Mem. Differential Equations Math. Phys. 23(2001), 155–158

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SOME SUFFICIENT CONDITIONS FOR ξ-EXPONENTIALLY ASYMPTOTICALLY STABILITY OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

(Reported on March 26, 2001)

Consider a linear homogeneus system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t), \tag{1}$$

where $A : [0, +\infty[\to \mathbb{R}^{n\times n}$ is a real matrix-function with locally bounded variation components.

We give some sufficient conditions quaranteeing stability in the Liapunov sense of the system (1), which follow from the those given in [1].

The following notations an difinitions will be used in the paper: $\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ and $]a, b[(a, b) \in \mathbb{R}$ are, respectively, a closed and open intervals;

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|; \quad |X| = (|x_{ij}|)_{i,j=1}^{n,m};$$

$R^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$;

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X; I_n is the indentity $n \times n$-matrix.

$\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$;

$V_0^{+\infty}(X) = \sup_{b\in\mathbb{R}_+} V_0^b(X)$, where $V_0^b(X)$ is the sum of total variations on [0,b] of the components x_{ij} $(i = 1, \dots, n; j = 1, \dots, m)$ of the matrix-function $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(0) = 0$ and $v(x_{ij})(t) = V_0^t(x_{ij})$ for $0 < t < +\infty$ $(i = 1, \dots, n; j = 1, \dots, m)$.

$\operatorname{Re} z$ and $\operatorname{Im} z$ are a real and an imaginary parts of the complex number z;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ at the point t; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$;

$BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variations on every closed interval from $\mathbb{R}_+$;

$s_0 : BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}) \to BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ is an operator defined by

$$s_0(x)(t) \equiv x(t) - \sum_{0<\tau\le t} d_1x(\tau) - \sum_{0\le\tau<t} d_2x(\tau).$$

2000 *Mathematics Subject Classification.* 34B05.

Key words and phrases. Stability in the Liapunov sense, linear homologeneous systems of generalized ordinary differential equations.

If $g : \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function, $x : \mathbb{R}_+ \to \mathbb{R}$ and $0 \le s < t < +\infty$, then

$$\int_s^t x(\tau)dg(\tau) = \int_{]s,t[} x(\tau)dg_1(\tau) - \int_{]s,t[} x(\tau)dg_2(\tau) + \\ + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) - \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $g_j : \mathbb{R}_+ \to \mathbb{R}$ $(j = 1, 2)$ are continuons nondecreasing functions, such that $g_1(t) - g_2(t) \equiv s_0(g)(t)$, and $\int_{]s,t[} x(\tau)dg_j(\tau)$ is Lebesgue-Stiltjes integral over the open interval]s,t[with respect to the measure corresponding to the function g_j $(j = 1, 2)$ (if $s = t$, then $\int_s^t x(\tau)dg(\tau) = 0$);

A matrix-function is said to be nondecreasing if each of its components is such.

If $G = (g_{ik})_{i,k=1}^{l,n} : \mathbb{R}_+ \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function, $X = (x_{ik})_{i,k=1}^{n,m} : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau) \cdot X(\tau) = \Big(\sum_{k=1}^n \int_s^t x_{kj}(\tau)dg_{ik}(\tau) \Big)_{i,j=1}^{l,m} \quad \text{for} \quad 0 \le s \le t < +\infty.$$

If $G_j : \mathbb{R}_+ \to \mathbb{R}^{l\times n}$ $(j = 1, 2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$ and $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau) \cdot X(\tau) = \int_s^t dG_1(\tau) \cdot X(\tau) - \int_s^t dG_2(\tau) \cdot X(\tau) \quad \text{for} \quad 0 \le s \le t < +\infty.$$

Under a solution of the system (1) we understand a vector-function $x \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) \quad (0 \le s \le t < +\infty).$$

We will assume that $A = (a_{ik})_{i,k=1}^n \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $A(0) = O_{n\times n}$ and

$$\det(I_n + (-1)^j d_jA(t)) \ne 0 \quad \text{for} \quad t \in \mathbb{R}_+ \quad (j = 1, 2).$$

Let $x_0 \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ be a solution of the system (1).

Definition 1. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that

$$\lim_{t\to+\infty} \xi(t) = +\infty. \tag{2}$$

The the solution x_0 of the system (1) is called ξ-exponentially asymptotically stable, if there exists a positive number η such that for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that an arbitrary solution x of the system (1), satisfying the inequality

$$\|x(t_0) - x_0(t_0)\| < \delta$$

for some $t_0 \in \mathbb{R}_+$, admits the estimate

$$\|x(t) - x_0(t)\| < \varepsilon \exp(-\eta(\xi(t) - \xi(t_0))) \quad \text{for} \quad t \ge t_0.$$

Stability, uniformly stability and asymptotically stability of the solution x_0 are defined analogously as for systems of ordinary differential equations (see [2]), i.e. in case when $A(t)$ is the diagonal matrix-function with diagonal elements equal t. Note that exponentially asymptotically stable ([2]) is particular case of ξ-exponentially asymptotically stability if we assume $\xi(t) \equiv t$.

Definition 2. The system (1) is called stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable) if every solution of this system is stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable).

Definition 3. The matrix-function A is called stable (uniformly stable, asymptotically stable or ξ-exponentially asymptotically stable) if the system (1) is stable (unformly stable, asymptotically stable or ξ-exponentially asymptotically stable).

If $X \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, then $\mathcal{A}(X,\cdot) : BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times m}) \to BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times m})$ an operator defined by

$$\mathcal{A}(X,Y)(t) = Y(t) + \sum_{0<\tau\leq t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1}\cdot d_1Y(\tau) -$$
$$- \sum_{0\leq\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\cdot d_2Y(\tau) \quad \text{for} \quad t\in\mathbb{R}_+;$$

If $G \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, then $\mathcal{B}(G,\cdot) : BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times m}) \to BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times m})$ is an operator difined by

$$\mathcal{B}(G,X)(t) = G(t)X(t) - G(0)X(0) - \int_0^t dG(\tau)\cdot X(\tau) \quad \text{for} \quad t\in\mathbb{R}_+.$$

Moreover, if $\det(G(t)) \neq 0$ $(t\in\mathbb{R}_+)$, then $\mathcal{L}(G,\cdot) : BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times m}) \to BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times m})$ is an operator given by

$$\mathcal{L}(G,X)(t) = \int_0^t d[G(\tau) + \mathcal{B}(G,X)(\tau)]\cdot G^{-1}(\tau) \quad \text{for} \quad t\in\mathbb{R}_+.$$

Theorem 1. *Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function satisfying the condition* (2)*. Then the matrix-function A is ξ-exponentially asymptotically stable if and only if there exist a positive number η and a matrix-function $H \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ such that the conditions*

$$\det(H(t)) \neq 0 \quad \textit{for} \quad t\in\mathbb{R}_+, \quad \sup\Big\{\|H^{-1}(t)H(s)\| : t\geq s\geq 0\Big\} < +\infty$$

and

$$\Bigg\|\int_0^{+\infty} dV(\mathcal{L}(H,A) + \eta\,\mathrm{diag}(\xi,\ldots,\xi))(t)\cdot|H(t)|\Bigg\| < +\infty$$

hold.

Theorem 2. *Let the matrix-function $Q\in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ be ξ-exponentially asymptotically stable,*

$$\det(I_n + (-1)^j d_jQ(t)) \neq 0 \quad \textit{for} \quad t\in\mathbb{R}_+ \ (j=1,2).$$

Let, moreover, there exists a positive number η such that

$$\Big\|\int\limits_0^{+\infty}|Z^{-1}(t)|dV(\mathcal{A}(Q,A-Q)+\eta\,\mathrm{diag}(\xi,\dots,\xi))(t)\Big\|<+\infty,$$

where $Z(Z(0)=I_n)$ is the fundamental matrix of the system

$$dz(t)=dQ(t)\cdot z(t),$$

and $\xi:\mathbb{R}_+\to\mathbb{R}_+$ is the continuons nondecreasing function satisfying the condition (2). Then the matrix-function A is ξ-exponentially asymptotically stable, as well.

Theorem 3. *Let the constant matrix $P=(p_{ik})_{i,k=1}^n\in\mathbb{R}^n$ be stable* (*asymptotically stable or ξ-exponentially asymptotically stable*). *Let, moreover, $\lambda_1,\dots,\lambda_n$ $(\lambda_i\neq\lambda_j$ for $i\neq j)$ be its eigenvalues with the multiplicities $n_1,\dots,n_m$, respectively, and*

$$\int\limits_0^{+\infty}t^{n_l-1}\exp(-t\,\mathrm{Re}\,\lambda_l)dv(b_{ik})(t)<+\infty,\ \ (l=1,\dots,m;i,k=1,\dots,n),$$

where $\xi:\mathbb{R}_+\to\mathbb{R}_+$ is the continuouns nondecreasing function satisfying the condition (2), *$b_{ik}(t)\equiv a_{ik}(t)-p_{ik}t$, $a_{ik}\in BV_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R})$, $(i,k=1,\dots,n)$. Then the matrix-function $A=(a_{ik})_{i,k=1}^n$ is uniformly stable* (*asymptotically stable or ξ-exponentially asymptotically stable*) *as well.*

References

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