Mem. Differential Equations Math. Phys. 21(2000), 146–150

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ON NONDECREASING SINGULAR SOLUTIONS OF THE EMDEN–FOWLER EQUATION

(Reported on May 22, 2000)

Consider the Emden–Fowler equation

$$u^{(n)} = p(t)|u|^{\lambda} sign u, \quad p(t) \geq 0, \quad \lambda > 1, \quad n \geq 2, \tag{1}$$

with a locally Lebesgue integrable on (a, b) function $p(t)$, which differs from zero on a set of positive measure in any left neighborhood of b.

A solution $u : [a, b) \to (0, +\infty)$ of the equation (1) is said to be a nonoscillatory singular solution of the second kind, if

$$u^{(i)}(t) > 0 \quad (i = 0, \dots, n-1) \quad \text{for} \quad t \in (a, b), \quad \lim_{t \to b} u(t) = +\infty. \tag{2}$$

Problems on the existence of such solutions and on its asymptotic estimation were studied for the equation (1) in [1, p. 323–325], where they are reduced to the similar ones for a proper strongly increasing solutions, which are more investigated (see [1, 2] and the bibliography therein). In particular, this approach allows to give the sufficient condition

$$J(a, b) < +\infty, \quad \text{where} \quad J(s, t) \equiv \int_s^t p(\tau)(b - \tau)^{n-1} d\tau, \tag{3}$$

for the equation (1) to have a solution (2).

In this paper new necessary conditions of solvability of the problem (1)–(2) and two-sided asymptotic estimates of its solutions are obtained. Here also it is established the necessity of the condition (3) in certain cases.

Begin with a simple assertion which presents some important properties of solutions of the problem under consideration.

Lemma 1. *Let $u(t)$ be a solution of the problem (1), (2) and $\varphi(t) > 0$ be a nondecreasing function on (a, b). Then $v_i(t) = \sum_{l=0}^{i} u^{(l)}(t)(b - t)^l / l!$ $(i = 0, \dots, n-1)$ are nondecreasing unbounded functions on (a, b), satisfying there conditions*

$$\frac{\dot v_{\varphi,n-1}(t)}{v_{\varphi,n-1}(t)} = \frac{p(t)(b-t)^{n-1} v^{\lambda}_{\varphi,0}(t)}{(n-1)! \varphi^{\lambda-1}(t) v_{\varphi,n-1}(t)} + \frac{\dot\varphi(t)}{\varphi(t)}, \quad \frac{\dot v_{\varphi,i}(t)}{v_{\varphi,i+1}(t)} \geq \varphi_M(t), \tag{4}$$

where $v_{\varphi,i}(t) = v_i(t)\varphi(t)$, $\varphi_M(t) = \min\{(b - t)^{-1}, \dot\varphi(t)/(M\varphi(t))\}$.

The lower asymptotic estimate of solutions of the problem (1), (2) gives

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2000 *Mathematics Subject Classification.* 34C05

Key words and phrases. Emden-Fowler equation, nondecreasing singular solution.

Theorem 1. *Every solution u of the problem* (1), (2) *admits the lower estimate*

$$v_{n-1}(t) > \Big((n-1)!(\lambda-1)^{-1}J(t,b)\Big)^{1/(1-\lambda)}, \quad t \in (a,b), \tag{5}$$

provided the condition (3) *holds.*

Proof. Let u be a solution of the problem (1), (2). From (1) we deduce $\dot v_{n-1}(t) = u^{(n)}(t)(b-t)^{n-1}/(n-1)! \geq p(t)(b-t)^{n-1}v_{n-1}^{\lambda}(t)/(n-1)!$ for $t \in (a,b)$. Integrate this inequality taking into acount (2):

$$\frac{v_{n-1}^{1-\lambda}(t)}{\lambda-1} = \int\limits_t^b \frac{\dot v_{n-1}(\tau)d\tau}{v_{n-1}^{\lambda}(\tau)} < \int\limits_t^b \frac{p(\tau)(b-\tau)^{n-1}d\tau}{(n-1)!}.$$

The last estimate is equivalent to (5). Thus the theorem is proved.
The main result of this article is contained in the following □

Theorem 2. *Let $u(t)$ be a solution of the problem* (1)–(2) *and let $\varphi : (a,b) \to (0,+\infty)$ be any nondecreasing function. Then for any numbers $\nu \in [0,1)$, $\mu \in ((1-\nu)/n_1, (1-\nu)/n)$, $M > 0$ and $\sigma > 0$ the equality*

$$\lim_{t\to b} F_{\nu,\mu,\sigma,M}(\varphi)(t) = 0 \tag{6}$$

is true and the upper estimate

$$u(t) < \gamma[F_{\nu,\mu,\sigma,M}(\varphi)(t)]^{1/((1-\lambda)\mu)}, \tag{7}$$

is fulfilled, where

$$F_{\nu,\mu,\sigma,M}(\varphi)(t) = \varphi^{\sigma}(t)\int\limits_t^b \frac{(p(\tau)(b-\tau)^{n-1})^{\mu}}{\varphi^{\sigma}(\tau)\varphi_M^{\mu+\nu-1}(\tau)}\left(\frac{\dot\varphi(\tau)}{\varphi(\tau)}\right)^{\nu} d\tau,$$

$n_1 = 1 + (n-1)\lambda$ and $\gamma > 0$ depends only on n, λ, μ, ν.

Proof. Let u be a solution of the problem (1), (2) and $\varphi(t) > 0$ be a nondecreasing function on (a,b). By lemma 1 and the inequality

$$\sum_{i=1}^{n}\beta_i x_i \geq \prod_{i=1}^{n} x_i^{\beta_i}, \quad x_i > 0, \quad \beta_i \geq 0, \quad \sum_{i=1}^{n}\beta_i = 1,$$

for the derivative of the function $\omega_\varphi(t) = \prod\limits_{i=0}^{n-1} v_{\varphi,i}(t)$ we have

$$\frac{\dot\omega_\varphi(t)}{\omega_\varphi(t)} = \sum_{i=0}^{n-1}\frac{\dot v_{\varphi,i}(t)}{v_{\varphi,i}(t)} > \frac{p(t)(b-t)^{n-1}v_{\varphi,0}^{\lambda}(t)}{(n-1)!\varphi^{\lambda-1}(t)v_{\varphi,n-1}(t)} + \varphi_M(t)\sum_{i=0}^{n-2}\frac{v_{\varphi,i+1}(t)}{v_{\varphi,i}(t)} +$$
$$+\frac{\dot\varphi(t)}{\varphi(t)} \geq \gamma\Big(\frac{p(t)(b-t)^{n-1}v_{\varphi,0}^{\lambda}(t)}{\varphi^{\lambda-1}(t)v_{\varphi,n-1}(t)}\Big)^{\mu_n}\prod_{i=0}^{n-2}\Big(\frac{v_{\varphi,i+1}(t)}{v_{\varphi,i}(t)}\varphi_M(t)\Big)^{\mu_{i+1}} \times$$
$$\times\Big(\frac{\dot\varphi(t)}{\varphi(t)}\Big)^{\mu_{n+1}} = \gamma\Big(\frac{\dot\varphi(t)}{\varphi(t)}\Big)^{\nu}\frac{\Big(p(t)(b-t)^{n-1}\Big)^{\mu}v_{\varphi,0}^{\lambda\mu-\mu_1}(t)}{\varphi^{\lambda-1}(t)\Big(\varphi_M(t)\Big)^{\mu+\nu-1}}\prod_{i=1}^{n-1}v_{\varphi,i}^{\mu_i-\mu_{i+1}}(t),$$

where the numbers μ_i are defined by the equalities $\mu_n = \mu$, $\mu_{n+1} = \nu$,

$$\mu_i = \begin{cases} \lambda\mu - i\varepsilon_1, \ \ \varepsilon_1 = \dfrac{2(\nu + n_1\mu - 1)}{n(n-1)}, & \mu \in \left(\dfrac{1-\nu}{n_1}, \dfrac{2(1-\nu)}{n+n_1}\right) \\ \mu + (n-i)\varepsilon_1, \ \varepsilon_1 = \dfrac{2(1-\nu-n\mu)}{n(n-1)}, & \mu \in \left(\dfrac{2(1-\nu)}{n+n_1}, \dfrac{1-\nu}{n}\right) \end{cases}$$

for $i = 1, \dots, n$. It is clear that these numbers satisfy the conditions

$$\mu_i - \mu_{i+1} \geq \varepsilon_1 \quad (i = 1, \dots, n-1), \quad \lambda\mu_n - \mu_1 \geq \varepsilon_1, \quad \sum_{i=1}^{n+1} \mu_i = 1. \tag{9}$$

Dividing (8) by $\omega_\varphi^\varepsilon(t)\varphi^\delta(t)$ with $\varepsilon < \min\{\varepsilon_1, \sigma/n\}$, $\delta = \sigma - n\varepsilon$ and integrating the result, we get

$$\varepsilon^{-1} w_\varphi^{-\varepsilon}(t)\varphi^{-\delta}(t) > \int_t^b \dot{\omega}_\varphi(s)\omega_\varphi^{-1-\varepsilon}(s)\varphi^{-\delta}(s)ds \geq$$

$$\geq \gamma \int_t^b \frac{((b-x)^{n-1}p(x))^\mu}{\varphi^\sigma(x)\varphi_M^{\mu+\nu-1}(x)} \left(\frac{\dot{\varphi}(x)}{\varphi(x)}\right) \nu v_{\varphi,0}^{\lambda\mu_n - \mu_1 - \varepsilon}(x) \prod_{i=1}^{n-1} v_{\varphi,i}^{\mu_i - \mu_{i+1} - \varepsilon}(x)dx.$$

Hence by Lemma 1 and (9) it follows (7) and (8). The theorem is proved. □

The following part of the report contains some applications of the above results, where they are applied to analysis of the condition (3) and its natural extensions.

Theorem 3. *If the equation* (1) *with the function* $p(t)$ *sutisfying*

$$p(t)(b-t)^n < cJ(a,t) \quad \textit{for} \quad t \in [t_c, b) \subset (a,b), \quad c > 0, \tag{10}$$

has a solution of the type (2), *then the condition* (3) *holds and in some left neighborhood of* b

$$u(t) < \gamma J^{1/(\mu(1-\lambda))}(t,b), \quad \textit{where} \quad \gamma = \gamma(n, \lambda, \mu). \tag{11}$$

If along with (3)

$$p(t)(b-t)^n < cJ(t,b) \quad \textit{for} \quad t \in [t_c, b) \subset (a,b), \quad c > 0 \tag{12}$$

holds then there takes place the two-sided estimate

$$0 < \gamma_1 < u(t)J^{1/(\lambda-1)}(t,b) < \gamma_2, \tag{13}$$

where γ_1 *and* γ_2 *depend on* n, λ.

Proof. Let the equation (1) with the function $p(t)$ satisfying (10) has a solution of the type (2). Suppose to the contrary that (3) is not true. Then the function $\varphi(t) = J(a,t)$ increases with no bound on (a,b) and by (10) $(b-t)\varphi_c(t) = p(t)(b-t)^n/J(a,t) < 1$ holds. Therefore, for any $\sigma > \mu$ we have the equality $F_{0,\mu,\sigma,M}(\varphi)(t) = J^\sigma(a,t)\int_t^b (p(x)(b-x)^{n-1}J^{\mu-\sigma-1}(a,x)dx = J^\mu(a,t)/(\mu-\sigma)$, which contradicts the conclusion of Theorem 2. This means the validity of (3) and the boundness of $\varphi(t)$. In view of this fact we obtain $F_{0,\mu,\sigma,M}(\varphi)(t) > \gamma J(t,b)$ for $t > t_\gamma > a$, which by Theorem 2 yields (11).

Now assume that (3) and (12) are fulfilled. Then $\varphi(t) = 1/J(t,b)$ increases with no bound on (a,b) and for any $\mu, \sigma > 0$

$$F_{\nu,\mu,\sigma,M}(\varphi)(t) = J^{-\sigma}(t,b)\int\limits_t^b p(x)(b-x)^{n-1}J^{\mu+\sigma-1}(x,b)dx = J^{\mu}(t,b)/(\mu+\sigma)$$

holds from which by Theorem 2 there immediatelly follows the estimate (13). The theorem is proved. □

Corollary 1. *The problem* (1), (2) *with function* $p(t)$, *satisfying on* (a,b) $0 < c_1 < p(t)(b-t)^{n-1}\ln_k^{\sigma}(1/(b-t))l_k(1/(b-t)) < c_2$, $k \geq 0$, *has a solutions if and only if* $\sigma > 0$, *and every such solution admits the estimate* $0 < \gamma_1 < u(t)\ln_k^{\sigma/(1-\lambda)}(1/(b-t)) < \gamma_2$, *where* $\ln_0 t = t$, $\ln_{k+1} t = \ln(\ln_k t)$, $l_k(t) = \prod\limits_{i=0}^{k} \ln_i t$ *and* γ_1, γ_2 *depend only on* n, λ, σ.

In the general case it is useful to introduce into consideration the nonnegative functions $p_{f*}(t) = \min\{p(t),\ f(t)(b-t)^{-n}\}$, $p_f^*(t) = p(t) - p_{f*}(t)$ and the integrals

$$J_{f*}(s,t) = \int\limits_s^t \frac{p_{f*}(x)(b-x)^{n-1}}{f(x)}dx, \quad J_f^*(s,t) = \int\limits_s^t \frac{p_f^*(x)(b-x)^{n-1}}{f(x)}dx,$$

where $f(x)$ is an arbitrary nondecreasing positive function.

Theorem 4. *If the equation* (1) *has a soluttion* u *of the type* (2), *then for an arbitrary nondecreasing positive function* $f(t)$ *and all* $\mu \in (0,1/n)$ $J_{f*}(a,b) < +\infty$, $\lim\limits_{t\to b} f^{\mu}(t)J_{f,*}(t,b) = 0$ *and in some neighborhood of* b *the estimate*

$$u(t) < \gamma(f(t)J_{f*}^{1/\mu}(t,b))^{1/(1-\lambda)}$$

is true.

If in addition $p_{f*}(t)(b-t)^n < cf(t)J_{f*}(t,b)$ *for* $t \in (a,b)$, *holds then there takes place the estimate* $u(t) < \gamma(f(t)J_{f*}(t,b))^{1/(1-\lambda)}$.

Corollary 2. *If the problem* (1), (2) *with the function* $p(t)$, *satisfying*

$$J_f(t,b) = \int\limits_t^b \frac{p(x)(b-x)^{n-1}}{f(x)}dx = +\infty \quad for \quad t \in (a,b),$$

where $f(x) > 0$ *is an arbitrary nondecreasing function, is solvable, then*

$$\varlimsup_{t\to b} p_f^*(t)(b-t)^{n-1}/f(t) = +\infty, \quad J_f^*(t,b) = +\infty.$$

Theorem 5. *Let the equation* (1) *with the function* $p(t)$ *satisfying the condition*

$$p_{f*}(t)(b-t)^n > cf(t)J_{f*}(t,b) \quad for \quad t \in (a,b)$$

holds, where $f(x)$ *increases on* (a,b), *have a solution* u *of the type* (2). *Then the estimate*

$$u(t) < \gamma\left(f(t)J_{f*}^{(1-\nu)/\mu}(t,b)\right)^{1/(1-\lambda)},$$

holds, where γ *depends only on* n, λ, μ.

Theorem 6. *If the equation* (1) *with the function* $p(t)$ *satisfying the condition*

$$p_f^*(t)(b-t)^n > cf(t)J_f^*(a,t) \quad for \quad t \in (a,b),$$

where $f(x)$ *increases on* (a,b), *has a solution* u *of the type* (2), *then the estimate*

$$u(t) < \gamma\Big(f(t)J_f^*(a,t)\Big(\int_t^b \tau^{-1} sgn p_f^*(\tau)d\tau\Big)^{1/\mu}\Big)^{1/(1-\lambda)},$$

holds, where γ *depends only on* n, λ, μ.

To prove the Theorem it is sufficient to apply Theorem 2 with the function $\varphi(t) = J_f^*(a,t)$, which is unbounded in the case where (3) is not fulfilled.

References

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