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# ON ZEROS OF SOLUTIONS OF A SECOND ORDER SINGULAR HALF-LINEAR EQUATION

**Abstract.** In the present paper we consider the equation

$$u'' = p(t)|u|^{\alpha}|u'|^{1-\alpha} \operatorname{sgn} u,$$

where $\alpha \in ]0,1]$, the function $p : ]a,b[ \to R$ is locally integrable, and $\int_a^b (s-a)^{\alpha}(b-s)^{\alpha}|p(s)|ds < +\infty$. Sufficient conditions for the existence of a solution having at least two zeros on the segment $[a,b]$ are established.

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**Key words and phrases**: second order singular half-linear equation, proper solutions, number of zero points.

**რეზიუმე.** განხილულია მეორე რიგის ნახევრად წრფივი სინგულარული განტოლება

$$u'' = p(t)|u|^{\alpha}|u'|^{1-\alpha} \operatorname{sgn} u,$$

სადაც $\alpha \in ]0,1]$, ხოლო ფუნქცია $p : ]a,b[ \to R$ აკმაყოფილებს პირობას $\int_a^b (s-a)^{\alpha}(b-s)^{\alpha}|p(s)|ds < +\infty$, დადგენილია საკმარისი პირობები იმისა, თუ როდის აქვს აღნიშნულ განტოლებას ერთი მაინც ნიშანმუდმივი წესიერი ამონახსნი და საკმარისი პირობები იმისა, თუ როდის აქვს ერთი მაინც წესიერი ამონახსნი, რომელსაც აქვს ორი მაინც ნული $[a,b]$ სეგმენტზე.

## Introduction

Below we will use the following notation:

$R$ is the set of real numbers;

$L_{\rm loc}(]a,b[)$ is the set of functions $p:]a,b[\to R$ which are Lebesgue integrable on each segment contained in $]a,b[$;

$\widetilde{C}([a,b])$ is the set of functions $u:[a,b]\to R$ absolutely continuous on the segment $[a,b]$;

$\widetilde{C}_{\rm loc}(I)$, where $I\subset R$, is the set of functions $u:I\to R$ absolutely continuous on each segment contained in $I$;

$\widetilde{C}'_{\rm loc}(I)$, where $I\subset R$, is the set of functions $u\in\widetilde{C}_{\rm loc}(I)$ for which $u'\in\widetilde{C}_{\rm loc}(I)$;

$u(s+)$ and $u(s-)$ are the limits of the function $u$ at the point $s$ from the right and from the left, respectively;

$[p(t)]_- = \frac{1}{2}(|p(t)|-p(t))$.

Consider the equation

$$u'' = p(t)|u|^\alpha|u'|^{1-\alpha}\operatorname{sgn} u, \tag{1.1}$$

where $\alpha\in]0,1]$, $a,b\in R$, and $p\in L_{\rm loc}(]a,b[)$. Under a solution of the equation (1.1) is understood a function $u\in\widetilde{C}'_{\rm loc}(]a_1,b_1[)$, where $a_1\in[a,b[$ and $b_1\in]a_1,b]$, which satisfies the equation (1.1) almost everywhere in $]a_1,b_1[$.

Throughout the paper we will assume that

$$\int_a^b (s-a)^\alpha(b-s)^\alpha|p(s)|ds<+\infty. \tag{1.2}$$

Below we will see (see Lemma 2.2) that all non-continuable solutions of the equation (1.1) are defined on the whole segment $[a,b]$; note that under the values of a solution $u$ at the points $a$ and $b$ we understand respectively the limits $u(a+)$ and $u(b-)$, whose existence (and finiteness) is quaranteed by the condition (1.2). Moreover, it is found that none of non-trivial solutions of the equation (1.1) may have an infinite number of zeros on the segment $[a,b]$ (see Remark 2.1 and Lemma 2.8).

In the case of the linear equation, i.e., for $\alpha=1$, the number of zeros of two arbitrary non-trivial solutions differ from each other by not more than 1. This fact does not, generally speaking, take place for the equation (1.1) with $\alpha\neq 1$, since any constant function turns out to be its solution; however it remains valid for a definite subset of the set of solutions, which in the sequel will be called the set of proper solutions.

**Definition 1.1.** A solution $u$ of the equation (1.1) is said to be proper, if there exists $A\subset]a,b[$ such that $\operatorname{mes} A=0$ and $\{t\in]a,b[: u'(t)=0\}\subset\{t\in ]a,b[: p(t)=0\}UA$.

Below we shall show (see Lemma 2.9) that the set of proper solutions of the equation (1.1) is non-empty and, moreover, almost every Cauchy problem has at least one proper solution (see Remark 2.4).

According to the above-said, the following definition is substantial and meaningful.

**Definition 1.2.** We say that the function $p$ belongs to the set $O_\alpha(]a,b[)$, if there exists a proper solution of the equation (1.1) having at least two zeros on the segment $[a,b]$.

In other words, $p \notin O_\alpha(]a,b[)$ if and only if there is no proper solution $u$ of the equation (1.1) satisfying for some $a_1 \in [a,b[$ and $b_1 \in ]a,b]$ the conditions

$$u(a_1+) = 0, \quad u(b_1-) = 0. \tag{1.3}$$

**Definition 1.3.** We say that the function $p$ belongs to the set $U_\alpha(]a,b[)$, if for any $a_1 \in [a,b[$ and $b_1 \in ]a_1,b]$ the problem (1.1), (1.3) has no non-zero (not necessarily proper) solution.

It is clear that if $p \in U_\alpha(]a,b[)$, then $p \notin O_\alpha(]a,b[)$. In the case, where $\alpha = 1$, or $\alpha \in ]0,1[$ and $p(t) \le 0$ for $a < t < b$, the converse assertion is valid, i.e., if $p \notin U_\alpha(]a,b[)$, then $p \in O_\alpha(]a,b[)$. The problem on a mutual complement ability of these two sets remains as yet unstudied in the general case.

Note that if $p \in O_1(]a,b[)$, then the (linear) equation (1.1) is called conjugate, but if $p \notin O_1(]a,b[)$ (and hence $p \in U_1(]a,b[)$), then it is disconjugate. A vast number of works (see, e.g., [1–6] and references therein) are devoted to the question of an effective description of the sets $O_1(]a,b[)$ and $U_1(]a,b[)$. As to the sets $O_\alpha(]a,b[)$ and $U_\alpha(]a,b[)$, they are studied not well enough even in the regular case, where the function $p$ is integrable on $[a,b]$.

The aim of the present work is to fill in the above-mentioned gap. Below we give some new integral criteria for belonging of the function $p$ to sets $O_\alpha(]a,b[)$ and $U_\alpha(]a,b[)$, not excepting the possibility for $p$ to have nonintegrable singularities at the points $a$ and $b$ (see condition (1.2)). The paper is organized as follows: the main results are formulated in Section 1; auxiliary propositions are given in Section 2; proofs of the main results can be found in Section 3.

## 1. Statement of the Main Results

**Theorem 1.1.** *Let there exist $\lambda \in ]a,b[$ and $\mu \in ]\lambda,b[$ such that*

$$-\alpha\int_\lambda^\mu p(s)ds \ge \frac{1}{(\lambda-a)^\alpha} + \frac{1}{(b-\mu)^\alpha} + \frac{\alpha}{(\lambda-a)^{\alpha+1}}\int_a^\lambda (s-a)^{\alpha+1}p(s)ds + \\ + \frac{\alpha}{(b-\mu)^{\alpha+1}}\int_\mu^b (b-s)^{\alpha+1}p(s)ds. \tag{1.4}$$

*Then $p \in O_\alpha(]a,b[)$.*

Introduce the notation

$$Q(t,t_0,\alpha)=\begin{cases}\alpha(t-a)^\alpha\displaystyle\int_t^{t_0}p(s)ds & \text{for}\quad a<t<t_0,\\ \alpha(b-t)^\alpha\displaystyle\int_{t_0}^{t}p(s)ds & \text{for}\quad t_0<t<b,\end{cases}$$
$$Q_*(t_0,\alpha)=\inf\{Q(t,t_0,\alpha):a<t<b\},$$
$$Q^*(t_0,\alpha)=\sup\{Q(t,t_0,\alpha):a<t<b\}.$$

**Corollary 1.1.** *Let*

$$-(\alpha+1)\int_a^b Q\Big(s,\frac{a+b}{2},\alpha\Big)ds>b-a. \tag{1.5}$$

*Then* $p\in O_\alpha(]a,b[)$.

**Corollary 1.2.** *Let* $t_0\in]a,b[$,

$$p(t)\le 0\quad for\quad a<t<b, \tag{1.6}$$

$$Q_*(t_0,\alpha)\le -1-\max\Big\{\Big(\frac{t_0-a}{b-t_0}\Big)^\alpha,\ \Big(\frac{b-t_0}{t_0-a}\Big)^\alpha\Big\}. \tag{1.7}$$

*Then* $p\in O_\alpha(]a,b[)$.

In the case where the condition (1.6) is satisfied, Theorem 1.1 can be somewhat improved; to be more exact, the following theorem is valid.

**Theorem 1.1′.** *Let* (1.6) *be satisfied and let there exist* $\lambda\in]a,b[$, $\mu\in]\lambda,b[$ *and a natural number* $n$ *such that*

$$\alpha\int_\lambda^\mu |p(s)|ds\ge\frac{1}{a_n^\alpha(\lambda)}+\frac{1}{b_n^\alpha(\mu)}, \tag{1.8}$$

*where*

$$a_1(t)=t-a,\quad a_{k+1}(t)=t-a+\int_a^t a_k^{\alpha+1}(s)|p(s)|ds\quad for\quad a<t<b,$$

$$b_1(t)=b-t,\quad b_{k+1}(t)=b-t+\int_t^b b_k^{\alpha+1}(s)|p(s)|ds\quad for\quad a<t<b.$$

*Then* $p\in O_\alpha(]a,b[)$.

**Theorem 1.2.** *Let there exist* $c\in]a,b[$ *and the functions* $f\in\widetilde{C}([a,c])$ *and* $g\in\widetilde{C}([c,b])$ *such that* $f(a)=0$, $g(b)=0$, $f(t)>0$ *for* $a<t<c$, $g(t)>0$ *for* $c<t<b$, $\frac{|f'|^{\alpha+1}}{f^\alpha}$ *and* $\frac{|g'|^{\alpha+1}}{g^\alpha}$ *be integrable on* $[a,c]$ *and* $[c,b]$, *respectively, and*

$$-\alpha\Big[g(c)\int_a^c f(s)p(s)ds+f(c)\int_c^b g(s)p(s)ds\Big]>$$

$$> \frac{1}{(\alpha+1)^{\alpha+1}}\left[g(c)\int_a^c \frac{|f'(s)|^{\alpha+1}}{f^\alpha(s)}ds + f(c)\int_c^b \frac{|g'(s)|^{\alpha+1}}{g^\alpha(s)}ds\right]. \quad (1.9)$$

*Then* $p \in O_\alpha(]a,b[)$.

**Corollary 1.3.** *Let at least one of the following three conditions be fulfilled:*

$$-\alpha\int_a^b (s-a)^{\alpha+1}(b-s)^{\alpha+1}p(s)ds > \frac{(b-a)^{\alpha+2}}{\alpha+2}, \quad (1.10)$$

$$-\alpha\left[\int_a^{\frac{a+b}{2}} (s-a)^\lambda p(s)ds + \int_{\frac{a+b}{2}}^b (b-s)^\lambda p(s)ds\right] >$$
$$> \frac{2\lambda^{\alpha+1}}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}}\left(\frac{b-a}{2}\right)^{\lambda-\alpha}, \quad \text{where} \quad \lambda > \alpha, \quad (1.11)$$

$$-\alpha\int_a^b \sin^{\alpha+1}\left[\frac{\pi(s-a)}{b-a}\right]p(s)ds > \frac{2(\alpha+2)^\alpha}{(\alpha+1)^{\alpha+1}}\left(\frac{\pi}{b-a}\right)^\alpha. \quad (1.12)$$

*Then* $p \in O_\alpha(]a,b[)$.

*Remark* 1.1. As we will see from the proofs below, in the conditions of the above-given results not only $p \in O_\alpha(]a,b[)$, but every proper solution of the equation (1.1) has at least one zero in the interval $]a,b[$.

Finally, we present theorems concerning the case where the function $p$ belongs to the set $U_\alpha(]a,b[)$ and also theorems with this function not belonging to the set $O_\alpha(]a,b[)$.

**Theorem 1.3.** *Let* $\sup\{A^\alpha(t)|B(t)|^{1-\alpha} : a < t < b\} < b-a$, *where*

$$A(t) = \frac{(b-t)^{1-\lambda}}{(t-a)^\lambda}\int_a^t (s-a)^{1+\alpha\lambda}(b-s)^{\alpha\lambda}[p(s)]_- ds +$$
$$+ \frac{(t-a)^{1-\lambda}}{(b-t)^\lambda}\int_t^b (s-a)^{\alpha\lambda}(b-s)^{1+\alpha\lambda}[p(s)]_- ds,$$
$$B(t) = \int_a^t (s-a)^{1+\alpha\lambda}(b-s)^{\alpha\lambda}[p(s)]_- ds -$$
$$- \int_t^b (s-a)^{\alpha\lambda}(b-s)^{1+\alpha\lambda}[p(s)]_- ds \quad \text{for} \quad a < t < b,$$

*and* $\lambda \in [0,1]$. *Then* $p \in U_\alpha(]a,b[)$.

**Corollary 1.4.** *Let* $C^{\frac{1-\alpha}{\alpha}} \int_a^b (s-a)^\alpha(b-s)^\alpha[p(s)]_- ds < (b-a)^{\frac{1}{\alpha}}$, *where*

$$C = \max\left\{\int_a^b (s-a)^{1+\alpha}(b-s)^\alpha[p(s)]_- ds,\ \int_a^b (s-a)^\alpha(b-s)^{1+\alpha}[p(s)]_- ds\right\}.$$

*Then* $p \in U_\alpha(]a,b[)$.

In particular, this corollary implies that if

$$\int_a^b (s-a)^\alpha (b-s)^\alpha [p(s)]_- ds \le (b-a)^\alpha,$$

then $p \in U_\alpha(]a,b[)$.

Denote by $\varphi(\lambda)$, where $\lambda > -\frac{1}{\alpha+1}(\frac{\alpha}{\alpha+1})^\alpha$, the largest positive root of the equation $x - |x|^{\frac{\alpha}{\alpha+1}} - \lambda = 0$.

**Theorem 1.4.** *Let* $Q_*(t_0,\alpha) \ge -\frac{1}{\alpha+1}(\frac{\alpha}{\alpha+1})^\alpha$ *and*

$$Q^*(t_0,\alpha) < \varphi(Q_*(t_0,\alpha)). \tag{1.13}$$

*Then* $p \in U_\alpha(]a,b[)$.

**Corollary 1.5.** *Let* $-\frac{1}{\alpha+1}(\frac{\alpha}{\alpha+1})^\alpha \le Q_*(t_0,\alpha)$, $Q^*(t_0,\alpha) < (\frac{\alpha}{\alpha+1})^{\alpha+1}$. *Then* $p \in U_\alpha(]a,b[)$.

**Theorem 1.5.** *Let* $Q_*(t_0,\alpha) \ge -\frac{1}{\alpha+1}(\frac{\alpha}{\alpha+1})^\alpha$ *and either*

$$Q_*(t_0,\alpha) \ne 0 \quad \text{and} \quad Q^*(t_0,\alpha) \le 2\varphi(Q_*(t_0,\alpha)) - Q_*(t_0,\alpha), \tag{1.14}$$

*or*

$$Q_*(t_0,\alpha) = 0 \quad \text{and} \quad Q^*(t_0,\alpha) < 2. \tag{1.15}$$

*Then* $p \notin O_\alpha(]a,b[)$.

**Corollary 1.6.** *Let* $-\frac{1}{\alpha+1}(\frac{\alpha}{\alpha+1})^\alpha \le Q_*(t_0,\alpha)$ *and* $Q^*(t_0,\alpha) \le \frac{2\alpha+1}{\alpha+1}(\frac{\alpha}{\alpha+1})^\alpha$. *Then* $p \notin O_\alpha(]a,b[)$.

**Corollary 1.7.** *Let* $\alpha = 1$, $Q_*(t_0,1) \ge -\frac{1}{4}$ *and either* $Q_*(t_0,1) \ne 0$, $Q^*(t_0,1) \le 1 + Q_*(t_0,1) + \sqrt{1+4Q_*(t_0,1)}$, *or* $Q_*(t_0,1) = 0$, $Q^*(t_0,1) < 2$. *Then* $p \notin O_1(]a,b[)$ (*i.e.*, $p \in U_1(]a,b[)$ ).

## 2. Some Auxiliary Propositions

In this section we establish some properties of solutions of the equation

$$u'' = p(t)|u|^\alpha |u'|^{1-\alpha} \operatorname{sgn} u. \tag{2.1}$$

Below, throughout all the paper, the function $p : ]a,b[ \to R$ will be zassumed to belong to the set $L_{\mathrm{loc}}(]a,b[)$ and to satisfy the condition (1.2).

First of all, for the convenience of reference we will quote one simple proposition without proving it.

**Proposition 2.1.** *The equalities*

$$\lim_{t\to a+} (t-a)^\alpha \int_t^{\frac{a+b}{2}} |p(s)| ds = 0, \quad \lim_{t\to b-} (b-t)^\alpha \int_{\frac{a+b}{2}}^t |p(s)| ds = 0$$

*take place.*

**Lemma 2.1.** *There exist solutions $u_1$ and $u_2$ of the equation* (2.1) *satisfying the initial conditions*

$$u_1(a+) = 0, \quad u_1'(a+) = 1, \tag{2.2}$$

$$u_2(b-) = 0, \quad u_2'(b-) = -1. \tag{2.3}$$

*Moreover, all non-continuable solutions of the problems* (2.1), (2.2) *and* (2.1), (2.3) *are defined on the entire segment* $[a, b]$.

*Proof.* We will prove only the existence of $u_1$. The existence of $u_2$ can be proved similarly.

Let $x \in ]a, b[$. Denote by $A_x$ the set of all non-continuable to the right solutions of the equation (2.1) satisfying the initial conditions

$$u(x) = 0, \quad u'(x) = 1. \tag{2.4}$$

Let $u(\cdot, x) \in A_x$, and let us show that this solution is defined in the interval $[x, b[$. Suppose $I_x = \{t \in ]x, b[ : |u(s, x)| < +\infty,\ |u'(s, x)| < +\infty \text{ for } x \leq s \leq t\}$.

Integrating (2.1) and taking into account (2.4), we obtain

$$u'(t, x) = 1 + \int_x^t p(s)|u(s, x)|^\alpha |u'(s, x)|^{1-\alpha} \operatorname{sgn} u(s, x) ds \quad \text{for } t \in I_x, \tag{2.5}$$

$$u(t, x) = t - x + \\ + \int_x^t (t - s)p(s)|u(s, x)|^\alpha |u'(s, x)|^{1-\alpha} \operatorname{sgn} u(s, x) ds \quad \text{for} \quad t \in I_x, \tag{2.6}$$

whence we readily find that

$$|u'(t, x)| \leq 1 + \int_x^t (s - a)^\alpha |p(s)| \Big| \frac{u(s, x)}{s - x} \Big|^\alpha |u'(s, x)|^{1-\alpha} ds \quad \text{for} \quad t \in I_x,$$

$$\Big| \frac{u(t, x)}{t - x} \Big| \leq 1 + \int_x^t (s - a)^\alpha |p(s)| \Big| \frac{u(s, x)}{s - x} \Big|^\alpha |u'(s, x)|^{1-\alpha} ds \quad \text{for} \quad t \in I_x.$$

If we add these inequalities and take into account the fact that

$$y^\alpha < 1 + y \quad \text{for} \quad y \geq 0, \tag{2.7}$$

we obtain

$$\Big| \frac{u(t, x)}{t - x} \Big| + |u'(t, x)| \leq 2 + 2 \int_x^t (s - a)^\alpha |p(s)| \Big[ \Big| \frac{u(s, x)}{s - x} \Big| + |u'(s, x)| \Big] ds \quad \text{for } t \in I_x.$$

According to the Gronwall-Bellmann lemma, we now have

$$\Big| \frac{u(t, x)}{t - x} \Big| + |u'(t, x)| \leq 2 \exp \Big[ 2 \int_a^t (s - a)^\alpha |p(s)| ds \Big] \quad \text{for} \quad t \in I_x. \tag{2.8}$$

Consequently, $\sup I_x = b$. Moreover, from (2.8) and (2.5) we also get

$$|u(t, x)| \leq 2(t - a) \exp \Big[ 2 \int_a^t (s - a)^\alpha |p(s)| ds \Big] \quad \text{for} \quad x \leq t < b, \tag{2.9}$$

$$|u'(t,x)| \le 2\exp\left[2\int_a^t (s-a)^\alpha |p(s)|ds\right] \quad \text{for} \quad x \le t < b, \tag{2.10}$$

$$|u'(t,x)-1| \le 2\int_a^t (s-a)^\alpha |p(s)|ds \times$$
$$\times \exp\left[2\int_a^t (s-a)^\alpha |p(s)|ds\right] \quad \text{for} \quad x \le t < b. \tag{2.11}$$

Let $x_k \in ]a,b[$, $x_{k+1} < x_k$ for $k = 1,2,\dots$, $\lim\limits_{k\to+\infty} x_k = a$. Suppose $v_k(t) = u(t,x_k)$ for $x_k \le t < b$, $k = 1,2,\dots$. By (2.9) and (2.10), the sequences $(v_k)_{k=1}^{+\infty}$ and $(v_k')_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous in $]a,b[$ (i.e., on each segment contained in $]a,b[$). Hence, without loss of generality, by the Arzela-Ascoli lemma we may assume that $\lim\limits_{k\to+\infty} v_k^{(i)}(t) = u_1^{(i)}(t)$, $i = 0,1$, uniformly in $]a,b[$. It is easily seen that the function $u_1$ is a solution of the equation (2.1). On the other hand, by (2.9) and (2.11) the function $u_1$ satisfies the conditions (2.2) as well. Thus we have proved that the set of solutions $A$ of the problem (2.1), (2.2) is non-empty. Let us now show that all functions from $A$ are defined on the entire segment $[a,b]$.

Let $u_1 \in A$ be a non-continuable solution. Analogously to that as it has been done above, we see that

$$u_1'(t) = 1 + \int_a^t (s-a)^\alpha p(s)\left|\frac{u_1(s)}{s-a}\right|^\alpha |u_1'(s)|^{1-\alpha} \times \operatorname{sgn} u_1(s)ds \quad \text{for} \quad t \in I,$$
$$\frac{u_1(t)}{t-a} = 1 + \frac{1}{t-a}\times$$
$$\times \int_a^t (s-a)^\alpha (t-s)p(s)\left|\frac{u_1(s)}{s-a}\right|^\alpha |u_1'(s)|^{1-\alpha} \operatorname{sgn} u_1(s)ds \quad \text{for} \quad t \in I, \tag{2.12}$$

where $I = \{t \in ]a,b[ : |u_1(s)| < +\infty,\ |u_1'(s)| < +\infty \ \text{ for } \ a \le s \le t\}$. These two equalities immediately yield

$$(b-t)|u_1'(t)| \le b-a + \int_a^t (s-a)^\alpha (b-s)^\alpha |p(s)|\left|\frac{u_1(s)}{s-a}\right|^\alpha |(b-s)u_1'(s)|^{1-\alpha} ds$$
$$\text{for} \quad t \in I,$$

$$\frac{|u_1(t)|}{t-a} \le 1 + \frac{1}{b-a}\int_a^t (s-a)^\alpha (b-s)^\alpha |p(s)|\left|\frac{u_1(s)}{s-a}\right|^\alpha |(b-s)u_1'(s)|^{1-\alpha} ds \ \text{for} \ t \in I.$$

Adding these inequalities, taking into account (2.7) and using the Gronwall-Bellman lemma, we obtain

$$|u_1(t)|(t-a)^{-1} + (b-t)|u_1'(t)| \le (1+b-a)\times$$
$$\times \exp\left[\frac{1+b-a}{b-a}\int_a^b (s-a)^\alpha (b-s)^\alpha |p(s)|ds\right] \quad \text{for} \quad t \in I. \tag{2.13}$$

Consequently, $\sup I = b$.

The equality (2.12) results in

$$u_1(t) = t - a - (b-t)\times$$
$$\times \int_a^t \frac{[(s-a)(b-s)]^\alpha}{b-s} p(s) \Big|\frac{u_1(s)}{s-a}\Big|^\alpha |(b-s)u_1'(s)|^{1-\alpha} \operatorname{sgn} u_1(s) ds +$$
$$+ \int_a^t [(s-a)(b-s)]^\alpha p(s) \Big|\frac{u_1(s)}{s-a}\Big|^\alpha |(b-s)u_1'(s)|^{1-\alpha} \operatorname{sgn} u_1(s) ds$$
$$\text{for } \ a < t < b,$$

from which by (2.13) and Proposition 2.1 we conclude that there exists a finite limit $u_1(b-)$. □

In a similar manner we can see that the following lemma is valid.

**Lemma 2.2.** *All non-continuable solutions of the equation* (2.1) *are defined on the entire segment* $[a, b]$.

*Remark* 2.1. We can be easily convinced that if $c \in ]a, b[$, and $u$ is a solution of the equation (2.1) satisfying the conditions $u(c) = 0$, $u'(c) = 0$, then $u$ is identically equal to zero (see also [5]).

**Lemma 2.3.** *Let $u$ be a solution of the equation* (2.1). *Then*

$$\lim_{t\to a+} (t-a)|u'(t)| = 0 \quad \big(\lim_{t\to b-} (b-t)|u'(t)| = 0\big). \tag{2.14}$$

*Proof.* First let us show that

$$\lim_{t\to a+} \inf (t-a)|u'(t)| = 0 \quad \big(\lim_{t\to b-} \inf (b-t)|u'(t)| = 0\big). \tag{2.15}$$

Assume to the contrary that (2.15) violated. Then there exist $c \in ]a, b[$ and $\varepsilon > 0$ such that

$$|u'(t)| > \frac{\varepsilon}{t-a} \quad \text{for} \quad a < t < c$$
$$\Big(|u'(t)| > \frac{\varepsilon}{b-t} \quad \text{for} \quad c < t < b\Big).$$

The integration of this inequality from $t$ to $c$ (from $c$ to $t$) yields

$$|u(t) - u(c)| > \varepsilon \ln \frac{c-a}{t-a} \quad \text{for} \quad a < t < c$$
$$\Big(|u(t) - u(c)| > \varepsilon \ln \frac{b-c}{b-t} \quad \text{for} \quad c < t < b\Big),$$

which is impossible because the function $u$ is bounded. Thus (2.15) is fulfilled.

Multiplying both parts of (2.1) by $t-a$ (by $b-t$) and integrating from $\tau$ to $t$ (from $t$ to $\tau$), we obtain

$$
\begin{gathered}
(t-a)u'(t)=(\tau-a)u'(\tau)+u(t)-u(\tau)+\\
+\int_\tau^t (s-a)p(s)|u(s)|^\alpha|u'(s)|^{1-\alpha}\operatorname{sgn}u(s)ds \quad \text{for} \quad a<\tau<t<b\\
\Big((b-t)u'(t)=(b-\tau)u'(\tau)+u(\tau)-u(t)-\\
-\int_t^\tau (b-s)p(s)|u(s)|^\alpha|u'(s)|^{1-\alpha}\operatorname{sgn}u(s)ds \quad \text{for} \quad a<t<\tau<b\Big),
\end{gathered}
\tag{2.16}
$$

from which, by (2.7), we can conclude that

$$
\begin{gathered}
(t-a)|u'(t)|\le(\tau-a)|u'(\tau)|+|u(t)-u(\tau)|+\\
+\int_\tau^t (s-a)^\alpha|p(s)|(|u(s)|+(s-a)|u'(s)|)ds \quad \text{for} \quad a<\tau<t<b\\
\Big((b-t)|u'(t)|\le(b-\tau)|u'(\tau)|+|u(t)-u(\tau)|+\\
+\int_t^\tau (b-s)^\alpha|p(s)|(|u(s)|+(b-s)|u'(s)|)ds \quad \text{for} \quad a<t<\tau<b\Big).
\end{gathered}
\tag{2.17}
$$

Suppose

$$
\begin{gathered}
c(x)=2\max\{|u(t)-u(a+)|: a\le t\le x\}+\\
+\int_a^x (s-a)^\alpha|p(s)|\,|u(s)|ds \quad \text{for} \quad a<x<b\\
\Big(c(x)=2\max\{|u(t)-u(b-)|: x\le t\le b\}+\\
+\int_x^b (b-s)^\alpha|p(s)|\,|u(s)|ds \quad \text{for} \quad a<x<b\Big),\\
M(\tau,x)=(\tau-a)|u'(\tau)|+c(x) \quad \text{for} \quad a<\tau\le x<b\\
(M(\tau,x)=(b-\tau)|u'(\tau)|+c(x) \quad \text{for} \quad a<x\le\tau<b).
\end{gathered}
\tag{2.18}
$$

From (2.17) we find that

$$
\begin{gathered}
(t-a)|u'(t)|\le M(\tau,x)+\int_\tau^t (s-a)^\alpha|p(s)|\big((s-a)|u'(s)|\big)ds \text{ for } a<\tau<t<x<b\\
\Big((b-t)|u'(t)|\le M(\tau,x)+\int_t^\tau (b-s)^\alpha|p(s)|\big((b-s)|u'(s)|\big)ds \text{ for } a<x<t<\tau<b\Big),
\end{gathered}
$$

which in its turn by the Gronwall–Bellman lemma gives

$$
(t-a)|u'(t)|\le M(\tau,x)\exp\left[\int_a^x (s-a)^\alpha|p(s)|ds\right] \quad \text{for} \quad a<\tau<t<x<b.
$$

$$\Big((b-t)|u'(t)|\leq M(\tau,x)\exp\Big[\int_x^b (b-s)^\alpha |p(s)|ds\Big] \ \text{ for } \ a<x<t<\tau<b\Big).$$

Taking now into account (2.15) and (2.18), we easily find that

$$(t-a)|u'(t)|\leq c(x)\exp\Big[\int_a^x (s-a)^\alpha |p(s)|ds\Big] \ \text{ for } \ a<t<x<b$$
$$\Big((b-t)|u'(t)|\leq c(x)\exp\Big[\int_x^b (b-s)^\alpha |p(s)|ds\Big] \ \text{ for } \ a<x<t<b\Big).$$

Since $\lim\limits_{x\to a+} c(x)=0$ ( $\lim\limits_{x\to b-} c(x)=0$), the above inequality makes it possible to conclude that (2.14) is fulfilled. □

*Remark* 2.2. In proving the above lemma (see the beginning of the proof), it has been shown that if $v\in \widetilde{C}'_{\mathrm{loc}}(]a,b[)$ is bounded and there exists a finite limit $v(a+)$ $(v(b-))$, then $\lim\limits_{x\to a+}\inf (t-a)|v'(t)|=0$ ( $\lim\limits_{x\to b-}\inf (b-t)|v'(t)|=0$).

**Lemma 2.4.** *Let $a_1\in ]a,b[, b_1\in ]a_1,b[$ and let $u$ and $w$ be solutions of the equation* (2.1) *satisfying the conditions*

$$\begin{gathered} u(a_1)=0, \ \ u'(a_1)>0 \quad \big(u(b_1)=0, \ \ u'(b_1)<0\big), \\ w(t)>0 \ \ \textit{for} \ \ a_1\leq t\leq b_1, \quad w'(b_1)=0 \ \ (w'(a_1)=0). \end{gathered} \tag{2.19}$$

*Let further $v\in \widetilde{C}'([a_1,b_1])$ be such that*

$$v(t)>0, \ \ v'(t)>0 \ \ \textit{for} \ \ a_1\leq t\leq b_1$$
$$\big(\, v(t)>0, \ \ v'(t)<0 \ \ \textit{for} \ \ a_1\leq t\leq b_1\,\big),$$
$$v''(t)\leq p(t)|v(t)|^\alpha |v'(t)|^{1-\alpha} \quad \textit{for} \quad a_1<t<b_1.$$

*Then*

$$u'(t)>0 \ \ \textit{for} \ \ a_1\leq t\leq b_1 \quad \big(u'(t)<0 \ \ \textit{for} \ \ a_1\leq t\leq b\big) \tag{2.20}$$

*and*

$$\frac{w'(t)}{w(t)}<\frac{v'(t)}{v(t)} \quad \textit{for} \quad a_1\leq t\leq b_1 \quad \Big(\frac{w'(t)}{w(t)}>\frac{v'(t)}{v(t)} \quad \textit{for} \quad a_1\leq t\leq b_1\Big). \tag{2.21}$$

*Proof.* Suppose that (2.20) violated. Then by (2.19) there exists $c\in ]a_1,b_1[$ such that $u'(t)>0$ for $a_1<t<c$ $(u'(t)<0$ for $c<t\leq b_1)$, and $u'(c)=0$. Assume $\rho(t)=|\frac{v'(t)}{v(t)}|^\alpha \operatorname{sgn} v'(t)$ for $a_1<t<b_1$ and $\sigma(t)=|\frac{u'(t)}{u(t)}|^\alpha \operatorname{sgn} u'(t)$ for $a_1<t\leq c$ (for $c\leq t<b_1$). Clearly,

$$\rho'(t)\leq \alpha p(t)-\alpha|\rho(t)|^{\frac{\alpha+1}{\alpha}} \quad \text{for} \quad a_1<t<b_1, \tag{2.22}$$

$$\sigma'(t)=\alpha p(t)-\alpha|\sigma(t)|^{\frac{\alpha+1}{\alpha}} \quad \text{for} \quad a_1<t<c \ \ (\text{for} \ \ c<t<b_1), \tag{2.23}$$

$$\sigma(a_1+)=+\infty, \ \ \sigma(c)=0 \quad (\sigma(b_1-)=-\infty, \ \sigma(c)=0). \tag{2.24}$$

By (2.24), there exist $c_1 \in ]a_1, c[$ and $c_2 \in ]c_1, c[$ ($c_1 \in ]c, b_1[$ and $c_2 \in ]c, c_1[$) such that

$$\begin{gathered}\rho(t) > \sigma(t) > 0 \quad \text{for} \quad c_1 < t < c_2, \ \ \rho(c_1) = \sigma(c_1) \\ \big(\rho(t) < \sigma(t) < 0 \quad \text{for} \quad c_2 < t < c_1, \ \ \rho(c_1) = \sigma(c_1)\big).\end{gathered} \tag{2.25}$$

On account of this fact, from (2.22) and (2.23) we have

$$\begin{aligned}\sigma(t) &= \sigma(c_1) + \alpha \int_{c_1}^{t} p(s)ds - \alpha \int_{c_1}^{t} |\sigma(s)|^{\frac{\alpha+1}{\alpha}} ds > \rho(c_1) + \\ &\quad + \alpha \int_{c_1}^{t} p(s)ds - \alpha \int_{c_1}^{t} |\rho(s)|^{\frac{\alpha+1}{\alpha}} ds \geq \rho(t) \quad \text{for} \quad c_1 < t < c_2 \\ \Bigg(\sigma(t) &= \sigma(c_1) - \alpha \int_{t}^{c_1} p(s)ds + \alpha \int_{t}^{c_1} |\sigma(s)|^{\frac{\alpha+1}{\alpha}} ds < \rho(c_1) - \\ &\quad - \alpha \int_{t}^{c_1} p(s)ds + \alpha \int_{t}^{c_1} |\rho(s)|^{\frac{\alpha+1}{\alpha}} \leq \rho(t) \quad \text{for} \quad c_2 < t < c_1\Bigg),\end{aligned}$$

which contradicts (2.25). Hence there takes place (2.20). Quite in a similar manner we can easily be convinced that (2.21) is valid. Thus the lemma is proved. □

**Lemma 2.5.** *Let $u$ be a nontrivial solution of the equation* (2.1) *satisfying the condition*

$$u(a+) = 0 \quad (u(b-) = 0), \tag{2.26}$$

*and let $v \in \widetilde{C}'_{\rm loc}(]a, c])$ ($v \in \widetilde{C}'_{\rm loc}([c, b[)$), where $c \in ]a, b[$, have a finite limit $v(a+) \geq 0$ ($v(b-) \geq 0$) and satisfy the conditions*

$$\begin{gathered}v'(t) > 0 \quad for \quad a < t \leq c \quad (v'(t) < 0 \quad for \quad c \leq t < b), \\ v''(t) \leq p(t)|v(t)|^{\alpha}|v'(t)|^{1-\alpha} \quad for \quad a < t < c \ \ (for \ \ c < t < b).\end{gathered}$$

*Then*

$$u'(t) \neq 0 \quad for \quad a < t \leq c \ \ (for \ \ c \leq t < b). \tag{2.27}$$

*Proof.* Let $u_1$ be a solution of the problem (2.1), (2.2). First let us show that

$$u_1'(t) > 0 \quad \text{for} \quad a < t \leq c. \tag{2.28}$$

Assume the contrary. Then there exists $c_1 \in ]a, c[$ such that $u_1'(t) > 0$ for $a < t < c_1$, $u_1'(c_1) = 0$. By Lemma 2.4 we have

$$\frac{u_1'(t)}{u_1(t)} < \frac{v'(t)}{v(t)} \quad \text{for} \quad a < t < c_1. \tag{2.29}$$

Suppose

$$w(t) = v'(t)u_1(t) - u_1'(t)v(t) \quad \text{for} \quad a < t < c_1,$$

$$h(t)=\frac{|v'(t)u_1(t)|^{1-\alpha}-|u_1'(t)v(t)|^{1-\alpha}}{v'(t)u_1(t)-u_1'(t)v(t)} \quad \text{for} \quad a<t<c_1,$$
$$g(t)=p(t)h(t)(u_1(t)v(t))^{\alpha} \quad \text{for} \quad a<t<c_1. \tag{2.30}$$

Since the function $f(x)=\frac{|x|^{1-\alpha}-1}{x-1}$ is bounded, there exists $M_0>0$ such that $|h(t)|\le\frac{M_0}{(v(t)u_1'(t))^{\alpha}}$ for $a<t<c_1$. Bearing this in mind, from (2.30) we have

$$|g(t)|\le M_0(t-a)^{\alpha}|p(t)|\Big(\frac{u_1(t)}{(t-a)u_1'(t)}\Big)^{\alpha}\le M(t-a)^{\alpha}|p(t)| \text{ for } a<t<\frac{a+c_1}{2},$$

where

$$M=M_0\cdot\sup\Big\{\Big(\frac{u_1(t)}{(t-a)u_1'(t)}\Big)^{\alpha}:a<t<\frac{a+c_1}{2}\Big\}.$$

Thus the function $g$ is integrable on the segment $[a,\frac{a+c_1}{2}]$.

It can be easily seen that $w'(t)\le g(t)w(t)$ for $a<t<(a+c_1)/2$. According to the theorem on differential inequalities, the above inequality yields

$$w(t)\ge w\Big(\frac{a+c_1}{2}\Big)\exp\Big[-\int_t^{\frac{a+c_1}{2}}g(s)ds\Big]\ge$$
$$\ge w\Big(\frac{a+c_1}{2}\Big)\exp\Big[-\int_a^{\frac{a+c_1}{2}}|g(s)|ds\Big] \text{ for } a<t\le(a+c_1)/2.$$

The latter inequality, owing to (2.29), results in

$$\lim_{t\to a+}\inf w(t)>0. \tag{2.31}$$

On the other hand, by Remark 2.2 and condition (2.2) we have

$$\lim_{t\to a+}\inf w(t)=\lim_{t\to a+}\inf\Big((t-a)v'(t)\frac{u_1(t)}{t-a}-u_1'(t)v(t)\Big)\le 0,$$

which contradicts (2.31). Thus we have proved that (2.28) is fulfilled.

Let us now show that (2.27) is satisfied. By Lemma 2.4, there exists $c_0\in]a,c[$ such that $u(t)\ne 0$ for $a<t<c_0$. Without loss of generality we will assume that

$$u(t)>0 \quad \text{for} \quad a<t<c_0. \tag{2.32}$$

Show that

$$u'(t)>0 \quad \text{for} \quad a<t<c_0. \tag{2.33}$$

Assume that (2.33) violated. Then there exists $c_2\in]a,c_0[$ such that $u'(c_2)=0$. By Lemma 2.4 we have $\frac{u'(t)}{u(t)}<\frac{u_1'(t)}{u_1(t)}$ for $a<t<c_2$. From what has been done above we can see that $\widehat{w}'(t)=\widehat{g}(t)\widehat{w}(t)$ for $a<t<c_2$, where

$$\widehat{w}(t)=u'(t)u_1(t)-u_1'(t)u(t) \quad \text{for} \quad a<t<c_2,$$

$$\widehat{g}(t) = p(t)\widehat{h}(t)(u(t)u_1(t))^\alpha \quad \text{for} \quad a < t < c_2,$$
$$\widehat{h}(t) = \frac{|u'(t)u_1(t)|^{1-\alpha} - |u_1'(t)u(t)|^{1-\alpha}}{u'(t)u_1(t) - u_1'(t)u(t)} \quad \text{for} \quad a < t < c_2,$$

which, as above, leads to the contradiction $0 > 0$.

Thus we have proved that if for some $c_0 \in ]a, c[$ the inequality (2.32) is fulfilled, then (2.33) also holds. On the basis of the above-said we readily conclude that $u(t) > 0$ for $a < t \le c$, and hence (2.27) takes place. □

*Remark* 2.3. Let $u_1(u_2)$ be a solution of the problem (2.1), (2.2) ((2.1), (2.3)). It is clear that there exists $c \in ]a, b[$ such that $u_1'(t) > 0$ for $a < t \le c$ ($u_2'(t) < 0$) for $c \le t < b$). Using Lemma 2.5 for the case $v(t) = u_1(t)$ for $a < t \le c$ ($v(t) = u_2(t)$ for $c \le t < b$), we find that if $u$ is a nontrivial solution of the equation (2.1) satisfying the condition (2.26), then there exists $c \in ]a, b[$ such that (2.27) holds.

**Lemma 2.6.** *Let $u$ be a nontrivial solution of the equation* (2.1) *satisfying the conditions*

$$\begin{gathered} u(a+) = 0, \quad \lim_{t \to a+} \inf (t-a)\Big|\frac{u'(t)}{u(t)}\Big| < +\infty, \\ \Big(u(b-) = 0, \quad \lim_{t \to b-} \inf (b-t)\Big|\frac{u'(t)}{u(t)}\Big| < +\infty\Big). \end{gathered} \tag{2.34}$$

*Then*

$$\lim_{t \to a+} (t-a)\frac{u'(t)}{u(t)} = 1, \quad \Big(\lim_{t \to b-} (b-t)\frac{u'(t)}{u(t)} = -1\Big). \tag{2.35}$$

*Proof.* By Remark 2.3, without loss of generality we can assume that for some $c_0 \in ]a, b[$

$$u'(t) > 0 \quad \text{for} \quad a < t < c_0 \quad (u'(t) < 0 \quad \text{for} \quad c_0 < t < b). \tag{2.36}$$

Multiplying both parts of (2.1) by $t - a$ (by $b - t$) and integrating from $\tau$ to $t$ (from $t$ to $\tau$), we obtain equality (2.16), which with account for (2.36) and (2.7) yields

$$\begin{gathered} \frac{(t-a)u'(t)}{u(t)} \le 1 + \frac{(\tau-a)u'(\tau)}{u(\tau)} + \\ + \int_\tau^t (s-a)^\alpha |p(s)| \Big[1 + \frac{(s-a)u'(s)}{u(s)}\Big] ds \quad \text{for} \quad a < \tau < t < c_0 \\ \Big(\frac{(b-t)|u'(t)|}{u(t)} \le 1 + \frac{(b-\tau)|u'(\tau)|}{u(\tau)} + \\ + \int_t^\tau (b-s)^\alpha |p(s)| \Big[1 + \frac{(b-s)|u'(s)|}{u(s)}\Big] ds \quad \text{for} \quad c_0 < t < \tau < b\Big) \end{gathered} \tag{2.37}$$

and

$$\begin{gathered}\Big|\frac{(t-a)u'(t)}{u(t)}-1\Big|\leq\frac{(\tau-a)u'(\tau)+u(\tau)}{u(t)}+\\ +\int_\tau^t(s-a)^\alpha|p(s)|\Big[1+\frac{(s-a)u'(s)}{u(s)}\Big]ds\ \text{ for }\ a<\tau<t<c_0\\ \Big(\Big|\frac{(b-t)u'(t)}{u(t)}+1\Big|\leq\frac{(b-\tau)|u'(\tau)|+u(\tau)}{u(t)}+\\ +\int_t^\tau(b-s)^\alpha|p(s)|\Big[1+\frac{(b-s)|u'(s)|}{u(s)}\Big]ds\ \text{ for }\ c_0<t<\tau<b\Big).\end{gathered}\tag{2.38}$$

From (2.37) we have

$$\begin{gathered}\frac{(t-a)u'(t)}{u(t)}\leq M(\tau)+\int_\tau^t(s-a)^\alpha|p(s)|\Big(\frac{(s-a)u'(s)}{u(s)}\Big)ds\text{ for }a<\tau<t<c_0\\ \Big(\frac{(b-t)|u'(t)|}{u(t)}\leq M(\tau)+\int_t^\tau(b-s)^\alpha|p(s)|\Big(\frac{(b-s)|u'(s)|}{u(s)}\Big)ds\text{ for }c_0<t<\tau<b\Big),\end{gathered}$$

where

$$\begin{gathered}M(\tau)=1+\frac{(\tau-a)u'(\tau)}{u(\tau)}+\int_a^{c_0}(s-a)^\alpha|p(s)|ds\ \text{ for }\ a<\tau<c_0\\ \Big(M(\tau)=1+\frac{(b-\tau)|u'(\tau)|}{u(\tau)}+\int_{c_0}^b(b-s)^\alpha|p(s)|ds\ \text{ for }\ c_0<\tau<b\Big).\end{gathered}$$

By the Gronwall–Bellman lemma the last inequality results in

$$\begin{gathered}\frac{(t-a)u'(t)}{u(t)}\leq M(\tau)\exp\Big[\int_a^t(s-a)^\alpha|p(s)|ds\Big]\ \text{ for }\ a<\tau<t<c_0\\ \Big(\frac{(b-t)|u'(t)|}{u(t)}\leq M(\tau)\exp\Big[\int_t^b(b-s)^\alpha|p(s)|ds\Big]\ \text{ for }\ c_0<t<\tau<b\Big).\end{gathered}$$

Taking now into account (2.34), it is not difficult to conclude that there exists $M_0>0$ such that

$$\frac{(t-a)u'(t)}{u(t)}\leq M_0\ \text{ for }\ a<t<c_0\Big(\frac{(b-t)|u'(t)|}{u(t)}\leq M_0\ \text{ for }\ c_0<t<b\Big).$$

On the basis of the above-said, from (2.38) we find that

$$\begin{gathered}\Big|\frac{(t-a)u'(t)}{u(t)}-1\Big|\leq\frac{(\tau-a)u'(\tau)+u(\tau)}{u(t)}+(1+M_0)\int_a^t(s-a)^\alpha|p(s)|ds\\ \text{for }\ a<\tau<t<c_0\end{gathered}$$

$$\left(\left|\frac{(b-t)u'(t)}{u(t)}+1\right| \le \frac{(b-\tau)|u'(\tau)|+u(\tau)}{u(t)}+(1+M_0)\int_t^b (b-s)^\alpha |p(s)|ds \quad \text{for} \quad c_0<t<\tau<b\right).$$

This, by virtue of Lemma 2.3, allows us to conclude that

$$\left|\frac{(t-a)u'(t)}{u(t)}-1\right| \le (1+M_0)\int_a^t (s-a)^\alpha |p(s)|ds \quad \text{for} \quad a<t<c_0$$
$$\left(\left|\frac{(b-t)u'(t)}{u(t)}+1\right| \le (1+M_0)\int_t^b (b-s)^\alpha |p(s)|ds \quad \text{for} \quad c_0<t<b\right).$$

Hence (2.35) is fulfilled. □

**Lemma 2.7.** *Let $u$ be a non-trivial solution of the equation* (2.1) *satisfying the condition* (2.26). *Then* (2.35) *is satisfied.*

*Proof.* By Remark 2.3, without loss of generality we can assume that (2.36), where $c_0 \in ]a,b[$ is some point, is fulfilled. Multiplying both parts of (2.1) by $t-a$ (by $b-t$), integrating from $a$ to $t$ (from $t$ to $b$) and taking Lemma 2.3 into account, we obtain

$$\begin{gathered}(t-a)u'(t)=u(t)+\int_a^t (s-a)p(s)|u(s)|^\alpha |u'(s)|^{1-\alpha}ds \quad \text{for} \quad a<t<c_0 \\ \left((b-t)|u'(t)|=u(t)+\int_t^b (b-s)p(s)|u(s)|^\alpha |u'(s)|^{1-\alpha}ds \quad \text{for} \quad c_0<t<b\right).\end{gathered} \tag{2.39}$$

By Lemma 2.6,

$$\lim_{t\to a+} \inf (t-a)\frac{u'(t)}{u(t)}>0 \quad \left(\lim_{t\to b-} \inf (b-t)\frac{|u'(t)|}{u(t)}>0\right).$$

Hence there exist $\varepsilon>0$ and $c_1 \in ]a,c_0[$, ($c_1 \in ]c_0,b[$) such that

$$u(t)<\varepsilon(t-a)u'(t) \quad \text{for} \quad a<t<c_1, \quad (u(t)<\varepsilon(b-t)|u'(t)| \quad \text{for} \quad c_1<t<b). \tag{2.40}$$

Owing to (2.36) and (2.40), it follows from (2.39) that

$$(t-a)u'(t) \le u(x)+\varepsilon^\alpha \int_a^t (s-a)^\alpha |p(s)|((s-a)|u'(s)|)ds \quad \text{for} \quad a<t\le x<c_1$$
$$\left((b-t)|u'(t)| \le u(x)+\varepsilon^\alpha \int_t^b (b-s)^\alpha |p(s)|((b-s)|u'(s)|)ds \quad \text{for} \quad c_1<x\le t<b\right),$$

from which by the Gronwall-Bellman lemma we find that

$$\frac{(x-a)u'(x)}{u(x)} \le \exp\left[\varepsilon^\alpha \int_a^x (s-a)^\alpha |p(s)|ds\right] \quad \text{for} \quad a<x<c_1$$
$$\left(\frac{(b-x)|u'(x)|}{u(x)} \le \exp\left[\varepsilon^\alpha \int_x^b (b-s)^\alpha |p(s)|ds\right] \quad \text{for} \quad c_1<x<b\right).$$

Consequently (2.34) is fulfilled and Lemma 2.6 allows one to conclude that the equality (2.35) holds. $\square$

**Lemma 2.8.** *Let $u$ be a solution of the equation* (2.1)*, satisfying the conditions*

$$u(a+)=0,\quad \liminf_{t\to a+}|u'(t)|=0 \quad \bigl(u(b-)=0,\quad \liminf_{t\to b-}|u'(t)|=0\bigr). \tag{2.41}$$

*Then $u$ is identically equal to zero.*

*Proof.* Assume to the contrary that $u$ is a non-trivial solution of the equation (2.1) satisfying the conditions (2.41). By Remark 2.3, without loss of generality we will assume that (2.36), where $c_0 \in ]a,b[$, is fulfilled. From (2.1) we find that

$$u'(t)=u'(\tau)+\int_\tau^t p(s)|u(s)|^\alpha |u'(s)|^{1-\alpha}\operatorname{sgn} u(s)ds \ \text{ for } \ a<\tau<t<b,$$

$$u(t)=u(\tau)+(t-\tau)u'(\tau)+\int_\tau^t (t-s)p(s)|u(s)|^\alpha |u'(s)|^{1-\alpha}\operatorname{sgn} u(s)ds$$
$$\text{for} \quad a<\tau<t<b.$$

This, with regard for (2.7), allows one to conclude that

$$|u'(t)|\le |u'(\tau)|+\int_\tau^t (s-a)^\alpha |p(s)|\Bigl[\frac{|u(s)|}{s-a}+|u'(s)|\Bigr]ds \quad \text{for} \quad a<\tau<t<b,$$

$$\frac{|u(t)|}{t-a}\le\frac{|u(\tau)|}{\tau-a}+|u'(\tau)|+\int_\tau^t (s-a)^\alpha |p(s)|\Bigl[\frac{|u(s)|}{s-a}+|u'(s)|\Bigr]ds \ \text{ for } a<\tau<t<b.$$

By Lemma 2.7, there exist $\varepsilon>0$ and $c_1 \in ]a,c_0[$ such that (2.40) is fulfilled. Combining the last two inequalities, using the Gronwall-Bellman lemma and condition (2.40), we obtain

$$\frac{|u(t)|}{t-a}+|u'(t)|\le(2+\varepsilon)u'(\tau)\exp\biggl[\int_a^t (s-a)^\alpha |p(s)|ds\biggr] \text{ for } a<\tau<t<b,\ \tau<c_1.$$

If we now take into account (2.41), then from the above inequality we will arrive at $|u(t)|+(t-a)|u'(t)|=0$ for $a<t<b$, which contradicts our assumption. $\square$

**Lemma 2.9.** *The set of proper solutions of the equation* (2.1) *is nonempty.*

*Proof.* Let $\gamma\in[a,b[$ and $\delta\in]\gamma,b]$. Denote by $B([\gamma,\delta])$ the set of all nontrivial solutions of (2.1) satisfying the condition

$$\operatorname{mes}\Bigl\{\{t\in]\gamma,\delta[:\, u'(t)=0\}\setminus\{t\in]\gamma,\delta[:\, p(t)=0\}\Bigr\}=0.$$

We have to show that $B([a,b])\ne\varnothing$.

First we show that if $u_0 \in B([\gamma,\delta])$, where $\delta < b$, then there exist $\delta_1 \in ]\delta, b[$ and a non-trivial solution $\overline{u}_0$ of (2.1) such that

$$\overline{u}_0(t) = u_0(t) \quad \text{for} \quad \gamma \le t \le \delta, \quad \overline{u}_0 \in B([\gamma,\delta_1]). \tag{2.42}$$

Indeed, by Remark 2.1, either

$$u_0(\delta) \ne 0, \tag{2.43}$$

or

$$u_0(\delta) = 0, \quad u_0'(\delta) \ne 0. \tag{2.44}$$

Suppose that (2.43) is fulfilled. Denote by $\rho_0$ the solution of the Cauchy problem

$$\rho' = \alpha p(t) - \alpha|\rho|^{\frac{\alpha+1}{\alpha}}; \quad \rho(\delta) = \left|\frac{u_0'(\delta)}{u_0(\delta)}\right|^\alpha \operatorname{sgn} u_0'(\delta).$$

Assume $\rho_0$ is defined on $[\delta,\delta_1]$. Let

$$u(t) = \begin{cases} u_0(t) & \text{for} \quad \gamma \le t \le \delta, \\ u_0(\delta)\exp\left[\int_\delta^t |\rho_0(s)|^{\frac{1}{\alpha}} \operatorname{sgn} \rho_0(s) ds\right] & \text{for} \quad \delta < t \le \delta_1. \end{cases}$$

The function $u$, as is easily seen, satisfies (2.1) for $t \in ]\gamma,\delta_1[$. By Lemma 2.2, there exists a solution $\overline{u}_0$ of (2.1) such that $\overline{u}_0(t) = u(t)$ for $t \in [\gamma,\delta_1]$. Clearly, $\overline{u}_0$ satisfies (2.42).

Suppose now that (2.44) is fulfilled. Denote by $v_0$ a solution of the Cauchy problem $u'' = p(t)|u|^\alpha|u'|^{1-\alpha}\operatorname{sgn} u$; $u(\delta) = 0$, $u'(\delta) = u_0'(\delta)$. Obviously, there exists $\delta_1 \in ]\delta, b[$ such that $v_0'(t) \ne 0$ for $t \in [\delta,\delta_1]$. Suppose

$$u(t) = \begin{cases} u_0(t) & \text{for} \quad \gamma \le t \le \delta, \\ v_0(t) & \text{for} \quad \delta < t \le \delta_1. \end{cases}$$

Further, as above, we can easily see that there exists a solution $\overline{u}_0$ of the equation (2.1) satisfying (2.42).

Let now $u_1$ be a solution of the problem (2.1), (2.2). It is not difficult to see that there exists $a_1 \in ]a, b[$ such that $u_1 \in B([a, a_1])$. Denote by $I$ the set of those $\delta \in [a_1, b]$ for which there exists a solution $\overline{u}$ of the equation (2.1), satisfying the conditions $\overline{u}(t) = u_1(t)$ for $a \le t \le a_1$, $\overline{u} \in B([a,\delta])$. Let $b_1 = \sup I$. By Lemma 2.2, to prove the lemma it suffices to show that $b_1 = b$.

Assume to the contrary that $b_1 < b$. Then, as is shown above, if $\delta \in I$, there exists $\delta_1 \in ]\delta, b[$ such that $\delta_1 \in I$. Consequently, $I \ne \varnothing$, and $I$ is a connected set whose every point, except the point $a_1$, is an interior point. In view of this fact,

$$b_1 \notin I. \tag{2.45}$$

Choose a sequence of points $(t_k)_{k=1}^{+\infty} \subset I$ and a sequence of functions $(v_k)_{k=1}^{+\infty}$ such that

$$\begin{gathered}
t_k < t_{k+1}, \quad k = 1, 2, \dots, \quad a_1 < t_1, \quad \lim_{k \to +\infty} t_k = b_1, \\
v_k \in B([a, t_k]), \quad k = 1, 2, \dots, \\
v_1(t) = u_1(t) \ \text{ for } \ a < t < a_1, \\
v_{k+1}(t) = v_k(t) \ \text{ for } \ a < t < t_k, \quad k = 1, 2, \dots .
\end{gathered} \tag{2.46}$$

Assume

$$u(t) = \begin{cases} v_1(t) & \text{for } \ a \le t < t_1, \\ v_k(t) & \text{for } \ t_{k-1} \le t < t_k, \quad k = 2, 3, \dots . \end{cases} \tag{2.47}$$

It is seen that $u \in \widetilde{C}'_{\mathrm{loc}}(]a, b_1[)$ and $u$ satisfies (2.1) for $t \in ]a, b_1[$. Hence by Lemma 2.2 there exists a solution $\overline{u}$ of (2.1) such that $\overline{u}(t) = u(t)$ for $a \le t < b_1$. From (2.46) and (2.47) we have

$$\begin{gathered}
\{t \in ]a, t_k[ : v_k'(t) = 0\} \subset \{t \in ]a, t_k[ : p(t) = 0\} \cup A_k, \quad \operatorname{mes} A_k = 0, \quad k = 1, 2. \dots, \\
\{t \in ]a, b_1[ : \overline{u}'(t) = 0\} \subset \{t \in ]a, b_1[ : p(t) = 0\} \cup A, \quad A_k = \bigcup_{k=1}^{\infty} A_k.
\end{gathered}$$

Since $\operatorname{mes} A = \sum\limits_{k=1}^{+\infty} \operatorname{mes} A_k = 0$, this implies that $\overline{u} \in B([a, b_1])$ and, consequently, $b_1 \in I$, which contradicts (2.45). $\square$

*Remark* 2.4. In the above lemma we have, in fact, proved that the problem (2.1), (2.2) has at least one proper solution. Similarly we can see that the problem (2.1), (2.3) has at least one proper solution, and for any $t_0 \in ]a, b[$ and $c_1$, $c_2 \in R$, $|c_1| + |c_2| \ne 0$, the Cauchy problem

$$u'' = p(t)|u|^{\alpha}|u'|^{1-\alpha} \operatorname{sgn} u, \quad u(t_0) = c_1, \quad u'(t_0) = c_2$$

has at least one proper solution.

**Lemma 2.10.** *Let there exist $c \in ]a, b[$ and a continuous function $v \in \widetilde{C}'_{\mathrm{loc}}(]a, c[ \cup ]c, b[)$ having finite limits $v(a+) \ge 0$, $v(b-) \ge 0$, $v'(c-) > 0$, $v'(c+) < 0$ and satisfying the conditions*

$$\begin{gathered}
v''(t) \le p(t)|v(t)|^{\alpha}|v'(t)|^{1-\alpha} \quad \textit{for} \quad a < t < b, \\
v(t) > 0 \ \textit{for} \ a < t < b, \quad v'(t) > 0 \ \textit{for} \ a < t < c, \quad v'(t) < 0 \ \textit{for} \ c < t < b.
\end{gathered}$$

*Then $p \in U_{\alpha}(]a, b[)$.*

*Proof.* Assume the contrary. Let $p \notin U_{\alpha}(]a, b[)$. Then there exist $a_1 \in [a, b[$, $b_1 \in ]a_1, b]$ and a solution $u_0$ of the equation (2.1) satisfying the conditions

$$u_0(t) > 0 \ \text{ for } \ a_1 < t < b_1, \quad u_0(a_1+) = 0, \quad u_0(b_1-) = 0. \tag{2.48}$$

By Lemma 2.8 and Remark 2.1, there exist $\tau_1 \in ]a_1, b_1[$ and $\tau_2 \in [\tau_1, b_1[$ such that

$$u_0'(t) > 0 \text{ for } a_1 < t < \tau_1, \quad u_0'(\tau_1) = 0,$$
$$u_0'(t) < 0 \text{ for } \tau_1 < t < b_1, \quad u_0'(\tau_2) = 0.$$

Let $u_1$ and $u_2$ be some solutions of the problems (2.1), (2.2) and (2.1), (2.3), respectively. Then by Lemmas 2.4 and 2.5 (with $v(t) = u_1(t)$, $u(t) = u_0(t)$ and $v(t) = u_2(t)$, $u(t) = u_0(t)$, respectively), there exist $t_1 \in ]a, \tau_1]$ and $t_2 \in [\tau_2, b[$ such that

$$u_1'(t) > 0 \text{ for } a < t < t_1, \quad u_1'(t_1) = 0,$$
$$u_2'(t) < 0 \text{ for } t_2 < t < b, \quad u_2'(t_2) = 0.$$

By Lemma 2.4 (with $u(t) = u_1(t)$ and $u(t) = u_2(t)$, respectively), we have $t_1 > c$ and $t_2 < c$, which is impossible, since $t_2 \geq t_1$. $\square$

**Lemma 2.11.** *Let there exist $c \in ]a, b[$ and a function $\sigma \in \widetilde{C}_{\rm loc}(]a, c[\cup]c, b[)$ having finite limits $\sigma(c-) \geq \sigma(c+)$ and satisfying the conditions*

$$\sigma'(t) \leq \alpha p(t) - \alpha|\sigma(t)|^{\frac{\alpha+1}{\alpha}} \quad \textit{for} \quad a < t < b, \tag{2.49}$$

$$\lim_{t\to a+} \inf (t-a)^\alpha \sigma(t) < 1, \quad \lim_{t \to b-} \sup (b-t)^\alpha \sigma(t) > -1. \tag{2.50}$$

*Then $p \notin O_\alpha(]a, b[)$.*

*Proof.* Assume to the contrary that $p \in O_\alpha(]a, b[)$. Then there exist $a_1 \in [a, b[$, $b_1 \in ]a_1, b]$ and a proper solution $u_0$ of the equation (2.1) satisfying the conditions (2.48). Suppose $a_1 = a$ and $b_1 = b$ (the lemma for the rest cases is proved analogously). Let us introduce the function $\rho$ by $\rho(t) = \left|\frac{u'(t)}{u(t)}\right|^\alpha \operatorname{sgn} u'(t)$ for $a < t < b$. Clearly,

$$\rho'(t) = \alpha p(t) - \alpha|\rho(t)|^{\frac{\alpha+1}{\alpha}} \text{ for } a < t < b. \tag{2.51}$$

By Lemma 2.6,

$$\lim_{t\to a+} (t-a)^\alpha \rho(t) = 1, \quad \lim_{t\to b-} (b-t)^\alpha \rho(t) = -1. \tag{2.52}$$

From (2.50) and (2.52) we readily conclude that there exist $t_1 \in ]a, b[$ and $\varepsilon \in ]0, b-t_1[$ such that $\sigma \in \widetilde{C}([t_1, t_1+\varepsilon])$ and

$$\sigma(t) > \rho(t) \text{ for } t_1 < t < t_1 + \varepsilon, \quad \sigma(t_1) = \rho(t_1). \tag{2.53}$$

Suppose $w(t) = \sigma(t) - \rho(t)$ for $t_1 \leq t \leq t_1 + \varepsilon$.

It is easy to see that there exists an integrable on $[t_1, t_1+\varepsilon]$ function $h : ]t_1, t_1+\varepsilon[ \to R$ such that $\left|\sigma(t)\right|^{\frac{\alpha+1}{\alpha}} - \left|\rho(t)\right|^{\frac{\alpha+1}{\alpha}} = \big(\sigma(t) - \rho(t)\big)h(t)$ for $t_1 < t < t_1+\varepsilon$. Owing to this fact, from (2.49) and (2.51) we obtain $w'(t) \leq -\alpha h(t) w(t)$ for $t_1 < t < t_1 + \varepsilon$, which allows us to conclude that $w(t) < 0$ for $t_1 < t < t_1 + \varepsilon$, since $w(t_1) = 0$. But this contradicts (2.53). $\square$

## 3. Proof of the Basic Results

*Proof of Theorem* 1.1. Let $u_0$ be a proper solution of the equation (1.1) (see Lemma 2.9). Let us show that $u_0$ has at least one zero in the interval $]a,b[$. Assume to the contrary that

$$u_0(t) > 0 \quad \text{for} \quad a < t < b. \tag{3.1}$$

Suppose

$$\rho(t) = \Big|\frac{u_0'(t)}{u_0(t)}\Big|^{\alpha} \operatorname{sgn} u_0'(t) \ \text{ for } \ a < t < b. \tag{3.2}$$

Clearly,

$$\rho'(t) = \alpha p(t) - \alpha\big|\rho(t)\big|^{\frac{\alpha+1}{\alpha}} \ \text{ for } \ a < t < b. \tag{3.3}$$

We can easily see that

$$\begin{gathered}\big\{t \in ]a,\lambda] : (t-a)^{\alpha}\rho(t) = 1\big\} \cup \\ \cup\big\{t \in [\mu,b[ : (b-t)^{\alpha}\rho(t) = -1\big\} \cup \big\{t \in [\lambda,\mu] : \rho(t) = 0\big\} \neq ]a,b[. \end{gathered} \tag{3.4}$$

Multiplying both parts of (3.3) by $(t-a)^{\alpha+1}$, integrating from $a$ to $\lambda$ and taking into account Lemmas 2.3 and 2.7, we obtain

$$\begin{gathered}(\lambda-a)^{\alpha}\rho(\lambda) = \frac{\alpha}{\lambda-a}\int_a^{\lambda}(s-a)^{\alpha+1}p(s)ds + \\ +\frac{1}{\lambda-a}\int_a^{\lambda}\Big[(\alpha+1)(s-a)^{\alpha}\rho(s) - \alpha(s-a)^{\alpha+1}\big|\rho(s)\big|^{\frac{\alpha+1}{\alpha}}\Big]ds. \end{gathered} \tag{3.5}$$

Multiplying now both parts of (3.3) by $(b-t)^{\alpha+1}$, integrating from $\mu$ to $b$ and taking into account Lemmas 2.3 and 2.7, we obtain

$$\begin{gathered}(b-\mu)^{\alpha}\rho(\mu) = \frac{\alpha}{b-\mu}\int_{\mu}^{b}(b-s)^{\alpha+1}p(s)ds + \\ +\frac{1}{b-\mu}\int_{\mu}^{b}\Big[(\alpha+1)(b-s)^{\alpha}\rho(s) + \alpha(b-s)^{\alpha+1}\big|\rho(s)\big|^{\frac{\alpha+1}{\alpha}}\Big]ds. \end{gathered} \tag{3.6}$$

Finally, integrating (3.3) from $\lambda$ to $\mu$, we obtain

$$-\alpha\int_{\lambda}^{\mu} p(s)ds = \rho(\lambda) - \rho(\mu) - \alpha\int_{\lambda}^{\mu}\big|\rho(s)\big|^{\frac{\alpha+1}{\alpha}}ds. \tag{3.7}$$

We readily see that

$$\begin{gathered}(\alpha+1)x - \alpha\big|x\big|^{\frac{\alpha+1}{\alpha}} < 1 \ \text{ for } \ x \in R\backslash\{1\}, \\ (\alpha+1)x + \alpha\big|x\big|^{\frac{\alpha+1}{\alpha}} > -1 \ \text{ for } \ x \in R\backslash\{-1\}. \end{gathered} \tag{3.8}$$

Taking into account (3.4) and (3.8), from (3.5)–(3.7) we obtain three inequalities

$$(\lambda - a)^\alpha \rho(\lambda) \le 1 + \frac{\alpha}{\lambda - a} \int_a^\lambda (s-a)^{\alpha+1} p(s)ds, \tag{3.9}$$

$$(b - \mu)^\alpha \rho(\mu) \ge -1 - \frac{\alpha}{b - \mu} \int_\mu^b (b-s)^{\alpha+1} p(s)ds, \tag{3.10}$$

$$-\alpha \int_\lambda^\mu p(s)ds \le \rho(\lambda) - \rho(\mu), \tag{3.11}$$

of which at least one is fulfilled in the strong sense. Owing to (3.9) and (3.10), the inequality (3.11) enables us to conclude that the condition (1.4) violated. Consequently, every proper solution of the equation (1.1) has at least one zero in the interval $]a,b[$. In particular, the proper solution of the problem (2.1), (2.2) (see Remark 2.4) has also at least one zero in the interval $]a,b[$, and hence $p \in O_\alpha(]a,b[)$. $\square$

*Proof of Corollary* 1.1. By the condition (1.5), one can choose $\lambda \in ]a, \frac{a+b}{2}]$ and $\mu \in [\frac{a+b}{2}, b[$ such that

$$-\alpha(\alpha+1)\Big[\frac{1}{(\lambda-a)^{\alpha+1}} \int_a^\lambda Q\Big(s, \frac{a+b}{2}, \alpha\Big)ds + \\ + \frac{1}{(b-\mu)^{\alpha+1}} \int_\mu^b Q\Big(s, \frac{a+b}{2}, \alpha\Big)ds\Big] \ge \frac{1}{(\lambda-a)^\alpha} + \frac{1}{(b-\mu)^\alpha}. \tag{3.12}$$

Taking into account Proposition 2.1, we can easily verify that

$$\int_a^\lambda Q\Big(s, \frac{a+b}{2}, \alpha\Big)ds = \frac{1}{\alpha+1}\Big[(\lambda-a)^{\alpha+1}\int_\lambda^{\frac{a+b}{2}} p(s)ds + \int_a^\lambda (s-a)^{\alpha+1} p(s)ds\Big],$$

$$\int_\mu^b Q\Big(s, \frac{a+b}{2}, \alpha\Big)ds = \frac{1}{\alpha+1}\Big[(b-\mu)^{\alpha+1}\int_{\frac{a+b}{2}}^\mu p(s)ds + \int_\mu^b (b-s)^{\alpha+1} p(s)ds\Big].$$

In view of that fact we find from (3.12) that the inequality (1.4) is fulfilled. Hence, by Theorem 1.1, $p \in O_\alpha(]a,b[)$. $\square$

*Proof of Corollary* 1.2. By (1.7), there exists $c \in ]a,b[ \setminus \{t_0\}$ such that

$$-Q(c, t_0, \alpha) \ge 1 + \max\Big\{\Big(\frac{t_0 - a}{b - t_0}\Big)^\alpha, \Big(\frac{b - t_0}{t_0 - a}\Big)^\alpha\Big\}. \tag{3.13}$$

Suppose

$$\lambda = \begin{cases} c & \text{if } c < t_0, \\ t_0 & \text{if } c > t_0, \end{cases} \qquad \mu = \begin{cases} t_0 & \text{if } c < t_0, \\ c & \text{if } c > t_0. \end{cases}$$

From (3.13) we easily conclude that $-\alpha \int_\lambda^\mu p(s)ds \ge \frac{1}{(\lambda-a)^\alpha} + \frac{1}{(b-\mu)^\alpha}$. This and the inequality (1.6) show that (1.4) is fulfilled, and hence by Theorem 1.1, $p \in O_\alpha(]a,b[)$. $\square$

*Proof of Theorem* 1.1′. Let $u_1$ be a proper solution of the problem (2.1), (2.2) (see Remark 2.4). Let us show that $u_1$ has at least one zero in the interval $]a,b[$. Assume the contrary. Then by (1.6) either

$$u_1'(t) > 0 \quad \text{for} \quad a < t < b, \tag{3.14}$$

or there exist $c_1 \in ]a,b[$ and $c_2 \in [c_1,b[$ such that

$$\begin{gathered} u_1'(t) > 0 \;\text{ for }\; a < t < c_1, \quad u_1'(t) < 0 \;\text{ for }\; c_2 < t < b, \\ u_1'(t) = 0 \;\text{ for }\; c_1 \le t \le c_2. \end{gathered} \tag{3.15}$$

Suppose $\rho(t) = |\frac{u_1'(t)}{u_1(t)}|^\alpha \operatorname{sgn} u_1'(t)$ for $a < t < b$. It is clear that (3.3) is fulfilled.

Assume that (3.15) is fulfilled. Multiplying both parts of (3.3) by $(t-a)^{\alpha+1}$ and integrating from $a$ to $t$, we obtain

$$(t-a)^\alpha \rho(t) = -\frac{\alpha}{t-a}\int_a^t (s-a)^{\alpha+1}|p(s)|ds +$$
$$+\frac{1}{t-a}\int_a^t \Big[(\alpha+1)(s-a)^\alpha \rho(s) - \alpha(s-a)^{\alpha+1}\big|\rho(s)\big|^{\frac{\alpha+1}{\alpha}}\Big]ds \;\text{ for }\; a<t<b,$$

which, according to (3.8), results in $\rho(t) \le \frac{1}{a_1^\alpha(t)}$ for $a < t < b$. Assume now that for some natural $k$

$$\rho(t) \le \frac{1}{a_k^\alpha(t)} \quad \text{for} \quad a < t < b. \tag{3.16}$$

From (3.3) with regard for (3.16) we have

$$\big(\rho^{-\frac{1}{\alpha}}(t)\big)' = |p(t)|\big(\rho(t)\big)^{-\frac{\alpha+1}{\alpha}} + 1 \ge 1 + a_k^{\alpha+1}(t)|p(t)| \quad \text{for} \quad a < t < c_1.$$

Integrating the above inequality from $a$ to $t$, we arrive at

$$\frac{1}{\rho^{\frac{1}{\alpha}}(t)} \ge t - a + \int_a^t a_k^{\alpha+1}(s)|p(s)|ds \;\text{ for }\; a < t < c_1.$$

Owing to (3.15), we find

$$\rho(t) \le \frac{1}{a_{k+1}^\alpha(t)} \quad \text{for} \quad a < t < b.$$

Thus it is proved by induction that for any natural $k$ the inequality (3.16) is fulfilled. Analogously, we can see that for any natural $k$

$$\rho(t) \ge -\frac{1}{b_k^\alpha(t)} \quad \text{for} \quad a < t < b. \tag{3.17}$$

Suppose now that (3.14) is fulfilled. Then, as above, we see that (3.16) holds and the inequality (3.17) is trivial in this case.

Integrating now (3.3) from $\lambda$ to $\mu$ and taking into account (3.16) and (3.17) (for $k=n$), we obtain

$$-\alpha\int_{\lambda}^{\mu}|p(s)|ds\le\frac{1}{a_n^{\alpha}(\lambda)}+\frac{1}{b_n^{\alpha}(\mu)},$$

which contradicts (1.8). $\square$

*Proof of Theorem* 1.2. Let $u_0$ be a proper solution of the equation (1.1). Let us show that $u_0$ has at least one zero in the interval $]a,b[$. Assume to the contrary that (3.1) is fulfilled. Introduce the function $\rho$ by (3.2). It is clear that (3.3) holds. We can easily verify that

$$(\alpha+1)f^{\frac{1}{\alpha+1}}(t)\le\int_a^t\frac{|f'(s)|}{f^{\frac{\alpha}{\alpha+1}}(s)}ds\le(t-a)^{\frac{\alpha}{\alpha+1}}\Big[\int_a^t\frac{|f'(s)|^{\alpha+1}}{f^{\alpha}(s)}ds\Big]^{\frac{1}{\alpha+1}}\text{ for } a<t<c,$$

$$(\alpha+1)g^{\frac{1}{\alpha+1}}(t)\le\int_t^b\frac{|g'(s)|}{g^{\frac{\alpha}{\alpha+1}}(s)}ds\le(b-t)^{\frac{\alpha}{\alpha+1}}\Big[\int_t^b\frac{|g'(s)|^{\alpha+1}}{g^{\alpha}(s)}ds\Big]^{\frac{1}{\alpha+1}}\text{ for } c<t<b.$$

By Lemmas 2.3 and 2.7 we have

$$\lim_{t\to a+}f(t)\rho(t)=0,\quad \lim_{t\to b-}g(t)\rho(t)=0. \tag{3.18}$$

Multiplying both parts of (3.3) by $f$ and integrating from $a+\varepsilon$ to $c$, where $\varepsilon\in]0,c-a[$, we obtain

$$\begin{aligned}-\alpha\int_{a+\varepsilon}^{c}&f(s)p(s)ds+f(c)\rho(c)-f(a+\varepsilon)\rho(a+\varepsilon)=\\ &=\int_{a+\varepsilon}^{c}\Big[f'(s)\rho(s)-\alpha f(s)\big|\rho(s)\big|^{\frac{\alpha+1}{\alpha}}\Big]ds\le\\ &\le\int_{a+\varepsilon}^{c}f(s)\Big[\frac{|f'(s)|}{f(s)}|\rho(s)|-\alpha\big|\rho(s)\big|^{\frac{\alpha+1}{\alpha}}\Big]ds.\end{aligned}\tag{3.19}$$

Analogously, multiplying both parts of (3.3) by $g$ and integrating from $c$ to $b-\eta$, where $\eta\in]0,b-c[$, we arrive at

$$\begin{aligned}-\alpha\int_{c}^{b-\eta}&g(s)p(s)ds+g(b-\eta)\rho(b-\eta)-g(c)\rho(c)\le\\ &\le\int_{c}^{b-\varepsilon}g(s)\Big[\frac{|g'(s)|}{g(s)}|\rho(s)|-\alpha\big|\rho(s)\big|^{\frac{\alpha+1}{\alpha}}\Big]ds.\end{aligned}\tag{3.20}$$

It can be easily verified that

$$\lambda|x|-\alpha|x|^{\frac{\alpha+1}{\alpha}}\le\Big(\frac{\lambda}{\alpha+1}\Big)^{\alpha+1}\quad\text{for}\quad x\in R.$$

Taking this and the equalities (3.18) into account, from (3.19) and (3.20) we conclude that

$$-\alpha g(c)\int_a^c f(s)p(s)ds + g(c)f(c)\rho(c) \le \frac{g(c)}{(\alpha+1)^{\alpha+1}}\int_a^c \frac{|f'(s)|^{\alpha+1}}{f^\alpha(s)}ds,$$
$$-\alpha f(c)\int_c^b g(s)p(s)ds - f(c)g(c)\rho(c) \le \frac{f(c)}{(\alpha+1)^{\alpha+1}}\int_c^b \frac{|g'(s)|^{\alpha+1}}{g^\alpha(s)}ds.$$

Adding the last two inequalities, we see that (1.9) violated. □

*Proof of Corollary* 1.3. The condition (1.10) follows from the condition (1.9) in the case where $f(t)=[(t-a)(b-t)]^{\alpha+1}$ and $g(t)=[(t-a)(b-t)]^{\alpha+1}$ for $a<t<b$, the condition (1.11) is obtained in the case where $f(t)=(t-a)^\lambda$, and $g(t)=(b-t)^\lambda$, for $a<t<b$, and the condition (1.12) in the case where $f(t)=\sin\frac{\pi(t-a)}{b-a}$ and $g(t)=\sin\frac{\pi(t-a)}{b-a}$ for $a<t<b$. □

*Proof of Theorem* 1.3. Suppose that $[p]_-\not\equiv 0$, since otherwise $p\in U_\alpha(]a,b[)$. Choose $\varepsilon>0$ such that $\sup\{(A(t)+\varepsilon)^\alpha\cdot|B(t)|^{1-\alpha} : a<t<b\}<b-a$. Let

$$v_0(t)=\varepsilon+A(t) \quad \text{for} \quad a<t<b. \tag{3.21}$$

Clearly, $v_0(t)>0$ for $a<t<b$, $v_0(a+)=\varepsilon$, $v_0(b-)=\varepsilon$, and there exists $c_0\in]a,b[$ such that $v_0'(t)\operatorname{sgn}(c_0-t)>0$ for $a<t<b$ and $t\ne c_0$ and

$$v_0'(t)=-\frac{1}{b-a}B(t) \quad \text{for} \quad a<t<b,$$
$$v_0''(t)=-[p(t)]_-\big[(t-a)(b-t)\big]^{\alpha\lambda}\le p(t)\big|v_0(t)\big|^\alpha\big|v_0'(t)\big|^{1-\alpha} \quad \text{for} \quad a<t<b. \tag{3.22}$$

Let $u_1$ and $u_2$ be some proper solutions of the problems (2.1), (2.2) and (2.1), (2.3), respectively. By Lemma 2.5 (in case $v(t)=v_0(t)$ for $a<t<b$), there exist $t_1\in[c_0,b[$ and $t_2\in]a,c_0]$ such that

$$\begin{aligned} &u_1'(t)>0 \quad \text{for} \quad a<t<t_1, \quad u_1'(t_1)=0,\\ &u_2'(t)<0 \quad \text{for} \quad t_2<t<b, \quad u_2'(t_2)=0. \end{aligned} \tag{3.23}$$

Suppose that $t_1=t_2$. Then by (3.23), the function

$$u(t)=\begin{cases} u_1(t) & \text{for} \quad a<t\le c_0,\\ u_2(t)\frac{u_1(c_0)}{u_2(c_0)} & \text{for} \quad c_0<t<b \end{cases}$$

will be a proper solution of the equation (1.1), and hence

$$p\in O_\alpha(]a,b[). \tag{3.24}$$

Suppose

$$\sigma(t)=\left|\frac{v_0'(t)}{v_0(t)}\right|^\alpha \operatorname{sgn} v_0'(t) \quad \text{for} \quad a<t<b.$$

By (3.21), (3.22) and Proposition 2.1 we can easily see that $\sigma$ satisfies the conditions of Lemma 2.11, and hence $p\notin O_\alpha(]a,b[)$, which contradicts (3.24).

Consequently, $t_1 > t_2$. Then by (3.23), there exists $c \in ]t_1, t_2[$ such that $u_1'(c) > 0 > u_2'(c)$. It can be easily verified that the function

$$v(t) = \begin{cases} u_1(t) & \text{for} \quad a < t \le c, \\ u_2(t)\frac{u_1(c)}{u_2(c)} & \text{for} \quad c < t < b \end{cases}$$

satisfies the conditions of Lemma 2.10, and hence $p \in U_\alpha(]a, b[)$. $\square$

*Proof of Theorem* 1.4. Let $k = \varphi(Q_*(t_0, \alpha))$,

$$\sigma(t) = \begin{cases} -\frac{Q(t,t_0,\alpha)-k}{(t-a)^\alpha} & \text{for} \quad a < t < t_0, \\ \frac{Q(t,t_0,\alpha)-k}{(b-t)^\alpha} & \text{for} \quad t_0 \le t < b. \end{cases} \tag{3.25}$$

By (1.13) we have

$$k - k^{\frac{\alpha}{\alpha+1}} \le Q(t, t_0, \alpha) < k \quad \text{for} \quad a < t < b, \tag{3.26}$$

from which we conclude that $\sigma(t) \operatorname{sgn}(t_0 - t) > 0$ for $a < t < b$, $t \ne t_0$ and

$$\big|\sigma(t)\big|^{\frac{\alpha+1}{\alpha}} \le kh(t) \quad \text{for} \quad a < t < b, \tag{3.27}$$

where

$$h(t) = \begin{cases} \frac{1}{(t-a)^{\alpha+1}} & \text{for} \quad a < t \le t_0, \\ \frac{1}{(b-t)^{\alpha+1}} & \text{for} \quad t_0 < t < b. \end{cases} \tag{3.28}$$

Suppose $v(t) = \exp\big[\operatorname{sgn}(t_0 - t)\int_{t_0}^t \big|\sigma(s)\big|^{\frac{1}{\alpha}} ds\big]$ for $a < t < b$. By (3.25)–(3.27) we can see that $v \in \widetilde{C}'_{\mathrm{loc}}(]a, t_0[\cup]t_0, b[)$, $v'(t)\operatorname{sgn}(t_0 - t) > 0$ for $a < t < b$, $t \ne t_0$, $v'(t_0+) > 0 > v'(t_0-)$, $v(a+) = 0$, $v(b-) = 0$ and

$$\begin{aligned} v''(t) = p(t)\big|v(t)\big|^\alpha \big|v'(t)\big|^{1-\alpha} + v(t)\big|\sigma(t)\big|^{\frac{1-\alpha}{\alpha}}\Big(\big|\sigma(t)\big|^{\frac{\alpha+1}{\alpha}} - kh(t)\Big) \le \\ \le p(t)\big|v(t)\big|^\alpha \big|v'(t)\big|^{1-\alpha} \quad \text{for} \quad a < t < b. \end{aligned}$$

Hence by Lemma 2.10 we have $p \in U_\alpha(]a, b[)$. $\square$

*Proof of Theorem* 1.5. Let

$$k = \begin{cases} \varphi(Q_*(t_0, \alpha)) & \text{for} \quad Q_*(t_0, \alpha) \ne 0, \\ k_1 & \text{for} \quad Q_*(t_0, \alpha) = 0, \end{cases}$$

where $k_1 \in ](\frac{\alpha}{\alpha+1})^{\alpha+1}, 1[$ is chosen in such a way that $Q^*(t_0, \alpha) < k_1 + k_1^{\frac{\alpha}{\alpha+1}}$. It is easy to see that $k \in ]0, 1[$.

Introduce the function $\sigma$ by the equality (3.25). Clearly,

$$\begin{gathered} \sigma \in \widetilde{C}_{\mathrm{loc}}(]a, t_0[\cup]t_0, b[), \quad \sigma(t_0-) = \frac{k}{(t_0 - a)^\alpha}, \quad \sigma(t_0+) = -\frac{k}{(b - t_0)^\alpha}, \\ \lim_{t \to a+}(t - a)^\alpha \sigma(t) = k < 1, \quad \lim_{t \to b-}(b - t)^\alpha \sigma(t) = -k > -1. \end{gathered}$$

According to (1.14) and (1.15) we readily conclude that $k - k^{\frac{\alpha}{\alpha+1}} \le \alpha Q(t, t_0, \alpha) \le k + k^{\frac{\alpha}{\alpha+1}}$ for $a < t < b$. Thus (3.27), where $h$ is the function defined by (3.28), is fulfilled. In view of this fact

$$\sigma'(t) = \alpha p(t) - \alpha k h(t) \le \alpha p(t) - \alpha |\sigma(t)|^{\frac{\alpha+1}{\alpha}} \quad for \quad a < t < b.$$

By Lemma 2.11 we now have $p \in O_\alpha(]a, b[)$. $\square$

The validity of Corollaries 1.5 and 1.6 follows from the fact that $\varphi(Q_*(t_0, \alpha)) \ge (\frac{\alpha}{\alpha+1})^{\alpha+1}$.

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