Mem. Differential Equations Math. Phys. 19(2000), 133–141

M. Alves

ABOUT A PROBLEM ARISING IN CHEMICAL REACTOR THEORY

(Reported on December 23, 1998)

1. Notation

Throughout this paper $C \equiv C[0,1]$ denotes the Banach space of continuous functions $x : [0,1] \longrightarrow \mathbf{R}^1$,

$$\|x\|_C \overset{def}{=} \max_{0 \le t \le 1} |x(t)|;$$

$L_p \equiv L_p[0,1]$ $(1 \le p < \infty)$ denotes the Banach space of summable in p-th degree functions $x : [0,1] \longrightarrow \mathbf{R}^1$,

$$\|x\|_{L_p} \overset{def}{=} \Big(\int_0^1 |x(t)|^p \, dt \Big)^{1/p};$$

$L_\infty \equiv L_\infty[0,1]$ denotes the Banach space of essentially bounded measurable functions $x : [0,1] \longrightarrow \mathbf{R}^1$,

$$\|x\|_{L_\infty} \overset{def}{=} \operatorname{vrai\,sup}_{0 \le t \le 1} |x(t)|;$$

$W_p^2 \equiv W_p^2[0,1]$ denotes the Banach space of continuous functions $x : [0,1] \longrightarrow \mathbf{R}^1$ with the absolutely continuous derivative $\dot{x}$ such that $\ddot{x} \in L_p$,

$$\|x\|_{W_p^2} \overset{def}{=} \|\ddot{x}\|_{L_p} + |x(0)| + |\dot{x}(0)|.$$

2. The Space of Solutions D_p

Consider the boundary-value problem

$$\begin{cases} (\Im_0 x)(t) \overset{def}{=} \ddot{x}(t) + \dfrac{k}{t}\dot{x}(t) = f(t), \quad t = \in [0,1], \\ \dot{x}(0) = 0, \quad x(1) = \alpha, \end{cases} \tag{1}$$

where $k > -\dfrac{1}{p'}$, $f \in L_p$, $1 < p \le \infty$, $\alpha \in \mathbf{R}^1$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $p' = 1$ if $p = \infty$.

Considering this problem on the traditional space W_p^2, we see that $\Im_0$ is not defined as an operator acting from W_p^2 into L_p. Following the scheme given in the monograph [1], we will investigate this problem on the space $D_p \subset W_p^2$ of functions $x : [0,1] \longrightarrow \mathbf{R}^1$, such that $\dot{x}(0) = 0$ and defined by

$$x(t) = \int_0^t (t-s) z(s) \, ds + \beta$$

1991 *Mathematics Subject Classification.* 34B15, 34K10.

Key words and phrases. Boundary value problem, functional differential equation, chemical reactor theory.

for each pair $\{z,\beta\} \in L_p \times \mathbf{R}^1$. The space D_p is isomorphic to the direct product $L_p \times \mathbf{R}^1$. The isomorphisms $\mathcal{J} : L_p \times \mathbf{R}^1 \longrightarrow D_p$ and $\mathcal{J}^{-1} : D_p \longrightarrow L_p \times \mathbf{R}^1$ we define by the equalities $\mathcal{J} = \{\Lambda, Y\}$, $\mathcal{J}^{-1} = [\delta, r]$, where

$$\begin{cases} (\Lambda z)(t) = \int\limits_0^t (t-s)z(s)\,ds, \quad (Y\beta)(t) = \beta, \\ \delta x = \ddot{x}, \quad rx = x(0). \end{cases}$$

The space D_p becomes a Banach one under the norm

$$\| x \|_{D_p} \stackrel{def}{=} \| \ddot{x} \|_{L_p} + |x(0)|.$$

The principal part of the operator $\Im_0 : D_p \longrightarrow L_p$ is

$$(Qz)(t) \stackrel{def}{=} (\Im_0 \Lambda z)(t) = z(t) + (\mathcal{P}z)(t),$$

where $(\mathcal{P}z)(t) \stackrel{def}{=} \frac{k}{t} \int\limits_0^t z(s)\,ds$ is the Cesàro operator [2] on the space L_p. The functions $u(t) \equiv 1$ and $v(t) = t^{1-k}$ satisfy the equation $\Im_0 x = 0$. Nevertheless the fundamental system of $\Im_0 x = 0$ consists only of $u(t) \equiv 1$, such as the other element $v(t) = t^{1-k}$ does not belong to the space D_p. By virtue of the results of [5, p. 102] it follows that, if $k > -\frac{1}{p'}$, the operator $Q : L_p \longrightarrow L_p$ has the bounded inverse

$$(Q^{-1}z)(t) = z(t) - kt^{-(1+k)} \int\limits_0^t s^k z(s)\,ds.$$

The solution of the problem (1) on the space D_p is given by the expression

$$x = \mathcal{W}f + \alpha,$$

where the Green operator $\mathcal{W} : L_p \longrightarrow D_p$ is defined by

$$(\mathcal{W}f)(t) \stackrel{def}{=} \int\limits_0^1 W(t,s)f(s)\,ds,$$

$$W(t,s) \stackrel{def}{=} \begin{cases} \dfrac{s^k(t^{1-k}-1)}{1-k} & \text{if } 0 \le s \le t \le 1, \\ \dfrac{s^k(s^{1-k}-1)}{1-k} & \text{if } 0 \le t < s \le 1, \end{cases}$$

for $k > -\frac{1}{p'}$, $k \neq 1$, or

$$W(t,s) \stackrel{def}{=} \begin{cases} s \ln t & \text{if } 0 \le s \le t \le 1, \\ s \ln s & \text{if } 0 \le t < s \le 1, \end{cases}$$

for $k = 1$. Really, using the equality

$$\dot{x}(t) = \int\limits_0^t \ddot{x}(s)\,ds$$

for $x \in D_p$ we rewrite the problem (1) on the space D_p in the form

$$\ddot{x}(t) = f(t) - kt^{-(1+k)} \int\limits_0^t s^k f(s)\,ds, \quad t \in [0,1], \quad x(1) = \alpha.$$

Immediate computations show that

$$\begin{aligned} x(t) \quad &= \int\limits_0^t (t-s)\Big[f(s) - ks^{-(1+k)} \int\limits_0^s \tau^k f(\tau)\,d\tau\Big]\,ds + x(0) = \\ &= \int\limits_0^t \Big[t - s - ks^k \int\limits_s^t (t-\tau)\tau^{-(1+k)}\,d\tau\Big] f(s)\,ds + x(0). \end{aligned}$$

The condition $x(1) = 0$ gives

$$x(0) = -\int\limits_0^1 \Big[1 - s - ks^k \int\limits_s^1 (1-\tau)\tau^{-(1+k)}\,d\tau\Big] f(s)\,ds.$$

Consequently

$$x(t) = \int\limits_0^1 W(t,s) f(s)\,ds + \alpha, \quad t \in [0,1].$$

Bellow we will use results of [4] about estimation of the spectral radius $\rho(\mathcal{H})$ of the isotonic operator $\mathcal{H} : C \longrightarrow C$. We formulate this result in the form satisfying our aims:

Lemma 1. *Suppose that the isotonic operator $\mathcal{H}$ enjoys the property $(\mathcal{H}\zeta)(1) = 0$ for each $\zeta \in C$. The following statements are equivalent:*

1) *There exists $y \in C$ such that*

$$y(t) > 0, \quad y(t) - (\mathcal{H}y)(t) > 0, \quad t \in [0,1);$$

2) $\rho(\mathcal{H}) < 1$. □

Lemma 2. *The integral operator $\mathcal{W} : L_p \longrightarrow C$ is completely continuous, for all $1 < p \le \infty$.*

Proof. We consider only the case $k > -\dfrac{1}{p'}$, $k \neq 1$, the case $k = 1$ can be proved analogously. To prove the compactness of the operator $\mathcal{W}$ it suffices to show [5, p. 102] that, for any $t_0 \in [0,1]$ the equality

$$\lim_{t \to t_0} \int\limits_0^1 |W(t,s) - W(t_0,s)|^{p'}\,ds = 0$$

holds. For $1 \le p' < \infty$, $0 < t_0 < t \le 1$ we have that

$$\int_0^1 |W(t,s) - W(t_0,s)|^{p'}\, ds = \int_0^{t_0} \left| \frac{s^k(t^{1-k}-1)}{1-k} - \frac{s^k(t_0^{1-k}-1)}{1-k} \right|^{p'} ds +$$

$$+ \int_{t_0}^{t} \left| \frac{s^k(s^{1-k}-1)}{1-k} - \frac{s^k(t_0^{1-k}-1)}{1-k} \right|^{p'} ds \le$$

$$\le \frac{t_0^{p'k+1}}{|1-k|^{p'}(p'k+1)} (t^{1-k} - t_0^{1-k})^{p'} + O(t^{p'+1} - t_0^{p'+1}) \to 0, \ t \to t_0^+.$$

Analogously we prove the respective statement for $0 = t_0 < t \le 1$ and $0 \le t < t_0 \le 1$. $\square$

3. The de la Vallée-Poussin Like Theorem

Consider the boundary-value problem

$$\begin{cases} (\Im x)(t) \stackrel{def}{=} (\Im_0 x)(t) - (Tx)(t) = f(t), \quad t \in [0,1], \\ \dot{x}(0) = 0, \quad x(1) = \alpha, \end{cases} \tag{2}$$

where $k > -\dfrac{1}{p'}$, $T : C \longrightarrow L_p$ is a linear antitonic operator, $f \in L_p$. Denote $\mathcal{A} \stackrel{def}{=} \mathcal{W}T : C \longrightarrow C$.

Lemma 3. *The following statements are equivalent:*

1) *There exists an element $y \in D_p$ such that*

$$y(t) > 0, \quad \phi(t) \stackrel{def}{=} (\Im_0 y)(t) - (Ty)(t) \le 0, \quad t \in [0,1), \quad \textit{and}$$

$$y(1) - \int_0^1 \phi(s)\, ds > 0;$$

2) $\rho(\mathcal{A}) < 1$;

3) *The boundary value-problem* (2) *is uniquely solvable on D_p for each $f \in L_p$, $\alpha \in \mathbf{R}^1$, and its Green operator $\mathcal{G}$ is antitonic;*

4) *There exists a positive solution $u \in D_p$ on $[0,1]$ of the homogeneous equation $\Im x = 0$.*

Proof. Since $y(\cdot)$ satisfies

$$(\Im_0 x)(t) - (Tx)(t) = \phi(t), \quad t \in [0,1], \quad x(1) = y(1)$$

on the space D_p, it follows that

$$y - \mathcal{A}y = \mathcal{W}\phi + y(1) > 0$$

on the space C. By virtue of Lemma 1 it follows that $\rho(\mathcal{A}) <= 1$. The implication 1) $\Longrightarrow$ 2) is proved.

Supposing $\alpha \ge 0$ we consider the problem (2), which is equivalent to the equation

$$x = \mathcal{A}x + g$$

on the space C. Here

$$g(\cdot) \stackrel{def}{=} \int_0^1 W(\cdot, s) f(s)\, ds + \alpha.$$

Since $\rho(\mathcal{A}) < 1$, it follows that

$$\mathcal{G} = (I + \mathcal{A} + \mathcal{A}^2 + \cdots)\mathcal{W}.$$

Consequently the implication 2) $\Longrightarrow$ 3) is proved.

The problem

$$\Im_0 x - Tx = 0, \quad x(1) = \alpha$$

is equivalent to the equation

$$x = \mathcal{A}x + \alpha.$$

Since $\rho(\mathcal{A}) < 1$, we have $x = \alpha + \mathcal{A}\alpha + \mathcal{A}^2\alpha + \cdots \geq 0$ if $\alpha > 0$. Thus the implication 3) $\Longrightarrow$ 4) is proved.

The implication 4) $\Longrightarrow$ 1), follows from Lemma 1 because the positive solution $u(t)$ of the equation $\Im x = 0$ satisfies the inequalities

$$u(t) > 0, \quad u(t) - (\mathcal{A}u)(t) = \alpha > 0, \quad t \in [0, 1]. \quad \square$$

4. The Main Result

Consider the nonlinear boundary-value problem

$$\Im_0 x = f(\cdot, \Theta x), \quad \dot{x}(0) = 0,\ x(1) = \alpha, \tag{3}$$

where $\Theta : C \longrightarrow L_p$ is a linear isotonic operator, $1 < p \leq \infty$, $k > -\dfrac{1}{p'}$, the function $f(\cdot, \cdot)$ satisfies the Carathéodory conditions. By definition put $\bar{v} = \Theta v$, $\bar{z} = \Theta z$, $[\bar{v}, \bar{z}] \stackrel{def}{=} \{x \in L_p : \bar{v} \leq x \leq \bar{z}\}$.

Following [5], we will say that the function $f(\cdot, \cdot)$ satisfies the condition $\mathcal{L}^i[\bar{v}, \bar{z}]$, $i = 1, 2$, if it is possible the decomposition

$$f[t, u(t)] = q_i(t)u(t) + M_i[t, u(t)], \quad u \in [\bar{v}, \bar{z}],$$

where $q_i \in L_\infty$, $i = 1, 2$, the operator $\mathcal{M}_i : [\bar{v}, \bar{z}]_{L_p} \longrightarrow L_p$ is defined by $(\mathcal{M}_i u)(\cdot) \stackrel{def}{=} M_i[\cdot, u(\cdot)]$, $\mathcal{M}_1$ is isotonic and $\mathcal{M}_2$ is antitonic.

Theorem 1. *Let $v, z \in D_p$ be a pair of functions such that $v(t) < z(t)$, $t \in [0, 1]$, and*

$$\Im_0 v \geq f(\cdot, \Theta v), \quad \Im_0 z \leq f(\cdot, \Theta z), \quad v(1) \leq \alpha \leq z(1). \tag{4}$$

Suppose that the function $f(\cdot, \cdot)$ satisfies the condition $\mathcal{L}^2[\bar{v}, \bar{z}]$ with $q_2 \in L_\infty$, $q_2(\cdot) \leq 0$. Then the problem (3) has at least one solution $x \in [v, z]_{D_p}$.

If besides the $\mathcal{L}^1[\bar{v}, \bar{z}]$ condition is fulfilled with a coefficient $q_1 \in L_\infty$, and the Green operator of the auxiliary problem

$$\Im_1 x \stackrel{def}{=} \Im_0 x - q_1 \Theta x = \varphi, \quad x(1) = 0, \tag{5}$$

is antitonic, then the problem (3) has only one solution $x \in [v, z]$.

Proof. Rewrite (3) in the form

$$(\Im_2 x)(\cdot) \overset{def}{=} (\Im_0 x)(\cdot) - q_2(\cdot)(\Theta x)(\cdot) = M_2[\cdot, (\Theta x)(\cdot)], \quad x(1) = \alpha$$

on the space D_p. This problem is equivalent to the equation

$$x = A_2 x \tag{6}$$

with the completely continuous isotonic operator $A_2 : [v, z]_C \longrightarrow C$, defined by

$$(A_2 x)(\cdot) \overset{def}{=} \int_0^1 G_2(\cdot, s) M_2[s, (\Theta x)(s)]\, ds + u_2(\cdot),$$

where $u_2(\cdot)$ is the solution of the semi-homogeneous problem

$$(\Im_2 x)(t) = 0, \quad t \in [0,1], \quad x(1) = \alpha,$$

$G_2(\cdot, \cdot)$ is the Green function of the problem

$$\Im_2 x = \xi, \quad x(1) = 0. \tag{7}$$

We use here the fact that the Green operator $\mathcal{G}_2$ of the problem (7) has the representation $\mathcal{G}_2 = \mathcal{W}\Gamma$ [1,p.19], where $\Gamma : L_p \longrightarrow L_p$ is a linear homeomorphism, consequently $\mathcal{G}_2$ is a completely continuous operator because of Lemma 2. Each continuous solution of the equation (6) belongs to the space D_p, because the operator A_2 is defined on the order interval $[v, z]_C$ of the space C and maps this interval into the space D_p. Obviously the isotonic operator $\Theta : C \longrightarrow L_p$ maps the order interval $[v, z]_C$ into order interval $[\bar{v}, \bar{z}]_{L_p}$. The operator $\mathcal{M}_2 : [\bar{v}, \bar{z}]_{L_p} \longrightarrow L_p$ is antitonic, therefore it maps the order interval $[\bar{v}, \bar{z}]_{L_p}$ into $[\mathcal{M}_2 \bar{z}, \mathcal{M}_2 \bar{v}]_{L_p}$. Let $y \overset{def}{=} z - v$. Then $y(t) > 0$, $t \in [0,1]$,

$$\Im_2 y \le \mathcal{M}_2 \Theta z - \mathcal{M}_2 \Theta v \le 0,$$

because of the antitonicity of $\mathcal{M}_2$ and

$$y(1) - \int_0^1 (\Im_2 y)(s)\, ds > 0.$$

Consequently, by Lemma 3 we have that the Green operator $\mathcal{G}_2 : L_p \longrightarrow D_p \subset C$ of the problem (7) is antitonic. Thus

$$[\mathcal{G}_2 \mathcal{M}_2 \bar{v}, \mathcal{G}_2 \mathcal{M}_2 \bar{z}]_{D_p} \subset [\mathcal{G}_2 \mathcal{M}_2 \bar{v}, \mathcal{G}_2 \mathcal{M}_2 \bar{z}]_C.$$

Therefore the equation (6) may be considered in the order interval $[v, z]_C$ of the space C. By virtue of the conditions (4) it follows that $z(t) \ge (A_2 z)(t)$ and $v(t) \le (A_2 v)(t)$ for all $t \in [0,1]$. Because of the isotonicity of the operator $A_2 : [v, z]_C \longrightarrow C$ this guarantees $A_2[v, z]_C \subset [v, z]_C$. For $1 < p \le \infty$ the operator $A_2 : [v, z]_C \longrightarrow [v, z]_C$ is completely continuous as a product of the operators $\Theta : [v, z]_C \longrightarrow [\bar{v}, \bar{z}]_{L_p}$, $\mathcal{M}_2 : [\bar{v}, \bar{z}]_{L_p} \longrightarrow [\mathcal{M}_2 \bar{z}, \mathcal{M}_2 \bar{v}]_{L_p}$ and the completely continuous $\mathcal{G}_2 : L_p \longrightarrow C$.

Thus, the operator A_2 maps the closed convex set $[v, z]_C$ of the Banach space C into itself. In accordance with the Schauder's fixed point theorem the equation (6) has at least one solution $x \in [v, z]_C$.

Let us show that the set of all solutions $x \in [v, z]_C$ has a superior element $\bar{x} \in [v, z]_C$ (the upper solution) and an inferior element $\underline{x} \in [v, z]_C$ (the lower solution). Let $x \in [v, z]_C$ be a solution of the equation (6). The sequence $\{z^i\}$, $z^{i+1} = A_2 z^i$, $z^0 = z$ monotonically decreases and is bounded by $x \in [v, z]_C$, because the operator A_2 maps the set $[v, z]_C$ into itself. A compact monotone sequence $\{z^i\}$ converges [2,

p. 38] to $\bar{x} = \lim\limits_{i\to\infty} z^i$. Since this limit is a solution, the inequality $\bar{x} \geq x$ for any solution $x \in [v,z]_C$ is proved. Analogously we show the existence of the inferior solution $\underline{x}$.

Now we have to show that if the condition $\mathcal{L}^1[\bar{v},\bar{z}]$ is fulfilled, the solution of the problem (3) is unique, i.e. $\bar{x} = \underline{x}$. Using the $\mathcal{L}^1[\bar{v},\bar{z}]$, condition we rewrite the problem (3) in the form

$$(\Im_1 x)(\cdot) = M_1[\cdot,(\Theta x)(\cdot)], \quad x(1) = \alpha.$$

This problem is equivalent to the equation

$$x = A_1 x$$

on the order interval $[v,z]_C$ of the space C with antitonic operator $A_1 : [v,z]_C \longrightarrow C$, defined by

$$(A_1 x)(\cdot) \stackrel{def}{=} \int_0^1 G_1(\cdot,s) M_1[s,(\Theta = x)(s)]\, ds + u_1(\cdot),$$

where $G_1(\cdot,\cdot)$ is the Green function of the problem (5), $u_1(\cdot)$ is the solution of semi-homogeneous problem

$$(\Im_1 x)(t) = 0, \quad t \in [0,1], \quad x(1) = \alpha.$$

Consider the equality $\bar{x} - \underline{x} = A_1\bar{x} - A_1\underline{x}$. The left-hand side of the equality is non negative and the right-hand side is non positive, thus we get $\underline{x} = \bar{x}$. □

5. Examples

Example 1. Consider the boundary-value problem

$$\begin{cases} \ddot{x}(t) + \dfrac{1}{t}\dot{x}(t) = -\beta \exp\left(-\dfrac{1}{|x(t)|}\right), & t \in [0,1], \\ \dot{x}(0) = 0, \quad x(1) = 0, \end{cases} \tag{8}$$

where $0 \leq \beta \leq e^2$. This problem describes processes arising in chemical reactor theory with cilindrical symmetry [7, p. 326], under the Arrhenius law. We consider this problem on the space D_∞.

As comparison functions we choose

$$v(t) \equiv 0, \quad z(t) = \frac{\beta}{4}(1-t^2) + \frac{1}{2}.$$

A trivial verification shows that the conditions (4) are fulfilled:

$$\ddot{v}(t) + \frac{\dot{v}(t)}{t} + \beta \exp\left(-\frac{1}{|v(t)|}\right) = 0,$$

$$\ddot{z}(t) + \frac{\dot{z}(t)}{t} + \beta \exp\left(-\frac{1}{|z(t)|}\right) \leq -\beta + \beta = 0, \quad t \in [0,1];$$

$$v(1) = 0 = x(1) < z(1) = \frac{1}{2}.$$

The function $f(\cdot,x) = -\beta \exp\left(-\dfrac{1}{|x|}\right)$ satisfies the condition $\mathcal{L}^2[v,z]$ with the coefficient $q_2 \equiv 0$. The boundary-value problem

$$\Im_0 x = \xi, \quad x(1) = 0,$$

has for each $\xi \in L_\infty$ a unique solution $x \in D_\infty$, and its Green function $W(t,s) \leq 0$ on the square $[0,1] \times [0,1]$.

Besides, the function $f(\cdot, x) = -\beta \exp\left(-\frac{1}{|x|}\right)$ satisfies the condition $\mathcal{L}^1[v, z]$ with the coefficient $q_1 = -4\beta e^{-2}$.

Taking the function $y(t) = \frac{\beta}{4}(1 - t^2)$ we have:

$$(\Im_1 y)(t) = (\Im_0 y)(t) + 4\beta e^{-2} y(t) = -\beta + \beta^2 e^{-2}(1 - t^2) < \beta(-1 + \beta e^{-2}) \leq 0$$

and

$$y(1) - \int_0^1 (\Im_1 y)(s)\, ds = \int_0^1 [\beta - \beta^2 e^{-2}(1 - s^2)]\, ds = (\beta - \frac{2}{3}\beta^2 e^{-2}) > 0,$$

since $\beta \leq e^2$. Consequently, by Lemma 3, the Green operator $\mathcal{G}_1$ of the problem

$$\Im_0 x + 4\beta e^{-2} x = \xi, \quad x(1) = 0$$

is antitonic. Then, because of Theorem 1 the problem (8) has a unique solution $x \in D_\infty$ such that

$$0 \leq x(t) \leq \frac{\beta}{4}(1 - t^2) + \frac{1}{2}, \quad t \in [0, 1]. \quad \square$$

Example 2. Let

$$\begin{cases} \ddot{x}(t) + \frac{2}{t}\dot{x}(t) = -\beta \exp\left(-\frac{1}{|x(t)|}\right), & t \in [0, 1], \\ \dot{x}(0) = 0, \quad x(1) = 0, \end{cases} \tag{9}$$

be a nonlinear boundary-value problem, $37.28 \leq \beta \leq \frac{12}{17}\, e^{17/2}$. This problem describes processes arising in chemical reactor with spherical symmetry [7, p. 326].

The problem (9) with such β has more than one solution on the space D_p, $1 < p \leq \infty$. Indeed, there are at least two pairs of functions

$$v_1(t) \equiv 0, \quad z_1(t) = \frac{2(1 - t^2)}{17}, \quad v_2(t) = 4[\text{erf}(1) - \text{erf}(t^2)], \quad z_2(t) = \frac{\beta}{6}(1 - t^2).$$

The conditions (4) are fulfilled:

$$\ddot{v}_1(t) + \frac{2\dot{v}_1(t)}{t} + \beta \exp\left(-\frac{1}{|v_1(t)|}\right) = 0,$$

$$\ddot{z}_1(t) + \frac{2\dot{z}_1(t)}{t} + \beta \exp\left(-\frac{1}{|z_1(t)|}\right) = \beta \exp\left(-\frac{17}{2(1 - t^2)}\right) - \frac{12}{17} < 0,$$

$$\ddot{v}_2(t) + \frac{2\dot{v}_2(t)}{t} + \beta \exp\left(-\frac{1}{|v_2(t)|}\right) = \frac{16\exp(-t^4)}{\sqrt{\pi}}(4t^4 - 3) + $$
$$+ \beta \exp\left(\frac{-0.25}{\text{erf}(1) - \text{erf}(t^2)}\right) > 0,$$

$$\ddot{z}_2(t) + \frac{2\dot{z}_2(t)}{t} + \beta \exp\left(-\frac{1}{|z_2(t)|}\right) = \beta \exp\left(-\frac{6}{\beta(1 - t^2)}\right) - \beta < 0,$$

since $37.28 \leq \beta \leq \frac{12}{17}\, e^{17/2}$, $t \in [0, 1]$. The existence of solution of the problem (9) on each interval $[v_i, z_i]$, $i = 1, 2$, follows from Theorem 1. Since the intervals $[v_1, z_1]$, $[v_2, z_2]$ are disjoint, the problem (9) has at least two solutions x_1, $x_2 \in D_p$, $1 < p \leq \infty$, such that $v_1 \leq x_1 \leq z_1$, $v_2 \leq x_2 \leq z_2$. $\square$

REFERENCES

1. N. AZBELEV AND L. RAKHMATULLINA, Theory of linear abstract functional differential equations and applications. *Mem. Differential Equations Math. Phys.* **8**(1996), 1–102.

2. P. ZABREĬKO AND OTHERS, Integral equations. (Russian) *Nauka, Moscow*, 1968.

3. I. MUNTEAN, The spectrum of the Cesàro operator. *Mathematica – Revue d'analyse numérique et de théorie de l'approximation* **45**(1980), 97–105.

4. N. AZBELEV AND L. RAKHMATULLINA, On the estimate for the spectral radius of a linear operator in the space of continuous functions. (Russian) *Izv. Vuzov. Matematika* **11**(1996), 23–28.

5. N. AZBELEV, V. MAKSIMOV, AND L. RAKHMATULLINA, Introduction to the theory of functional differential equations. (Russian) *Nauka, Moscow*, 1991.

6. M. KRASNOSEL'SKIĬ, Positive solutions of operator equations. (Russian) *Izv. Vuzov. Matematika* **11**(1996), 23–28.

7. D. FRANK-KAMENETSKIĬ, Diffusion and heat exchange in chemical kinetics. (Russian) *Nauka, Moscow*, 1987.

Author's address:
Eduardo Mondlane University
Department of Mathematics
P.O. Box 257 - Maputo
Mozambique