solution of the problem (4.22)–(4.26), while u^0 satisfies the problem

$$Au^0 = f, \qquad u^0 \in \mathcal{H}^1(\Omega); \tag{4.34}$$
$$u^0|_{\Gamma_2} = \psi; \quad Tu^0|_{\Gamma_1} = P; \tag{4.35}$$
$$(Tu^0)_s|_{\Gamma_0} = 0; \quad (Tu^0)_\nu|_{\Gamma_0} \geq 0; \tag{4.36}$$
$$u^0_\nu|_{\Gamma_0} \geq h; \tag{4.37}$$
$$\langle (Tu^0)_\nu, u^0_\nu - h\rangle_{\Gamma_0} = 0, \tag{4.38}$$

where the data are taken from the problem (4.20).

Prove the following

Theorem 4.5. *In order that the problem* (4.20) *to have a unique solution under the conditions* (4.21)*, it is necessary and sufficient that the following problem*

$$Av = 0, \qquad v \in \mathcal{H}^1(\Omega); \tag{4.39}$$
$$v|_{\Gamma_2} = 0; \quad Tv|_{\Gamma_1} = 0; \tag{4.40}$$
$$(Tv)_s|_{\Gamma_0} = 0; \quad (Tv)_\nu|_{\Gamma_0} \geq 0; \tag{4.41}$$
$$v_\nu|_{\Gamma_0} \geq g; \tag{4.42}$$
$$\langle (Tv)_\nu, v_\nu - g\rangle_{\Gamma_0} = 0; \tag{4.43}$$
$$v_\nu|_{\widetilde{\Gamma}} = g \tag{4.44}$$

has the solution under the following conditions: $\Gamma_0, \Gamma_1, \Gamma_2$ *satisfy the conditions* (4.7) *and*

$$\begin{gathered} \widetilde{\Gamma} \subset \Gamma_0, \quad \operatorname{mes} \widetilde{\Gamma} > 0, \quad \gamma \in \mathbb{R}, \quad \gamma \geq 0, \\ g \in H^1_{\text{loc}}(\Gamma_0) \cap H^{\frac{1}{2}}(\Gamma_0), \quad g|_{\Gamma_0} \leq \gamma, \quad g|_{\widetilde{\Gamma}} = \gamma. \end{gathered} \tag{4.45}$$

Proof. Show that the solvability of the problem (4.39)–(4.44) is equivalent to (4.33) when

$$f \in \mathcal{L}_2(\Omega), \quad \psi \in \overset{1}{\mathcal{H}}(\Gamma_2), \quad P \in \mathcal{L}_2(\Gamma_1), \quad h \in H^1_{\text{loc}}(\Gamma_0). \tag{4.46}$$

Indeed, suppose that (4.33) holds for (4.46). Choose arbitrary $\widetilde{\Gamma}, g$ and γ from the conditions (4.45) and take

$$\begin{gathered} f \in \mathcal{L}_2(\Omega), \quad \psi \in \overset{1}{\mathcal{H}}(\Gamma_2), \quad P \in \mathcal{L}_2(\Gamma_1), \\ h = g + w_\nu|_{\Gamma_0}, \quad \lambda = \gamma, \end{gathered} \tag{4.47}$$

where w satisfies the following conditions

$$\begin{aligned} & Aw = f, \qquad w \in \mathcal{H}^1(\Omega) \cup \mathcal{H}^1_{\mathrm{loc}}(\Gamma_1 \cup \overline{\Gamma}_0); \\ & w|_{\Gamma_2} = \psi, \quad Tw|_{\Gamma_1} = P, \quad Tw \in \mathcal{L}^2(\Gamma_0), \\ & (Tw)_s|_{\Gamma_0} = 0; \quad (Tw)_\nu|_{\widetilde{\Gamma}} > 0, \quad (Tw)_\nu|_{\Gamma_0 \setminus \widetilde{\Gamma}} = 0. \end{aligned} \tag{4.48}$$

Due to the regularization theorem (see[3]), there obviously exists such a function w, and therefore, $h \in H^1_{\mathrm{loc}}(\Gamma_0)$.

Suppose that u^0 and u^λ are the solutions of the problems (4.34)–(4.38) and (4.22)–(4.26), respectively, with the data (4.47),(4.48). Show, that $u^\lambda = w$. Indeed, it is clear that the conditions (4.22)–(4.25) are true for w. Check the last condition (4.26). From (4.45),(4.47) and (4.48), we have

$$\langle (Tw)_\nu, w_\nu - h + \lambda \rangle_{\Gamma_0} = \langle (Tw)_\nu, \gamma - g \rangle_{\Gamma_0} = \int\limits_{\Gamma_0} (Tw)_\nu (\gamma - g) d\Gamma = 0.$$

Hence, w is the solution of the problem (4.22)–(4.26) with the data (4.47) and $u^\lambda = w$. Thus

$$h = g + u^\lambda_\nu|_{\Gamma_0}. \tag{4.49}$$

Show that $Tu^0 \in \mathcal{L}^2_{\mathrm{loc}}(\Gamma_0)$. First we prove the following general fact:

$$\text{“If } u \in \mathcal{H}^{\frac{3}{2}}(\Omega), \ Au \in \mathcal{L}^2(\Omega), \text{ then } Tu \in \mathcal{L}_2(\Gamma)\text{”}.$$

Indeed, if $u \in \mathcal{H}^{\frac{3}{2}+\varepsilon}(\Omega)$, $\varepsilon > 0$ and $Au \in \mathcal{L}_2(\Omega)$, then, in view of the form (4.2) of the operator $T(x, \partial, \nu)$, we get $Tu \in \mathcal{H}^\varepsilon(\Gamma)$. If $u \in \mathcal{H}^1(\Omega)$, $Au \in \mathcal{L}_2(\Omega)$, then Tu is defined from the formula (4.5) and $Tu \in \mathcal{H}^{-\frac{1}{2}}(\Gamma)$. Hence, when $s = 1$, and $s > \frac{3}{2}$, then, $Tu \in \mathcal{H}^{s-\frac{3}{2}}(\Gamma)$, when $u \in \mathcal{H}^s(\Omega)$, $Au \in \mathcal{L}_2(\Omega)$. Due to the interpolation theorem (see[6]), the same is true when $s \in [1, \frac{3}{2}]$, which proves the above mentioned fact. Due to the regularization theorem for the Signorini problem of [3], from (4.47), (4.48) we have $u^0 \in \mathcal{H}^{\frac{3}{2}}(\Omega')$, where Ω' is any open set with $\overline{\Omega'} \subset \overline{\Omega} \setminus \partial\Gamma_1$, which, as it is already proved, means that $Tu^0 \in \mathcal{L}^2_{\mathrm{loc}}(\Gamma \setminus \partial\Gamma_1)$. Hence,

$$Tu^0 \in \mathcal{L}^2_{\mathrm{loc}}(\Gamma_0), \quad Tu^\lambda \in \mathcal{L}_2(\Gamma_0). \tag{4.50}$$

Show that the function $v = u^0 - u^\lambda$ is a solution of the problem (4.39)–(4.44). Indeed, due to the conditions (4.22)–(4.24), (4.34)–(4.36) and the assumption (4.33), v satisfies the conditions (4.39)–(4.41). From (4.37) and (4.49)

$$v_\nu|_{\Gamma_0} \geq h - u^\lambda_\nu|_{\Gamma_0} = g.$$

To prove (4.43), we note that from the already proved conditions (4.41), (4.42) and (4.49), we have

$$0 \leq \langle (Tv)_\nu, v_\nu - g \rangle_{\Gamma_0} = \langle (Tu^0)_\nu - (Tu^\lambda)_\nu, u^0_\nu - h \rangle_{\Gamma_0} =$$

$$= -\langle (Tu^\lambda)_\nu, u^0_\nu - h\rangle_{\Gamma_0} \leq 0,$$

i.e., $\langle (Tv)_\nu, v_\nu - g\rangle_{\Gamma_0} = 0$.

Prove (4.44). As it was mentioned, the inequalities $(Tu^0)_\nu|_{\Gamma_0} \geq 0$ and $(Tu^\lambda)_\nu|_{\Gamma_0} \geq 0$ have the same meaning as in section 1, but, by (4.50) and by Lemma 4.2, these inequalities, as well as (4.25) and (4.37), hold almost everywhere on the part Γ_0 and so, due to Lemma 4.1 and (4.50), (4.26) and (4.38), can be rewritten as follows

$$\int\limits_{\Gamma_0} (Tu^\lambda)_\nu (u^\lambda_\nu - h + \lambda) d\Gamma = 0, \quad \int\limits_{\Gamma_0} (Tu^0)_\nu (u^0_\nu - h) d\Gamma = 0.$$

from these equalities and from (4.33), we have

$$\begin{aligned} (u^\lambda_\nu - h + \lambda)\big|_{\{(Tu^\lambda)_\nu > 0\}\cap\Gamma_0} &= 0, \\ (u^0_\nu - h)\big|_{\{(Tu^0)_\nu > 0\}\cap\Gamma_0} &= 0, \\ (Tu^0)_\nu\big|_{\{(Tu^\lambda)_\nu > 0\}\cap\Gamma_0} &> 0. \end{aligned} \tag{4.51}$$

As we have shown, $u^\lambda = w$. Then, from the conditions (4.48), we get:

$$\widetilde{\Gamma} = \{x \in \Gamma_0;\ (Tu^\lambda)_\nu > 0\}.$$

On the other hand, from (4.51) we have

$$\begin{aligned} (u^\lambda_\nu - h + \lambda)|_{\widetilde{\Gamma}} = 0, \quad (u^0_\nu - h)|_{\widetilde{\Gamma}} = 0, \\ v_\nu|_{\widetilde{\Gamma}} = (u^0_\nu - u^\lambda_\nu)|_{\widetilde{\Gamma}} = \lambda = \gamma. \end{aligned}$$

Thus, due to the assumption (4.33) under the conditions (4.46), the problem (4.39)–(4.44), has a solution under the conditions (4.45).

Show vice versa. Let the problem (4.39)–(4.44) have a solution under the conditions (4.45) and f, P, ψ, h satisfy the conditions (4.46), u^0 and u^λ be solutions of the problems (4.34)–(4.38) and (4.22)–(4.26), respectively, for this data and for $\lambda \geq 0$. Set

$$g = h - u^\lambda_\nu|_{\Gamma_0}, \quad \gamma = \lambda, \quad \widetilde{\Gamma} = \{x \in \Gamma_0;\ (Tu^\lambda)_\nu > 0\}, \tag{4.52}$$

Since the data f, P, ψ, h are taken from the conditions (4.46), by the regularization theorem, analogously to the above conducted reasoning, it is easy to see that $Tu^\lambda \in \mathcal{L}^2_{\mathrm{loc}}(\Gamma_0)$, i.e., $\widetilde{\Gamma}$ is defined within a set of measure zero. Suppose, $mes\widetilde{\Gamma} = 0$. This means that $(Tu^\lambda)_\nu = 0$ and so, the condition (4.33) is valid. Consider the case $mes\widetilde{\Gamma} > 0$. Let us show that the data $g, \widetilde{\Gamma}, \gamma$, defined in (4.52), satisfy the conditions (4.45). Due to the conditions (4.24)–(4.26) and Lemma 4.1, we get

$$g = h - u^\lambda_\nu|_{\Gamma_0} \leq \lambda = \gamma, \quad (\gamma - g)|_{\widetilde{\Gamma}} = (u^\lambda_\nu - h + \lambda)|_{\widetilde{\Gamma}} = 0.$$

Hence, $g, \widetilde{\Gamma}, \gamma$ satisfy the conditions (4.45) and the problem (4.39)–(4.44) under the conditions (4.52) has a solution v. Let us show that $v + u^\lambda$, satisfies conditions of the problem (4.34)–(4.38). Indeed, in view of the properties of the functions v and u^λ, it is evident that $v + u^\lambda$ satisfies the conditions (4.34)–(4.36). Due to (4.42) and (4.52) we have

$$(v_\nu + u_\nu^\lambda)|_{\Gamma_0} \geq g + u_\nu^\lambda|_{\Gamma_0} = h.$$

Prove (4.38). From (4.43),(4.44) and (4.52)

$$\langle (Tv)_\nu + (Tu^\lambda)_\nu, v_\nu + u_\nu^\lambda - h\rangle_{\Gamma_0} = \langle (Tv)_\nu + (Tu^\lambda)_\nu, v_\nu - g\rangle_{\Gamma_0} = \\ = \int\limits_{\Gamma_0} (Tu^\lambda)_\nu (v_\nu - g) d\Gamma = 0,$$

i.e., $v + u^\lambda = u^0$ and $(Tu^0)_\nu - (Tu^\lambda)_\nu = (Tv)_\nu \geq 0$.

Finally, we prove that the validity of (4.33) under the conditions (4.46) is equivalent to the solvability of the problem (4.39)–(4.44) with the data (4.45). If (4.33) holds in the conditions (4.46), then it is valid for any data f, ψ, P, h, from (4.11). Indeed, let ψ_k, P_k and h_k be taken from (4.46) such that

$$P_k \xrightarrow{\mathcal{H}^{-\frac{1}{2}}(\Gamma_1)} P, \quad \psi_k \xrightarrow{\mathcal{H}^{\frac{1}{2}}(\Gamma_2)} \psi, \quad h_k \xrightarrow{H^{\frac{1}{2}}(\Gamma_0)} h.$$

Then (4.33) yields for f, ψ_k, P_k, h_k and for stability of the classical Signorini's problem, which is expressed in (4.19), it is valid also for the data f, ψ, P, h. As we have mentioned, (4.33) is equivalent to the uniqueness of solution of (4.20) under the conditions (4.21). □

Let us formulate some assumption, similar to "maximum principle" (see Lemma 1.4), for the operator $A(x, \partial)$:

Assumption M. *If the function u satisfies the following conditions*

$$\begin{aligned} &Au = 0, \quad u \in \mathcal{H}^1(\Omega), \ Tu \in \mathcal{L}^2_{\text{loc}}(\Gamma_0). \\ &u|_{\Gamma_2} = 0, \quad Tu|_{\Gamma_1} = 0, \\ &(Tu)_s|_{\Gamma_0} = 0, \ (Tu)_\nu|_{\Gamma_0} \geq 0, \ (Tu)_\nu|_{\Gamma_0} \neq 0, \end{aligned} \tag{4.53}$$

where $\Gamma_0, \Gamma_1, \Gamma_2$ satisfy the conditions (4.7)*, then*

$$\operatorname*{essup}_{\Gamma_0 \cap \{(Tu)_\nu > 0\}} u_\nu = \operatorname*{essup}_{\Gamma} u_\nu. \tag{4.54}$$

From the conditions (4.53), $Tu \in L^2_{\text{loc}}(\Gamma_0)$, $\text{mes}\{x \in \Gamma_0; (Tu)_\nu > 0\} > 0$. The mechanical meaning of this fact is the following: if the body is fixed by its part Γ_2 and on this body only surface forces act, which are directed along the outer normal, then the normal component of displacement reachs its maximal value on that part on Γ_0, where the forces are nonzero.

Prove the following

Theorem 4.6. *For the solvability of the problem* (4.39)–(4.44) *under the conditions* (4.45)*, it is necessary and sufficient that the "Assumption M" is fulfilled.*

Proof. Assume that the "Assumption M" is fulfilled and v is the solution of Signorini's classical problem (4.39)–(4.43) with data (4.45). If $\gamma = 0$, then $v = 0$ and (4.44) yields. Let $\gamma > 0$. From the regularization theorem and from (4.43) we have

$$Tv \in \mathcal{L}^2_{\rm loc}(\Gamma_0), \quad v_\nu = g|_{\{x\in\Gamma_0;\,(Tv)_\nu>0\}}.$$

It is clear that v satisfies the conditions (4.53) and from (4.42) and (4.54), in view of the $g|_{\widetilde{\Gamma}} = \gamma$ we obtain

$$\operatorname*{essup}_{\Gamma} v_\nu = \operatorname*{essup}_{\Gamma_0\cap\{(Tv)_\nu>0\}} g \le \gamma, \quad v_\nu|_{\widetilde{\Gamma}} \ge \gamma,$$

and $v_\nu|_{\widetilde{\Gamma}} = \gamma$. So, v is a solution of the problem (4.39)–(4.44).

Let the "Assumption M" be false, i.e., there exist $\Gamma_0, \Gamma_1, \Gamma_2$, satisfying the conditions (4.7) and a function u from the conditions (4.53) such that

$$\operatorname*{essup}_{\Gamma} u_\nu > \operatorname*{essup}_{\Gamma_0\cap\{(Tu)_\nu>0\}} u_\nu = M.$$

Then there exist an open set Γ_0' and a number M' such that

$$\Gamma_1 \cup \overline{\Gamma}_0 \supset \overline{\Gamma}_0' \supset \overline{\Gamma}_0, \quad \operatorname*{essup}_{\Gamma_0'} u_\nu = M', \quad M' > M. \tag{4.55}$$

Suppose $M \le 0$. Then from (4.53), applying Lemma 4.1, we have

$$0 \ge \int\limits_{\Gamma_0} (Tu)_\nu u_\nu d\Gamma = \langle Tu, u\rangle_\Gamma = B(u,u) \ge 0,$$

whence $u = 0$, which contradicts to the conditions (4.53), i.e., $M > 0$.

Consider the following data

$$\begin{aligned} &\Gamma_0', \quad \Gamma_1' = \overline{\Gamma}_0 \cup \Gamma_1 \setminus \overline{\Gamma}_0', \quad \Gamma_2' = \Gamma_2,\\ &\widetilde{\Gamma}' = \Big\{x \in \Gamma_0';\ u_\nu > \frac{M'+M}{2}\Big\},\\ &g' = \inf\Big(u_\nu|_{\Gamma_0'}, \frac{M'+M}{2}\Big), \quad \gamma = \frac{M'+M}{2}. \end{aligned} \tag{4.56}$$

Due to Lemma1.1 (ii), since $u_\nu|_{\Gamma_0'} \in H^1_{\rm loc}(\Gamma_0')$ and $g' = u_\nu|_{\Gamma_0'} - \max(u_\nu|_{\Gamma_0'} - \frac{M'+M}{2}, 0)$, we have $g' \in H^1_{\rm loc}(\Gamma_0)$ and from (4.55) it is easy to see that the data (4.56) satisfy the conditions (4.45). Consider the Signorini problem (4.39)–(4.43) with the data (4.56). It has a unique solution. Show that this solution coincide with u. Indeed, it is evident, that u satisfies the conditions (4.39)–(4.42). From (4.55) and (4.56),

$$g' - u_\nu|_{\Gamma_0' \setminus \overset{\sim}{\Gamma}'} = 0, \quad (Tu)_\nu|_{\overset{\sim}{\Gamma}'} = 0; \quad u_\nu|_{\overset{\sim}{\Gamma}'} > \gamma.$$

Hence, from Lemma 4.1,

$$\langle (Tu)_\nu, u_\nu - h \rangle_{\Gamma_0'} = \int\limits_{\Gamma_0'} (Tu)_\nu (u_\nu - g') d\Gamma = 0.$$

Therefore, u is the unique solution for the problem (4.39)–(4.43) with the data (4.56), which does not satisfy the condition (4.44). This implies that, if the "Assumption M" is not true, then, there can be constructed such data in conditions (4.45), that the problem (4.39)–(4.44) will have no solution. □

Let us prove one more theorem.

Theorem 4.7. *In order that the "Assumption M" to be true, it is necessary and sufficient that for the solution u of the following classical Signorini problem*

$$\begin{aligned} &Au = 0, \qquad u \in \mathcal{H}^1(\Omega); \\ &u|_{\Gamma_2} = 0, \quad Tu|_{\Gamma_1} = 0, \\ &(Tu)_s|_{\Gamma_0} = 0; \quad (Tu)_\nu|_{\Gamma_0} \geq 0; \\ &u_\nu|_{\Gamma_0} \geq h, \quad \langle (Tu)_\nu, u_\nu - h \rangle_{\Gamma_0} = 0 \end{aligned} \tag{4.57}$$

with

$$h \in H^{\frac{1}{2}}(\Gamma_0), \quad \operatorname*{essup}_{\Gamma_0} h \geq 0 \tag{4.58}$$

and with $\Gamma_0, \Gamma_1, \Gamma_2$ satisfying the conditions (4.7), the following assertion is true

$$\operatorname*{essup}_{\Gamma} u_\nu = \operatorname*{essup}_{\Gamma_0} h. \tag{4.59}$$

Proof. Let the "Assumption M" be true. Prove (4.59). Take the sequence h_k as follows:

$$h_k \in H^1(\Gamma_0), \quad \operatorname*{essup}_{\Gamma} h_k \geq 0, \quad h_k \xrightarrow{H^{\frac{1}{2}}(\Gamma_0)} h,$$

where h is any function from the conditions (4.58) and u, u^k are the solutions of the problem (4.57) with data h and h^k, respactively. Then, due to the regularization theorem, $(Tu^k)_\nu \in L^2_{\text{loc}}(\Gamma_0)$ and u^k satisfies the conditions (4.53), so

$$\operatorname*{essup}_{\Gamma_0 \cap \{(Tu^k)_\nu > 0\}} u^k_\nu = \operatorname*{essup}_{\Gamma} u^k_\nu.$$

By Lemma 4.1, we have $u_\nu^k = h_k\big|_{\Gamma_0\cap\{(Tu^k)_\nu>0\}}$. Thus

$$\operatorname*{essup}_{\Gamma} u_\nu^k = \operatorname*{essup}_{\Gamma_0} h_k. \tag{4.60}$$

For (4.19) we have $u_\nu^k \xrightarrow{H^{\frac{1}{2}}(\Gamma)} u_\nu$, i.e., $\operatorname*{essup}_{\Gamma} u_\nu^k \to \operatorname*{essup}_{\Gamma} u_\nu$, $\operatorname*{essup}_{\Gamma_0} h_k \to \operatorname*{essup}_{\Gamma_0} h$. Therefore, from the equality (4.60) we get (4.59). If the maximum type principle fails, then in proof of Theorem 4.6, we have constructed such data in the conditions (4.45), that for the unique solution v of the Signorini problem (4.39)–(4.43) the condition $\operatorname*{essup}_{\Gamma} v_\nu = M' > \operatorname*{essup}_{\Gamma_0} g = \frac{M'+M}{2}$ holds, which completes the proof. □

Finally, due to Theorem 4.6 and Theorem 4.7, we have proved the following

Theorem 4.8. *In order for the implicit Signorini problem* (4.20)*, under the conditions* (4.21)*, to have a unique solution, it is necessary and sufficient, that*

(i) *the "Assumption M" be true.*

(ii) *for the solution of the classical Signorini problem* (4.57)*, under the conditions* (4.58)*, the equality* (4.59) *be valid.*

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