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SOME QUALITATIVE PROBLEMS OF THE LINEAR ELASTICITY THEORY

Abstract. The first and the second boundary value problems of statics are considered. The dependence of the solutions and of the corresponding eigenfrequencies of these problems on the elastic constants and density is investigated. The same dependence is studied for the total deformation energy and for Green's operators. The following theorem is proved: among anisotropic elastic convex bodies of a given volume there exists one for which the first eigenfrequency of the first boundary value problem is minimal.

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რეზიუმე. განხილულია სტატიკის პირველი და მეორე სასაზღვრო ამოცანები. გამოკვლეულია ამ ამოცანების ამონახსნებისა და შესაბამისი ფუნდამენტური სიხშირეების დამოკიდებულება დრეკად მუდმივებზე და სხეულის სიმკვრივეზე. იგივე დამოკიდებულება შესწავლილია დეფორმაციის სრული ენერგიისა და გრინის ოპერატორებისათვის. დამტკიცებულია შემდეგი თეორემა: მოცემული მოცულობის ანიზოტროპულ დრეკად ამოზნექილ სხეულებს შორის არსებობს ისეთი, რომლისთვისაც პირველი სასაზღვრო ამოცანის პირველი საკუთრივი სიხშირე მინიმალურია.

1. Dependence of the Solutions and of the Deformation Energy of the First and the Second Boundary Value Problems Both on the Elastic Constants and on the Density of the Medium

1.1. On the Continuous Dependence. Let $\mathbb{R}^n (n \geq 2)$ be an n-dimensional Euclidean space, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be its points and let D be a bounded domain in $\mathbb{R}^n$ with the compact connected boundary S.

Consider the matrix differential operator of the elasticity theory (see [1])

$$A(x, \partial) = \|A_{jk}(x, \partial)\|_{n\times n}, \quad A_{jk}(x, \partial) = \frac{\partial}{\partial x_i}\Big(a_{ijek}(x)\frac{\partial}{\partial x_e}\Big), \tag{1}$$

where $a_{ijek} \in C(\overline{D}, \mathbb{R})$ are elastic constants satisfying

$$a_{ijek}(x) = a_{jiek}(x) = a_{ekij}(x), \quad \forall x \in \overline{D}, \tag{2}$$

and

$$\exists m > 0, \quad \forall \xi_{ij} \in \mathbb{R}^1 (\xi_{ij} = \xi_{ji}) : \ a_{ijek}(x)\xi_{ij}\xi_{ek} \geq m\xi_{ij}\xi_{ij}. \tag{3}$$

Here and in what follows, the repetition of an index denotes summation with respect to that index from 1 to n.

Introduce an n'-dimensional vector function $a : \overline{D} \to \mathbb{R}^{n'} (n' = \frac{1}{8}n(n+1) \times \times (n^2 + n + 2))$ defined by

$$\forall x \in \overline{D} : a(x) = (a_{1111}(x), a_{1112}(x), \dots, a_{nnnn}(x)) \tag{4}$$

and assume

$$\forall a \in C(\overline{D}; \mathbb{R}^{n'}) : \|a\|_{C(\overline{D};\mathbb{R}^{n'})} = \sup_{x\in\overline{D}} \|a(x)\|_{\mathbb{R}^{n'}},$$

where

$$\|\xi\|_{\mathbb{R}^k} = \Big\{\sum_{i=1}^{k} \xi_i^2\Big\}^{1/2}.$$

Denote by K the set of the vector functions (4) satisfying (2) and (3) and show that this set is an open convex cone in the space $C(\overline{D}; \mathbb{R}^{n'})$.

Indeed, let $0 \leq t \leq 1$ and $a^{(q)} \in K (q = 0, 1)$. Then, by (3), there exists $m > 0$ such that $\forall x \in \overline{D}$ and $\forall \xi_{ij} \in \mathbb{R}(\xi_{ij} = \xi_{ji})$:

$$a^{(q)}_{ijek}(x)\xi_{ij}\xi_{ek} \geq m\xi_{ij}\xi_{ij} \quad (q = 0, 1).$$

The vector function $ta^{(0)} + (1 - t)a^{(1)}$ obviously satisfies (2) and (3), i.e., $ta^{(0)} + (1 - t)a^{(1)} \in K$. Moreover, $\forall t > 0$: $ta \in K$ if $a \in K$.

Let $\varepsilon > 0$ and $\|\Delta a\|_{C(\overline{D};\mathbb{R}^{n'})} < \frac{\varepsilon}{n^2}$. Then for any $x \in \overline{D}$ and $\xi_{ij}\xi_{ij} = 1$, we have $|\Delta a_{ijek}(x)\xi_{ij}\xi_{ek}| < \varepsilon$, whence by virtue of (3) $\forall x \in \overline{D}$, $\xi_{ij}\ \xi_{ij} = 1$: $(a_{ijek}(x) + \Delta a_{ijek}(x))\xi_{ij}\xi_{ek} \geq m - \varepsilon$, that is, $a + \Delta a \in K$. Thus the set K is an open, convex cone in $C(\overline{D}; \mathbb{R}^{n'})$.

It is known that the elastic medium (D, ρ, a) is the set consisting of a domain D and of functions ρ and a_{ijek}, where a_{ijek} satisfies (2) and (3) and ρ (the density of the medium) satisfies $\rho(x) > 0, \quad \forall x \in \overline{D}$.

Consider the vector function $b = (\rho, a)$, $b : \overline{D} \to \mathbb{R}^m$, $m = n' + 1$, and the space $C(\overline{D}; \mathbb{R}^m)$ with the norm $||b||_{C(\overline{D};\mathbb{R}^m)} = \sup\limits_{x \in \overline{D}} ||b(x)||_{\mathbb{R}^m}$.

As above, we can show that the set $\mathcal{K}$ of vector functions b determining the elastic medium (D, ρ, a) is an open convex cone in the space $C(\overline{D}; \mathbb{R}^m)$.

We consider the following boundary value

Problem 1 (see [1]). Find a vector function $u : D \to \mathbb{R}^n$, satisfying

$$\forall x \in D : A(x, \partial)u(x) + f(x; \rho, a) = 0, \quad \forall x \in S : \lim_{D \ni z \to x} u(z) = 0,$$

where $f \in C(\mathcal{K}; (H_0(D))^n)$, and $A(x, \partial)$ is defined by (1). It should be noted that the setting of Problem 1 is understood in the generalized variational sense (for the spaces $H_m(D)$ and $\overset{0}{H}_m(D)$, see [2], [3]).

As is well known, if $b \in \mathcal{K}$ and $f(\cdot, b) \in (H_0(D)^n)$, then there exists a unique vector function $u(\cdot, b)$ which belongs to the class $(\overset{0}{H}_1(D))^n$ and minimizes the functional $\frac{1}{2}B(v, v, a)$–$(f(\cdot, b), v)$ on the set $(\overset{0}{H}_1(D))^n$ (that is, $u(\cdot, b)$ is a solution of Problem 1 in the generalized variational sense), where

$$B(u, v, a) = \int\limits_D a_{ijek}(x)\varepsilon_{ij}(u)\varepsilon_{ek}(v)dx, \quad \varepsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \quad (5)$$

and (f, v) is the scalar product in $(H_0(D))^n$.

Given $M \subset \mathcal{K}$, we consider the following problem: find a vector function $b^{(0)} = (\rho^{(0)}, a^{(0)})$ such that

$$B(u(\cdot, b^{(0)}), u(\cdot, b^{(0)}), a^{(0)}) = \inf_{b \in M} B(u(\cdot, b), u(\cdot, b), a).$$

To investigate this problem, we first show that if $f \in C(\mathcal{K}, (H_0(D))^n)$, then the solution $u(\cdot, b)$ of Problem 1 depends continuously on b.

Choose Δb such that $b + \Delta b \in \mathcal{K}$. Then

$$\begin{aligned}\forall v \in (\overset{0}{H}_1(D))^n : \; & B(u(\cdot, b), v, a) = (f(\cdot, b), v), \\ & B(u(\cdot, b + \Delta b), v, a + \Delta a) = (f(\cdot, b + \Delta b), v).\end{aligned}$$

Taking into account (5), we find that

$$\begin{aligned}\forall v \in (\overset{0}{H}_1(D))^n \; & : B(\Delta u(\cdot, b), v, a) = (\Delta f(\cdot, b), v) - \\ & - \int\limits_D \Delta a_{ijek}(x)\varepsilon_{ij}(u(\cdot, b + \Delta b))\varepsilon_{ek}(v)dx.\end{aligned}$$

If instead of v we write $\Delta u(\cdot, b)$, then due to (3), Korn's first inequality (see [2], [4]) yields

$$\|\Delta u(\cdot, b)\|_1 \leq m_1(\|\Delta f(\cdot, b)\|_0 + \|\Delta a\|_{C(\overline{D};\mathbb{R}^{n'})}\|u(\cdot, b + \Delta b)\|_1), \qquad (6)$$

where m_1 does not depend on Δb. Moreover, we note that if $\|\Delta b\|_{C(\overline{D};\mathbb{R}^m)}$ is sufficiently small, then $\|u(\cdot, b + \Delta b)\|_1 \leq m_2\|f(\cdot, b + \Delta b)\|_0$ (m_2 does not depend on Δb).

Consequently, (6) implies that

$$\|\Delta u(\cdot, b)\|_1 \leq m_0(\|\Delta f(\cdot, b)\|_0 + \|\Delta a\|_{C(\overline{D};\mathbb{R}^{n'})}\|f(\cdot, b + \Delta b)\|_0). \qquad (7)$$

The latter proves that $u(\cdot, b)$ is continuous in b, i.e., $u \in C(\mathcal{K}; (\overset{0}{H}_1(D))^n)$.

Finally, consider the function $\varphi_f : \mathcal{K} \to \mathbb{R}$ defined as follows:

$$\forall b \in \mathcal{K} : \varphi_f(b) = B(u(\cdot, b), u(\cdot, b), a).$$

Taking into account (5) and the fact that $u \in C(\mathcal{K}; (\overset{0}{H}_1(D))^n)$, we can easily conclude that $\varphi_f \in C(\mathcal{K})$.

Thus the following theorem is valid.

Theorem 1. *If $M \subset \mathcal{K}$ is a compact set in the space $C(\overline{D}; \mathbb{R}^m)$, then there exists at least one vector function $b^{(0)} \in M$ such that*

$$B(u(\cdot, b^{(0)}), u(\cdot, b^{(0)}), a^{(0)}) = \inf_{b \in M} B(u(\cdot, b), u(\cdot, b), a).$$

Remark 1. The theorem has the following mechanical meaning: of the elastic media (D, ρ, a), $(\rho, a) \in M$, we can choose at least one medium $(D, \rho^{(0)}, a^{(0)})$ for which the total deformation energy $\varphi_f(\rho^{(0)}, a^{(0)})$ corresponding to the solution $u(\cdot, \rho^{(0)}, a^{(0)})$ of Problem 1 is minimal.

We will now proceed to the consideration of the second boundary value

Problem 2 (see [1]). Find a vector function $u : D \to \mathbb{R}^n$ satisfying

$$\forall x \in D : A(x, \partial)u(x) + f(x, \rho, a) = 0,$$
$$\forall x \in S : \lim_{D \ni z \to x} T(\partial_z, \nu(x))u(z) = 0,$$

where

$$T(\partial_z, \nu) = \|T_{jk}(\partial_z, \nu)\|_{n \times n}, \quad T_{jk}(\partial_z, \nu) = a_{ijek}(z)\nu_i \frac{\partial}{\partial z_e},$$

and $\nu(x)$ is the unit normal at the point $x \in S$, exterior with respect to D (the problem is posed in the generalized sense).

For Korn's second inequality to be valid (see [2], [3]), the boundary S is assumed to be sufficiently smooth.

It is known that if $f \in C(\mathcal{K}; (H_0(D))^n)$ and the conditions

$$\int_D f(x,b)dx = 0, \quad \int_D (x_i f_j - x_j f_i)dx = 0, \tag{8}$$

are fulfilled, then Problem 2 has a solution $u(\cdot, b)$ of the class $(H_1(D))^n$. Any two solutions of this problem differ by the rigid displacement vector $Lx + C$, where $L = \|L_{ij}\|_{n\times n}$, $L_{ij} = -L_{ji}$ and $C = (c_1, \dots, c_n)$.

Similarly to the previous problem, the following equalities are valid:

$$\begin{aligned} \forall v \in (H_1(D))^n \ : & B(u(\cdot,b), v, a) = (f(\cdot,b), v), \\ & B(u(\cdot, b+\Delta b), v, a+\Delta a) = (f(\cdot, b+\Delta b), v), \end{aligned}$$

where $b \in \mathcal{K}$ and $b + \Delta b \in \mathcal{K}$.

This implies that

$$\begin{aligned} \forall v \in (H_1(D))^n \ : & B(\Delta u(\cdot,b), v, a) = (\Delta f(\cdot,b), v) - \\ & - \int_D \Delta a_{ijek}(x) \varepsilon_{ij}(u(x, b+\Delta b))\varepsilon_{ek}(v)dx. \end{aligned} \tag{9}$$

If in (9) instead of v we write $\Delta u(\cdot, b)$, then by virtue of (3), the left-hand side of (9) will admit the estimate

$$B(\Delta u(\cdot,b), \Delta u(\cdot,b), a) \geq m \sum_{i,j=1}^{n} \int_D \varepsilon_{ij}^2(\Delta u(x,b))dx. \tag{10}$$

Let $\mathcal{R}$ be the set of all rigid displacement vectors and P be the operator of orthogonal projection of $(H_1(D))^n$ onto $\mathcal{R}$ in the sense of $H_0(D)$. Then, taking into consideration the conditions (8) (which are valid for the vector function Δf) as well, we obtain

$$(\Delta f(\cdot,b), \Delta u(\cdot,b)) = (\Delta f(\cdot,b), \Delta u(\cdot,b) - P\Delta u(\cdot,b)),$$

whence

$$|(\Delta f(\cdot,b), \Delta u(\cdot,b))| \leq \|\Delta f(\cdot,b)\|_0 \|\Delta u(\cdot,b) - P\Delta u(\cdot,b)\|_0. \tag{11}$$

Moreover, it is known (see [4]) that

$$\forall v \in (H_1(D))^n : \sum_{ij=1}^{n} \int_D \varepsilon_{ij}^2(v)dx \geq c\|v - Pv\|_0^2, \tag{12}$$

where C does not depend on v. Thus from (11) we find that

$$|(\Delta f(\cdot,b), \Delta u(\cdot,b))| \leq c_1 \|\Delta f(\cdot,b)\|_0 \left(\sum_{ij=1}^{n} \int_D \varepsilon_{ij}^2(\Delta u(x,b))dx \right)^{1/2}. \tag{13}$$

Estimate the last term in the right-hand side of (9):

$$\int_D \Delta a_{ijek}(x)\varepsilon_{ij}(u(x,b+\Delta b))\varepsilon_{ek}(\Delta u(x,b))dx \le$$

$$\le C_2\|\Delta a\|_{C(\overline{D};R^{n'})}\sum_{ij=1}^{n}\|\varepsilon_{ij}(u(\cdot,b+\Delta b))\|_0\left\{\sum_{e,k=1}^{n}\int_D \varepsilon_{ek}^2(\Delta u(\cdot,b))dx\right\}^{1/2}. \quad (14)$$

Since

$$B(u(\cdot,b+\Delta b),u(\cdot,b+\Delta b),a+\Delta a)=\big(f(\cdot,b+\Delta b),u(\cdot,b+\Delta b)-Pu(\cdot,b+\Delta b)\big),$$

from (12) we have

$$\sum_{ij=1}^{n}\int_D \varepsilon_{ij}^2(u(x,b+\Delta b))dx \le \frac{1}{m\sqrt{c}}\|f(\cdot,b+\Delta b)\|_0\left\{\sum_{ij=1}^{n}\int_D \varepsilon_{ij}^2(u(\cdot,b+\Delta b))dx\right\}^{1/2},$$

that is,

$$\sum_{ij=1}^{n}\|\varepsilon_{ij}(u(\cdot,b+\Delta b))\|_0 \le \frac{1}{m\sqrt{c}}\|f(\cdot,b+\Delta b)\|_0.$$

Hence from (14) we get

$$\int_D \Delta a_{ijek}(x)\varepsilon_{ij}(u(x,b+\Delta b))\varepsilon_{ek}(\Delta u(x,b))dx \le$$

$$\le m_0\|\Delta a\|_{C(\overline{D};\mathbb{R}^{n'})}\|f(\cdot,b+\Delta b)\|_0\left\{\sum_{ij=1}^{n}\int_D \varepsilon_{ij}^2(\Delta u(x,b))dx\right\}^{1/2}.$$

The latter together with (9) and with regard for (10) and (13) finally gives

$$\left\{\sum_{ij=1}^{n}\int_D \varepsilon_{ij}^2(\Delta u(x,b))dx\right\}^{1/2} \le m_0\big(\|\Delta f(\cdot,b)\|_0+\|\Delta a\|_{C(\overline{D};\mathbb{R}^{n'})}\|f(\cdot,b+\Delta b)\|_0\big),$$

where m_0 does not depend on Δb. That is, $\varepsilon_{ij}(\Delta u(\cdot,b))=\Delta\varepsilon_{ij}(u(\cdot,b))\to 0$ in the sense of H_0 as $\Delta b\to 0$ in the space $C(\overline{D};\mathbb{R}^m)$. Consequently, $\varphi_f\in C(\mathcal{K})$, and we can conclude that Theorem 1 is valid for the second boundary value problem as well.

Remark 2. Let f in the first boundary value problem be a homogeneous function of order α with respect to b. Then it can be easily shown that

$$\forall t>0\ :\ u(\cdot,tb)=t^{\alpha-1}u(\cdot,b),\quad \varphi_f(tb)=t^{2\alpha-1}\varphi_f(b).$$

Clearly

1) if $\alpha<\frac{1}{2}$, then $\lim\limits_{t\to\infty}\varphi_f(tb)=0,\quad \lim\limits_{t\to 0}\varphi_f(tb)=\infty;$

2) if $\alpha>\frac{1}{2}$, then $\lim\limits_{t\to\infty}\varphi_f(tb)=\infty,\quad \lim\limits_{t\to 0}\varphi_f(tb)=0;$

3) if $\alpha=\frac{1}{2}$, then $\varphi_f(tb)=\varphi(b)$.

Remark 3. Let f be independent of b. Then

$$\forall t > 0 \ : \ u(\cdot, tb) = \tfrac{1}{t} u(\cdot, b), \quad \varphi_f(tb) = \tfrac{1}{t}\varphi_f(b).$$

Analogous remarks are valid for the second boundary value problem.

1.2. Differentiability of the Total Deformation Energy with Respect to the Parameters ρ and a. Let the elastic constants a_{ijek} and the density ρ be independent of x, $x \in \overline{D}$, i.e., the elastic medium is homogeneous and anisotropic. Denote the above introduced cones K and $\mathcal{K}$ by K_0 and $\mathcal{K}_0$, respectively. Moreover, assume that $f \in C^1(\mathcal{K}_0; (H_0(D))^n)$.

Consider Problem 1. If $u(\cdot, b)$ is a solution of this problem, then

$$\forall v \in (\overset{0}{H}_1(D))^n \ : \ \int\limits_D a_{ijek}\varepsilon_{ij}(u(x,b))\varepsilon_{ek}(v)dx = (f, v).$$

Let $t' = (0, \dots, 0, \Delta t', 0, \dots, 0)$ be an m-dimensional vector ($\Delta t'$ is the $(s+1)$-component of the vector t'). Then

$$\forall v \in (\overset{0}{H}_1(D))^n : \int\limits_D (a_{ijek} + t'_{ijek})\varepsilon_{ij}(u(x, b+t'))\varepsilon_{ek}(v)dx = (f(\cdot, b+t'), v),$$

where the set of the numbers t'_{ijek} is obtained from t' (see the definition of the space $C(\overline{D}; \mathbb{R}^{n'})$).

From these two equalities we find that

$$\forall v \in (\overset{0}{H}_1(D))^n \ : \ \int\limits_D a_{ijek}\varepsilon_{ij}(\Delta' u(x,b)\varepsilon_{ek}(v)dx =$$

$$= (\Delta' f(\cdot, b), v) - \Delta t' \int\limits_D \varepsilon_{i_0 j_0}(u(x, b+t'))\varepsilon_{e_0 k_0}(v)dx. \tag{15}$$

Note that here the summation takes place only for those indices i_0, j_0, k_0, l_0 for which $t'_{ijek} = \Delta t'$. Similarly, for the vector $t'' = (0, \dots, 0, \Delta t'', 0, \dots, 0)$,

$$\forall v \in (\overset{0}{H}_1(D))^n \ : \ \int\limits_D a_{ijek}\varepsilon_{ij}(\Delta'' u(x,b)\varepsilon_{ek}(v)dx =$$

$$= (\Delta'' f(\cdot, b), v) - \Delta t'' \int\limits_D \varepsilon_{i_0 j_0}(u(x, b+t''))\varepsilon_{e_0 k_0}(v)dx. \tag{16}$$

It follows from (15) and (16) that

$$\int\limits_D a_{ijek}\varepsilon_{ij}\Big(\frac{\Delta'' u(x,b)}{\Delta t''} - \frac{\Delta' u(x,b)}{\Delta t'}\Big)\varepsilon_{ek}(v)dx = \Big(\frac{\Delta'' f(\cdot,b)}{\Delta t''} - \frac{\Delta' f(\cdot,b)}{\Delta t'}, v\Big) +$$

$$+ \int\limits_D \Big[\varepsilon_{i_0 j_0}(u(x, b+t'')) - \varepsilon_{i_0 j_0}(u(x, b+t'))\Big]\varepsilon_{e_0 k_0}(v)dx.$$

Substituting here $\frac{\Delta'' u}{\Delta t''} - \frac{\Delta' u}{\Delta t'}$ instead of v and taking into account (3), from Korn's first inequality we get

$$\sum_{ij=1}^{n} \left\| \frac{\Delta''\varepsilon_{ij}(u)}{\Delta t''} - \frac{\Delta'\varepsilon_{ij}(u)}{\Delta t'} \right\|_0 \le C\Big(\| \frac{\Delta'' f}{\Delta t''} - \frac{\Delta' f}{\Delta t'} \|_0 + $$
$$+ \sum_{ij=1}^{n} \|\varepsilon_{ij}(u(\cdot, b+t'')) - \varepsilon_{ij}(u(\cdot, b+t'))\|_0 \Big). \qquad (17)$$

From the above inequality it is clear that there exists

$$\lim_{\Delta t \to 0} \frac{\Delta \varepsilon_{ij}(u)}{\Delta t} = \frac{\partial \varepsilon_{ij}(u)}{\partial a_s}$$

in the sense of H_0.

We can also show that the function φ_f has all partial derivatives with respect to a_s, $s = 1, \dots, n'$ (the differentiability of φ_f with respect to ρ is easily proved).

Suppose now that $u(\cdot, b) \in (H_1(D))^n$ is a solution of the second boundary value problem of elasticity. Then, analogously to the previous case,

$$\forall v \in (H_1(D))^n \ : \int\limits_D a_{ijek} \varepsilon_{ij}(W) \varepsilon_{ek}(v) dx = (F, v) + $$
$$+ \int\limits_D \varepsilon_{i_0 j_0}(u(x, b+t'') - u(x, b+t') \varepsilon_{e_0 k_0}(v) dx,$$

where

$$W(\cdot, b) = \frac{\Delta'' u(\cdot, b)}{\Delta t''} - \frac{\Delta' u(\cdot, b)}{\Delta t'}, \quad F(\cdot, b) = \frac{\Delta'' f(\cdot, b)}{\Delta t''} - \frac{\Delta' f(\cdot, b)}{\Delta t'}$$

and the summation takes place only for those indices i_0, j_0, l_0, k_0 for which $t'_{i_0 j_0 e_0 k_0} = \Delta t'$ or $t'_{i_0 j_0 e_0 k_0} = \Delta t''$.

If in the latter inequality we substitute W instead of v and apply the operator of orthogonal projection P, then we obtain

$$m \sum_{ij=1}^{n} \int\limits_D \varepsilon_{ij}^2(W(x,b)) dx \le$$
$$\le (F, W - PW) + \int\limits_D \varepsilon_{i_0 j_0}(u(x, b+t'') - u(x, b+t')) \varepsilon_{p_0 k_0}(W) dx$$

which together with (12) yields

$$\sum_{ij=1}^{n} \left\| \frac{\Delta''\varepsilon_{ij}(u)}{\Delta t''} - \frac{\Delta'\varepsilon_{ij}(u)}{\Delta t'} \right\|_0 \le$$

$$\leq c\Big(\Big\|\frac{\Delta'' f}{\Delta t''}-\frac{\Delta' f}{\Delta t'}\Big\|_0+\sum_{ij=1}^{n}\|\varepsilon_{ij}(u(\cdot,b+t'')-u(\cdot,b+t'))\|_0\Big).$$

Consequently, $\varepsilon_{ij}(u)$ is differentiable in the sense of H_0, and the total deformation energy φ_t is differentiable with respect to b.

1.3. The Continuity of Green's Operator of the First and the Second Boundary Value Problems with Respect to the Parameter. Let $G^{(1)}(a){:}(H_0(D))^n \to (\overset{0}{H}_1(D))^n$ be Green's operator of the first boundary value problem for the corresponding medium (D,ρ,a) (from the definition of Green's operator it follows that it depends only on a).

Estimate the norm of the operator $G^{(1)}(a+\Delta a)-G^{(1)}(a)$, where a, $a+\Delta a\in K$.

Using the estimate (7), we can write

$$\forall v\in(H_0(D))^n:\big\|(G^{(1)}(a+\Delta a)-G^{(1)}(a))v\big\|_1=\|u(\cdot,a+\Delta a)-$$
$$-u(\cdot,a)\|_1=\|\Delta u(\cdot,a)\|_1\leq m_0\|\Delta a\|_{C(\overline{D},\mathbb{R}^{n'})}\|v\|_0,$$

where $u(\cdot,a)=G^{(1)}(a)v$, $\forall a\in K$. This implies that

$$\sup_{\substack{v\in(H_0(D))^n\\ v\neq 0}}\frac{\|(G^{(1)}(a+\Delta a)-G^{(1)}(a))v\|_1}{\|v\|_0}\leq m_0\|\Delta\|_{C(\overline{D};\mathbb{R}^{n'})},$$

that is, the operator $G^{(1)}$ depends continuously on a.

Consider the problem of differentiability of the operator $G^{(1)}$ with respect to the parameter a, $a\in K_0$.

Using Korn's first inequality, from (17) and (7) we obtain

$$\Big\|\frac{(\Delta'' G^{(1)})v}{\Delta t''}-\frac{(\Delta' G^{(1)})v}{\Delta t'}\Big\|_1\leq C\|u(\cdot,a+t'')-u(\cdot,a+t')\|_1\leq C\|t'-t''\|\|v\|_0.$$

It is clear that the operator $G^{(1)}$ is differentiable with respect to the parameter a. Moreover, we can show that

$$G^{(1)}\in C^\infty(K_0;L((H_0(D))^n,(\overset{0}{H}_1(D))^n)).$$

Let as before $\mathcal{R}=\{Lx+C\}$, where $L=\|L_{ij}\|_{n\times n}$, $L_{ij}=-L_{ji}$ and $C=(C_1,\dots,C_n)$.

By the definition,

$$\mathcal{R}^\perp=\{f\in(H_0(D))^n:(f,B)=0,\ \forall B\in\mathcal{R}\}.$$

Introduce a norm in the space $(H_1(D))^n$ as follows:

$$\forall u\in(H_1(D))^n:\|u\|_1^2=\sum_{ij=1}^{n}\int_D\varepsilon_{ij}^2(u)dx+\|u\|_0^2.$$

By virtue of Korn's second inequality, the above introduced norm is equivalent to the standard norm of the space $(H_1(D))^n$.

Consider the quotient space $\mathcal{H}(D)=(H_1(D))^n/\mathcal{R}$ and define the norm by

$$\forall \widetilde{u} \in \mathcal{H}(D) : \|\widetilde{u}\|_{\mathcal{H}} = \inf_{B\in\mathcal{R}} \|u+B\|_1,$$

i.e.,

$$\|\widetilde{u}\|^2_{\mathcal{H}} = \inf_{B\in\mathcal{R}} \left(\|u+B\|_0^2 + \sum_{ij=1}^{n} \int_D \varepsilon_{ij}^2(u+B)dx \right) = \inf_{B\in\mathcal{R}} \|u+B\|_0^2 + \varepsilon(\widetilde{u}),$$

where $\varepsilon(\widetilde{u}) = \varepsilon(u)$, $\forall u \in \widetilde{u}$.

Therefore

$$\|\widetilde{u}\|^2_{\mathcal{H}} = \|\widetilde{u} - P\widetilde{u}\|_0^2 + \varepsilon(\widetilde{u}),$$

where P is the operator of orthogonal projection of $(\mathcal{H}_1(D))^n$ onto $\mathcal{R}$ in the sense of H_0, and

$$\|\widetilde{u} - P\widetilde{u}\|_0 = \|u - Pu\|_0, \quad \forall u \in \widetilde{u}.$$

For $\widetilde{u}$, $\widetilde{v} \in \mathcal{H}$ we define

$$B(\widetilde{u}, \widetilde{v}) = B(u, v), \quad u \in \widetilde{u}, \quad v \in \widetilde{v}.$$

The following inequality is valid: there is $C_1 > 0$ such that $\forall \widetilde{u} \ni \mathcal{H}(D)$: $\varepsilon(\widetilde{u}) \geq C_1 \|\widetilde{u} - P\widetilde{u}\|_0^2$ (see [4]), which implies that

$$\exists C > 0 : B(\widetilde{u}, \widetilde{u}) \geq C \|\widetilde{u}^2_{\mathcal{H}}, \quad \forall \widetilde{u} \in \mathcal{H}(D).$$

Consider the functional equation

$$\forall \widetilde{v} \in \mathcal{H}(D) : B(\widetilde{u}, \widetilde{v}) = (f, \widetilde{v}), \quad f \in \mathcal{R}^{\perp}.$$

By Lax-Milgram's theorem, the functional equation has a unique solution $\widetilde{u}$ of the class $\mathcal{H}$, and

$$\|\widetilde{u}\|_{\mathcal{H}} \leq C_1^{-1} \|f\|_0.$$

Below we will denote the elements of the quotient space $\mathcal{H}$ by $u, v, w, \ldots$.

Define now Green's operator for the second boundary value problem.

Let (D, ρ, a) be an elastic medium. Consider the functional equation

$$\forall v \in \mathcal{H}(D) : B(u(\cdot, a), v, a) = (g, v),$$

where $g \in \mathcal{R}^{\perp}$ and $a \in K$.

Denote the unique solution $u(\cdot, a)$ of this equation by $G^{(2)}(a)g$, and consider the mapping

$$G^{(2)}(a) : \mathcal{R}^{\perp} \to \mathcal{H}(D)$$

defined by

$$\forall g \in \mathcal{R}^{\perp} : G^{(2)}(a)g = u(\cdot, a).$$

Show that $G^{(2)} \in C(K; L(\mathcal{R}^{\perp}; \mathcal{H}(D)))$. Indeed,

$$\frac{\|G^{(2)}(a+\Delta a)g - G^{(2)}(a)g\|_{\mathcal{H}}^2}{\|g\|_0^2} = \frac{\|u(\cdot, a+\Delta a) - u(\cdot, a)\|_{\mathcal{H}}^2}{\|g\|_0^2} =$$

$$= \frac{\|\Delta u(\cdot, a) - P\Delta u(\cdot, a)\|_0^2 + \varepsilon(\Delta u(\cdot, a))}{\|g\|_0^2} \leq \frac{C\varepsilon(|\Delta u(\cdot, a))}{\|g\|_0^2} \leq m_0^2 \|\Delta a\|_{C(\overline{D}; \mathbb{R}^{n'})},$$

where

$$\varepsilon(v) = \int_D \varepsilon_{ij}(v)\varepsilon_{ij}(v)dx.$$

Thus

$$\|G^{(2)}(a+\Delta a) - G^{(2)}(a)\| \leq m_0 \|\Delta a\|_{C(\overline{D}; \mathbb{R}^{n'})},$$

i.e.,

$$G^{(2)} \in C(K; L(\mathcal{R}^{\perp}; \mathcal{H}(D))).$$

Show now that the operator $G^{(2)}$ is differentiable with respect to the parameter a, $a \in K_0$.

Indeed,

$$\Big\| \frac{\Delta'' G^{(2)}(a)}{\Delta t''} - \frac{\Delta' G^{(2)}(a)}{\Delta t'} \Big\|_{L(\mathcal{R}^{\perp}; \mathcal{H})} =$$

$$= \sup_{g \in \mathcal{R}^{\perp}, g \neq 0} \frac{\big\| \big(\frac{\Delta'' G^{(2)}(a)}{\Delta t''} - \frac{\Delta' G^{(2)}(a)}{\Delta t'} \big) g \big\|_{\mathcal{H}}}{\|g\|_0} = \sup_{g \in \mathcal{R}^{\perp}, g \neq 0} \frac{\big\| \frac{\Delta'' u}{\Delta t''} - \frac{\Delta' u}{\Delta t'} \big\|_{\mathcal{H}}}{\|g\|_0} \leq$$

$$\leq \sup_{g \in \mathcal{R}^{\perp}, g \neq 0} \frac{C \big[\sum_{ij=1}^{n} \int_D \varepsilon_{ij}^2 \big(u(\cdot, a+t'') - u(\cdot, a+t') \big) dx \big]^{1/2}}{\|g\|_0} \leq C|t' - t''|.$$

From the above we can conclude that there exists

$$\lim_{\Delta t \to 0} \frac{\Delta G^{(2)}(a)}{\Delta t} = \frac{\partial G^{(2)}}{\partial a_s}, \quad s = 1, \ldots, n'.$$

We can easily show that $G^{(2)} \in C^{\infty}(K_0; L(\mathcal{R}^{\perp}; \mathcal{H}(D)))$.

From the definition of Green's operator

$$\forall t > 0, \quad \forall a \in K : G^{(2)}(at) = \frac{1}{t} G^{(2)}(a).$$

Moreover, if $f \in C^2(\mathcal{K}_0, \mathcal{R}^{\perp})$, then $\varphi_f(b) = (f(\cdot, b), G^{(2)}(a) f(\cdot, b))$ and

$$\frac{\partial \varphi_f(b)}{\partial a_i} = 2\Big(\frac{\partial f(\cdot, b)}{\partial a_i}, G^{(2)}(a) f(\cdot, b) \Big) + \Big(f(\cdot, b), \frac{\partial G^{(2)}(a)}{\partial a_i} f(\cdot, b) \Big);$$

$$\frac{\partial \varphi_f(b)}{\partial \rho} = 2\Big(\frac{\partial f(\cdot, b)}{\partial \rho}, G^{(2)}(a) f(\cdot, b) \Big),$$

$$\frac{\partial^2 \varphi_f(b)}{\partial a_i \partial a_j} = \Big(\frac{\partial^2 f(\cdot, b)}{\partial a_i \partial a_j}, G^{(2)}(a) f(\cdot, b) \Big) + 2\Big(\frac{\partial f(\cdot, b)}{\partial a_i}, G^{(2)}(a) \frac{\partial f(\cdot, b)}{\partial a_j} \Big) +$$

$$+2\Big(\frac{\partial f(\cdot,b)}{\partial a_i},\frac{\partial G^{(2)}(a)}{\partial a_j}f(\cdot,b)\Big)+2\Big(\frac{\partial f(\cdot,b)}{\partial a_j},\frac{\partial G^{(2)}(a)}{\partial a_i}f(\cdot,b)\Big)+$$
$$+\Big(f(\cdot,b),\frac{\partial^2 G^{(2)}(a)}{\partial a_i\partial a_j}f(\cdot,b)\Big),$$
$$\frac{\partial^2\varphi_f(b)}{\partial a_i\partial\rho}=\Big(\frac{\partial^2 f(\cdot,b)}{\partial a_i\partial\rho},G^{(2)}(a)f(\cdot,b)\Big)+2\Big(\frac{\partial f(\cdot,b)}{\partial a_i},G^{(2)}(a)\frac{\partial f(\cdot,b)}{\partial\rho}\Big).$$

1.4. About Equivalent Forces. Let $f,\ g\in C(\mathcal{K}_0;(H_0(D))^n)$ in the case of Problem 1 and $f,\ g\in C(\mathcal{K}_0;\mathcal{R}^\perp)$ in the case of Problem 2.

Definition 1. The vector functions f and g are said to be weakly equivalent if the sets of local minimum points of the functions $\varphi_f^{(i)}$ and $\varphi_g^{(i)}$, $i=1,2$, coincide.

Definition 2. The vector functions f and g are said to be equivalent on the compact $M\subset\mathcal{K}$ if for any $i\in\{1,2\}$, the sets of the minimum points of the functions $\varphi_f^{(i)}$ and $\varphi_g^{(i)}$, $i=1,2$, coincide.

Theorem 2. *If the functions $f,\ g\in C^1(\mathcal{K}_0;(H_0(D))^n)$ are homogeneous with respect to b of order $\alpha\neq\frac{1}{2}$ and $\beta\neq\frac{1}{2}$, respectively, and moreover*

$$\{b\in\mathcal{K}_0: f(\cdot,b)=0\}=\{b\in\mathcal{K}_0: g(\cdot,b)=0\},$$

then f and g are weakly equivalent vector functions.

Proof. Indeed, since $\varphi_f^{(1)}$ and $\varphi_g^{(1)}$ are differentiable, at a point of local minimum we have

$$\frac{\partial\varphi_f^{(1)}(b)}{\partial b_i}=0,\quad i=1,\ldots,m.$$

Moreover, using Euler's formula, we obtain

$$\sum_{i=1}^{m}\frac{\partial\varphi_f^{(1)}(b)}{\partial b_i}b_i=(2\alpha-1)\varphi_f(b)$$

because $\varphi_f^{(1)}$ is homogeneous of order $2\alpha-1$.

This implies that $\varphi_f^{(1)}(b)=0\Rightarrow f(\cdot,b)=0$. Thus, the set of the local minimum points of $\varphi_f^{(1)}$ coincides with the set of zeros of f. This means that the vector functions f and g are weakly equivalent. ■

Theorem 3. *If f,$g\in C^1(\mathcal{K}_0;\mathcal{R}^\perp)$ are homogeneous vector functions with respect to b of order $\alpha\neq\frac{1}{2}$ and $\beta\neq 12$, respectively, and moreover*

$$\{b\in\mathcal{K}_0: f(\cdot,b)=0\}=\{b\in\mathcal{K}_0: g(\cdot,b)=0\},$$

then f and g are weakly equivalent vector functions.

1.5. On the Equivalence of Radial Forces in the Case of a Sphere for an Isotropic Homogeneous Elastic Medium. Let $B(0,r) = \{x \in \mathbb{R}^n; |x| < r\}$ and $f : B(0,r) \to \mathbb{R}^n$.

Definition 3. We call f a radial vector function (radial force) if

$$\forall Q \in O(n) \quad \text{and} \quad \forall x \in B(0,r) : f(Qx) = Qf(x),$$

where $O(n)$ is the group of orthogonal matrices.

Consider the differential system of equations of statics of the theory of elasticity for an anisotropic homogeneous elastic medium:

$$\forall x \in B(0,r) : a_{ijek} \frac{\partial^2 u_k(x)}{\partial x_i \partial x_e} = f_j(x). \tag{18}$$

It is known that if u is a solution of the system (18) and $Q \in O(n)$, then the vector function $v(x) = Q'u(Qx)$ is a solution of the system

$$\forall x \in B(0,r) : a'_{smhp} \frac{\partial^2 v_p}{\partial x_s \partial x_h} = \alpha_{pm} f_p(Qx),$$

where $Q = \|\alpha_{ij}\|_{n\times n}$, Q' is the transpose to Q and

$$a'_{smhp} = a_{ijek}\alpha_{is}\alpha_{jm}\alpha_{eh}\alpha_{kp}.$$

If the medium is isotropic and homogeneous, that is, if

$$a_{ijek} = \lambda\delta_{ij}\delta_{ek} + \mu(\delta_{ie}\delta_{jk} + \delta_{ik}\delta_{je}),$$

then $a'_{smhp} = a_{smhp}$.
Note that λ and μ satisfy

$$n\lambda + 2\mu > 0, \quad \mu > 0.$$

Consider the first boundary value problem for a sphere:

$$\forall x \in B(0,r) : \mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u(x) + f(x) = 0;$$
$$\lim_{|x|\to r} u(x) = 0. \tag{19}$$

Theorem 4. *If $u : B(0,r) \to \mathbb{R}^n$ is a solution of the problem* (19) *for the radial vector function $f \in (H_0(D))^n$, then u is a radial vector function.*

Note that the general form of the radial vector function is

$$\forall x \in B(0,r) : f(x) = \rho x g(|x|).$$

Consequently, if $f(x) = xg(|x|)$ is a solid force, then by Theorem 4, $u(x) = xv(|x|)$ and it satisfies the ordinary differential equation

$$\forall s \in (0;r) : \frac{d^2 v(s)}{ds^2} + \frac{n+1}{s}\frac{dv(s)}{ds} = \frac{\rho}{\lambda + 2\mu} g(s) \tag{20}$$

with the boundary conditions

$$v(r) = 0 \quad \text{and} \quad \lim_{s \to 0} \frac{dv(s)}{ds} = 0. \tag{21}$$

After elementary calculations, we obtain that if v is a solution of the problem (20), (21), then

$$B(u,u) = \frac{\omega_n}{\lambda + 2\mu} \int_0^r s^{n+1} \psi^2(s) ds,$$

where ω_n is the volume of the unit sphere, and

$$\psi(s) = -\frac{1}{s^{n+1}} \int_0^s g(t) t^{n+1} dt.$$

Hence the total energy of deformation is of the form

$$\varphi_f(\rho, \lambda, \mu) = \frac{c\rho^2}{\lambda + 2\mu}.$$

It is easily seen that given a compact subset M of the cone $K_1 = \{(\rho, \lambda, \mu) : \rho > 0,\ n\lambda + 2\mu > 0,\ \mu > 0\}$, then for any two radial forces f_1 and f_2 of the form $f_i = \rho x g_i(|x|)$, $i = 1, 2$, the set of minimum points of the functions φ_{f_1} and φ_{f_2} coincide, i.e., the radial forces are equivalent on every compact M, $M \subset K_1$.

Let G be an arbitrary compact in the cone $\{\lambda + 2\mu > 0,\ \mu > 0\}$ and $M = [\rho_0, \rho_1] \times G$. Then the minimum points of the function $\varphi_f(\rho, \lambda, \mu)$ are such $(\rho_0, \lambda_0, \mu_0)$ for which (λ_0, μ_0) satisfy the equation $\lambda + 2\mu = t_0$, where $t_0 = \max\{t; \{(\lambda, \mu) : \lambda + 2\mu = t\} \cap G \neq \varnothing\}$.

2. On an Isoperimetric Problem in the Elasticity Theory

In this section, we study the following isoperimetric problem: from the set of elastic bodies of a given volume find the one for which the first fundamental frequency of the first boundary value problem is minimal. In mathematical physics, many works have been devoted to isoperimetric problems. Among them we should point out [5].

As above, let $\mathbb{R}^n (n \geq 2)$ be the Euclidean n-dimensional space. A subset T of the space $\mathbb{R}^n$ is said to be convex if together with any two of its points, it contains the linear segment connecting them. A closed convex bounded set containing interior points is called a convex n-dimensional body.

Let T be a closed convex set. Consider the mapping $d(\cdot, T) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\forall x \in \mathbb{R}^n : d(x, T) = \inf_{y \in T} \|x - y\|_{R^n}.$$

We can easily see that $d(\cdot, T)$ is a convex function satisfying the Lipschitz condition (with the Lipschitz constant 1).

Denote by $\mathcal{A}(\mathbb{R}^n)$ the set of convex bodies in the space $\mathbb{R}^n$ and define the mapping $\Omega : \mathcal{A}(\mathbb{R}^n) \to \mathcal{L}ip(\mathbb{R}^n)$ which with $\forall T \in \mathcal{A}(\mathbb{R}^n)$ associates the function $\Omega_T(x)$ defined by

$$\forall x \in \mathbb{R}^n : \Omega_T(x) = d(x, T).$$

As is easily seen, this mapping is injective. Indeed, if $\Omega_{T_1} = \Omega_{T_2}$, then $\forall x \in \mathbb{R}^n : d(x, T_1) = d(x, T_2)$. Let $x \in T_1$. Then $d(x, T_1) = 0$, and therefore $d(x, T_2) = 0$, i.e., $x \in T_2$. Hence $T_1 \subset T_2$. The contrary can be proved analogously, and therefore $T_1 = T_2$.

Define $\forall T_1 \in \mathcal{A}(\mathbb{R}^n)$ and $\forall T_2 \in \mathcal{A}(\mathbb{R}^n)$: $\rho(T_1, T_2) = \sup_{x\in\mathbb{R}^n} |\Omega_{T_1}(x) - \Omega_{T_2}(x)|$.

Clearly, ρ is a metric (well-known as the Hausdorff distance). Denote by $\mathcal{A}(\mathbb{R}^n; \rho)$ the metric space defined by the above-introduced metric.

It is easy to show that

$$\rho(T_1, T_2) = \max\{\sup_{x\in T_1} d(x, T_2); \sup_{x\in T_2} d(x, T_1)\}.$$

The metric space $\mathcal{A}(\mathbb{R}^n; \rho)$ is complete (see [6]).

Let μ be a Lebesgue measure in $\mathbb{R}^n$. To every $T \in \mathcal{A}(\mathbb{R}^n; \rho)$, $\mu(T) > 0$, we put into correspondence a Hilbert space $H(T)$ so that the following conditions are fulfilled:

if $T_1 \subset T_2$, then there exists $I_{T_1T_2} \in L(H(T_1); H(T_2))$ such that

$$\|I_{T_1T_2}u\|_{H(T_2)} = \|u\|_{H(T_1)}, \quad \forall u \in H(T_1).$$

Introduce the continuous bilinear forms

$$\forall T \in \mathcal{A}(\mathbb{R}^n; \rho), \quad B_j(T) : H(T) \times H(T) \to \mathbb{C}, \quad j = 1, 2,$$

and the functional

$$\forall u \in H(T) : F(T)(u) = \frac{\operatorname{Re} B_1(T)(u, u)}{\operatorname{Re} B_2(T)(u, u)},$$

where $\mathbb{C}$ is the set of complex numbers.

Suppose that the bilinear forms satisfy the following conditions:

1. $\forall T \in \mathcal{A}(\mathbb{R}^n; \rho) : B_1(T)$ is coercive, i.e., there exists $M_T > 0$ such that $\forall u \in H(T)$: $\operatorname{Re} B_1(T)(u, u) \geq M_T \|u\|^2$;

2. $\forall u \in H(T)\backslash 0 : \operatorname{Re} B_2(T)(u, u) > 0$;

3. $\forall (u, v) \in H(T_2) \times H(T_2) : B_j(T_2)(u, v) = B_j(T_1)(I_{T_2T_1}u, I_{T_2T_1}v)$, if $T_2 \subset T_1$;

4. Let $\mathcal{H}_0^k$ be the homothety with the center 0 and the coefficient k, and let 0 be an interior point of the convex body T, $\mu(T) > 0$. There exists an isomorphism J_0^k of the space $H(T)$ onto the space $H(\mathcal{H}_0^k(T))$ such that

$$B_j(\mathcal{H}_0^k(T))(J_0^k u, J_0^k v) = k^{\alpha_j} B_j(T)(u, v), \quad \alpha_1 \leq \alpha_2, \quad \forall (u, v) \in H(T) \times H(T).$$

Definition 4.

$$\forall T \in \mathcal{A}(\mathbb{R}^n;\rho) : \Lambda(T) = \inf_{u\in H(T)\setminus\{0\}} F(T)(u). \tag{22}$$

Establish some properties of $\Lambda(T)$.

(I). If $T_2 \subset T_1$, then $\Lambda(T_1) \le \Lambda(T_2)$.

Indeed,

$$\Lambda(T_2) = \inf_{\substack{u\in H(T_2)\\ u\neq 0}} \frac{\operatorname{Re} B_1(T_2)(u,u)}{\operatorname{Re} B_2(T_2)(u,u)} = \inf_{\substack{u\in H(T_2)\\ u\neq 0}} \frac{\operatorname{Re} B_1(T_1)(I_{T_2T_1}u, I_{T_2T_1}u)}{(\operatorname{Re} B_2(T_1)(I_{T_2T_1}u, I_{T_2T_1}u)} \ge$$

$$\ge \inf_{\substack{v\in H(T_1)\\ v\neq 0}} \frac{\operatorname{Re} B_1(T_1)(v,v)}{\operatorname{Re} B_2(T_1)(v,v)} = \Lambda(T_1).$$

(II). $\forall T \in \mathcal{A}(\mathbb{R}^n;\rho) : \Lambda(\mathcal{H}_0^k(T)) = k^{\alpha_1-\alpha_2}\Lambda(T)$.

Taking into account the fourth property of the bilinear forms, we obtain

$$\Lambda(\mathcal{H}_0^k(T)) = \inf_{\substack{u\in H(T)\\ u\neq 0}} \frac{\operatorname{Re} B_1(\mathcal{H}_0^k(T))(J_0^k u, J_0^k u)}{\operatorname{Re} B_2(\mathcal{H}_0^k(T)(J_0^k u, J_0^k u)} =$$

$$= k^{\alpha_1-\alpha_2} \inf_{u\in H(T)\setminus\{0\}} \frac{\operatorname{Re} B_1(T)(u,u)}{\operatorname{Re} B_2(T)(u,u)} = k^{\alpha_1-\alpha_2}\Lambda(T).$$

Note that if $\alpha_1 = \alpha_2$, then $\forall T_j \in \mathcal{A}(\mathbb{R}^n;\rho)$:

$$\mu(T_j) > 0, \quad j = 1,2 : \Lambda(T_1) = \Lambda(T_2).$$

(III). If $T_0 \in \mathcal{A}(\mathbb{R}^n;\rho)$, $\mu(T_0) > 0$ and

$$\lim_{p\to\infty} T_p = T_0, \quad T_p \in \mathcal{A}(\mathbb{R}^n;\rho), \quad \text{then} \quad \lim_{p\to\infty} \Lambda(T_p) = \Lambda(T_0).$$

By virtue of (II), $\forall\varepsilon > 0$ there exist numbers $k_1 > 1$ and $0 < k_2 < 1$ such that

$$|\Lambda(\mathcal{H}_0^{k_j}(T_0)) - \Lambda(T_0)| < \varepsilon/2, \quad j = 1,2,$$

whence

$$|\Lambda(\mathcal{H}_0^{k_2}(T_0)) - \Lambda(\mathcal{H}_0^{k_1}(T_0))| < \varepsilon.$$

Let $\delta = \min\{d(\partial\mathcal{H}_0^{k_1}(T_0), \partial T_0),\ d(\partial\mathcal{H}_0^{k_2}(T_0), \partial T_0)\}$.

For $\delta > 0$, there exists p_0 (natural) such that $\rho(T_p, T_0) < \frac{\delta}{3}$, $\forall p \ge p_0$. Denote $B(T_0;\frac{\delta}{2}) = \{x \in \mathbb{R}^n : d(x,T_0) < \frac{\delta}{2}\}$. Then $\forall p \ge p_0 : T_p \subset B(T_0;\frac{\delta}{2}) \subset \mathcal{H}_0^{k_1}(T_0)$. Therefore $\forall p \ge p_0 : \Lambda(T_p) \ge \Lambda(\mathcal{H}_0^{k_1}(T_0))$.

Let $\mathbb{R}^n\setminus\mathring{T}_0 = C\mathring{T}_0$ and consider the set

$$B(C\mathring{T}_0;\delta/2) = \{x \in \mathbb{R}^n : d(x, C\mathring{T}_0) < \delta/2\}.$$

Since $\partial T_p \subset B(C\mathring{T}_0 T_0;\frac{\delta}{2})$, we have $T_p \supset B(C\mathring{T}_0;\frac{\delta}{2}) \supset \mathcal{H}_0^{k_2}(T_0)$, $\forall p > p_0$, i.e.,

$$\Lambda(T_0) \le \Lambda(\mathcal{H}_0^{k_2}(T_0)).$$

Hence we finally get

$$\Lambda(\mathcal{H}_0^{k_1}(T_0)) \leq \Lambda(T_p) \leq \Lambda(\mathcal{H}_0^{k_2}(T_0)), \quad \forall p \geq p_0.$$

Clearly the above inequality is valid for $\Lambda(T_0)$ as well. Therefore

$$|\Lambda(T_p) - \Lambda(T_0)| < \varepsilon, \quad \forall p \geq p_0.$$

Thus the property (III) is proved.

Suppose $\forall (u,v) \in (\overset{\circ}{H}_1(\overset{\circ}{T}))^n \times (\overset{\circ}{H}_1(\overset{\circ}{T}))^n$:

$$B_1(T)(u,v) = \int\limits_{\overset{\circ}{T}} a_{ijek}\varepsilon_{ij}(u)\varepsilon_{ek}(v)dx, \quad B_2(T)(u,v) = \int\limits_{\overset{\circ}{T}} u \cdot v dx.$$

Let us prove that $\Lambda(T)$ defined by the formula (22) is the first fundamental frequency for the first boundary value oscillation problem of the theory of elasticity.

Consider the following boundary value problem: find a vector function $u : \overset{\circ}{T} \to \mathbb{R}^n$ satisfying

$$\forall x \in \overset{\circ}{T} : A(\partial x)u(x) + \omega^2 u(x) = 0, \quad \forall x \in \partial \overset{\circ}{T} : \lim_{\overset{\circ}{T} \ni z \to x} u(z) = 0,$$

where

$$A(\partial x) = \|a_{ijek} \frac{\partial^2}{\partial x_i \partial x_e}\|_{n \times n}.$$

It is known that this problem has a countable set of fundamental frequencies $\{\omega_1^2, \ldots, \omega_p^2, \ldots\}$ such that $\omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_p^2 \leq \ldots$ (see [1], [2]).

Let $\varphi_1, \varphi_2, \ldots, \varphi_p, \ldots$ be the corresponding sequence of fundamental vector functions which form a complete orthonormal system in $(H_0(T))^n$.

The Fourier expansion of u and $A(\partial x)u$ in terms of eigenfunctions shows that

$$\int\limits_{\overset{\circ}{T}} a_{ijek}\varepsilon_{ij}(u)\varepsilon_{ek}(u)dx = \sum_{m=1}^{\infty} \omega_m^2 C_m^2,$$

where

$$C_m = \int\limits_{\overset{\circ}{T}} u \cdot \varphi_m dx.$$

Since

$$\frac{B_1(T)(u,u)}{B_2(T)(u,u)} = \frac{\sum_{m=1}^{\infty} \omega_m^2 c_m^2}{\sum_{m=1}^{\infty} C_m^2} \geq \omega_1^2,$$

we obtain

$$\Lambda(T) \geq \omega_1^2.$$

Taking into account the fact that in the above inequality the equality occurs for $u = \varphi_1$, we have

$$\Lambda(T) = \omega_1^2.$$

To obtain the final result, we have to estimate $\Lambda(T)$ as $\operatorname{diam} T \to \infty$.

Lemma 1. *Let $T \in \mathcal{A}(\mathbb{R}^n;\rho)$, $\mu(T) > 0$ and let E be the ellipsoid of minimal volume with the center O, containing the convex body T. Then $\mathcal{H}_0^{1/n}(E) \subset T$ (see [7]).*

Lemma 2. *Let E be an ellipsoid with the major axis $2a$. Then there exists a parallelepiped Π, $E \subset \Pi$, whose one edge is $2a$ and*

$$\mu(\Pi) \le C(n)\mu(E).$$

From these lemmas it follows that $\forall T \in \mathcal{A}(\mathbb{R}^n;\rho)$ there exists a parallelepiped Π, $T \subset \Pi$, such that $\frac{2a}{n} \le \operatorname{diam} T \le 2a$, where $2a$ is the major edge of Π and

$$\mu(\Pi) \le n^n C(n)\mu(T). \tag{23}$$

Note that the function $\Lambda : \mathcal{A}(\mathbb{R}^n;\rho) \to \mathbb{R}$ is invariant with respect to a parallel translation. Therefore we may suppose that the middle point of a diameter of the convex body T coincides with the origin of coordinates. Consider a new coordinate system $(y_1,\dots,y_n)$ in which the coordinate axes are parallel to the edges of the parallelepiped Π and the Oy-axis is parallel to the edge equal to $2a$.

Let $Q = \|\alpha_{ij}\|_{n\times n}$ be the corresponding orthogonal transformation and let in the new coordinate system the parallelepiped Π have the form

$$\Pi = \{y \in \mathbb{R}^n : a_i \le y_i \le b_i,\ i = 1,\dots,n\}, \quad b_1 - a_1 = 2a.$$

It can be easily shown that under such a transformation the coefficients of the equation are transformed by the formulas

$$a'_{mpsh} = a_{ijek}\alpha_{im}\alpha_{jp}\alpha_{es}\alpha_{kh}.$$

If we denote $Q'u(Qy) = v(y)$, then

$$\Lambda(\Pi) = \inf_{\substack{v \in \overset{\circ}{H}_1(\overset{\circ}{\Pi}) \\ v \ne 0}} \frac{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} a'_{mpsh}\varepsilon_{mp}(v)\varepsilon_{sh}(v)dy}{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} |v(y)|^2 dy}.$$

The condition (3) and Korn's first inequality result in

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} a'_{mpsh}\varepsilon_{mp}(v)\varepsilon_{sh}(v)dy \ge C_1 \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \frac{\partial v_e}{\partial y_j}\frac{\partial v_e}{\partial y_j}dy,$$

where C_1 does not depend on $a_1, b_1 \dots, a_n, b_n$.

By virtue of (23), $2a(b_2-a_2)\dots(b_n-a_n) < C(n)n^n\mu(T)$. This implies that the estimate

$$b_{j_0} - a_{j_0} \le \sqrt[n-1]{\frac{c(n)n^n\mu(T)}{2a}}$$

holds for some j_0, $1 < j_0 \le n$.

From the obvious identity

$$v(y)=\int\limits_{a_{j_0}}^{y_{j_0}}\frac{\partial v}{\partial z_{j_0}}dz_{j_0},\quad \forall y\in\Pi,$$

which is valid for every $v\in C_0^\infty(\overset{\circ}{\Pi})$, we have

$$|v(y)|^2\le (b_{j_0}-a_{j_0})\int\limits_{a_{j_0}}^{b_{j_0}}\Big|\frac{\partial v(y)}{\partial y_{j_0}}\Big|^2 dy_{j_0};$$

$$\int\limits_{a_1}^{b_1}\cdots\int\limits_{a_n}^{b_n}|v(y)|^2dy\le (b_{j_0}-a_{j_0})^2\int\limits_{a_1}^{b_1}\cdots\int\limits_{a_n}^{b_n}\Big|\frac{\partial v}{\partial y_{j_0}}\Big|^2dy\le (b_{j_0}-a_{j_0})^2\int\limits_{a_1}^{b_1}\cdots\int\limits_{a_n}^{b_n}\frac{\partial v_e}{\partial y_j}\frac{\partial v_e}{\partial y_j}dy.$$

Thus

$$\forall v\in C_0^\infty(\Pi):\frac{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}a'_{mpsh}\varepsilon_{mp}(v)\varepsilon_{sh}(v)dy}{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}|v(y)|^2dy}\ge$$

$$\ge\frac{c}{(b_{j_0}-a_{j_0})^2}\ge C\sqrt[n-1]{\frac{4a^2}{n^{2n}C(n)\mu^2(T)}},$$

and consequently

$$\Lambda(T)\ge C\sqrt[n-1]{\frac{4R^2}{n^{2n}C(n)\mu^2(T)}},\tag{24}$$

where $2R=\operatorname{diam}T$.

From the above we can see that $\Lambda(T)\to\infty$ as $\operatorname{diam}T\to\infty$.

Theorem 5. *Let $\mathcal{F}=\{T\in\mathcal{A}(\mathbb{R}^n;\rho),\ \ \mu(T)=V\}$, $V>0$. Then there exists $T_0\in\mathcal{F}$ such that*

$$\Lambda(T_0)=\inf_{T\in\mathcal{F}}\Lambda(T).$$

Proof. Consider the sphere $\overline{B}(0,r)$ with $\mu(\overline{B}(0,r))>V$ and let

$$M=\max_{T\subset\overline{B}(0,r)\cap\mathcal{F}}\Lambda(T).\tag{25}$$

Since Λ is continuous at every point $T\in\mathcal{A}(\mathbb{R}^n;\rho)$, $\mu(T)>0$, and the set $\{T\in\mathcal{A}(\mathbb{R}^n;\rho)\}$ due to Blaschke's theorem (see [8]) is compact, there exists M satisfying (25).

It follows from (25) that there exists $R_0>0$ such that $\Lambda(T)>M$, if only $R\ge R_0$ and $\operatorname{diam}T=2R$.

Consider the sphere $\overline{B}(0;2R_0)$. Then for $\forall T\in\mathcal{F}$, with $\operatorname{diam}T<2R_0$, there exists $T_1\in\mathcal{F}$ congruent to T and obtained by a parallel tranlation such that the middle point of a diameter of T_1 coincides with the origin coordinates, and $T_1 \subset \overline{B}(0; 2R_0)$. Then there exists $T_0 \in \mathcal{F}$ such that $T_0 \subset \overline{B}(0; 2R_0)$ and

$$\Lambda(T_0) = \inf_{T \in \mathcal{F}} \Lambda(T). \quad \blacksquare$$

Consider the mapping $J : \mathcal{A}(\mathbb{R}^n; \rho) \to \mathcal{A}(\mathbb{R}^n; \rho)$, where for $\forall T \in \mathcal{A}(\mathbb{R}^n; \rho)$, $J(T)$ is the convex body obtained from T by the parallel translation and whose middle point of a diameter coincides with the origin of coordinates. We have the following

Theorem 6. *Let $\mathcal{F} \in \mathcal{A}(\mathbb{R}^n; \rho)$, $\forall T \in \mathcal{F}$: $\mu(T) = V$, and let $J(\mathcal{F})$ be a closed set. Then there exists $T_0 \in \mathcal{F}$ such that*

$$\Lambda(T_0) = \inf_{T \in \mathcal{F}} \Lambda(T).$$

Denote by $\Lambda(T, a)$ the fundamental frequency of the first boundary value problem for the elastic body $(T; \rho; a)$ and show that it is continuous with respect to the parameter a. The use will be made of the well-known theorem due to Weyl and Courant (see [9]): if A_1 and A_2 are completely symmetric operators in a Hilbert space and if $A = A_1 + A_2$, then the eigenvalues of the operators A_1 and A with the same numbers differ not more than by $\|A_2\|$.

Let $\widetilde{G}^{(1)}(a) : (H_0(D))^n \to (H_0(D))^n$, $a \in \mathcal{K}$, where $\widetilde{G}^{(1)}(a) = IG^{(1)}(a)$ and I is the embedding operator, $(\overset{\circ}{H}_1(D))^n \to (H_0(D))^n$. Then $\widetilde{G}^{(1)}(a)$ is completely continuous and the eigenvalues of the operator $\widetilde{G}^{(1)}(a)$ coincide with the inverse values of the corresponding fundamental frequencies of the first boundary value problem. It follows from the continuity of the operator $\widetilde{G}^{(1)}$ and from the above -mentioned theorem that all fundamental frequencies of the first boundary value problem are continuous. Analogous conclusion is valid for the second boundary value problem.

Theorem 7. $\Lambda(T; a)$ *is continuous at every point* $(T_0; a^0) \in \mathcal{A}(\mathbb{R}^n; \rho) \times K_0$, $\mu(T_0) > 0$.

Proof. Let $0 \in \overset{\circ}{T}_0$, $k > 1$ and suppose that $\mathcal{H}_0^k(T_0) = T_2$, $\mathcal{H}_0^{1/k}(T_0) = T_1$.

Then, owing to the property (II), $\forall a \in K_0$:

$$\Lambda(T_1; a) = k^{\alpha} \Lambda(T_0; a) \quad \text{and} \quad \Lambda(T_2, a) = k^{-\alpha} \Lambda(T_0; a).$$

Therefore $\forall T$, $T_1 \subset T \subset T_2$ and $\forall a \in K_0$:

$$k^{-\alpha} \Lambda(T_0; a) \le \Lambda(T; a) \le k^{\alpha} \Lambda(T_0; a). \tag{26}$$

In particular, we have

$$k^{-\alpha} \Lambda(T_0; a^0) \le \Lambda(T_0; a^0) \le k^{\alpha} \Lambda(T_0; a^0). \tag{27}$$

Since $\Lambda(T_0; a)$ is continuous at the point $\overset{\circ}{a} \in K_0$, for $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $\Lambda(T_0; a^0) - \varepsilon < \Lambda(T_0; a) < \Lambda(T_0; a^0) + \varepsilon$, if only $\|a - a^0\|_{R^{n'}} < \delta$.

From the above arguments and from the formulas (26) and (27) we obtain

$$k^{-\alpha}\Lambda(T_0;a^0)-k^{-\alpha}\varepsilon\le\Lambda(T;a)\le k^{\alpha}\Lambda(T_0;a^0)+k^{\alpha}\varepsilon,$$
$$k^{-\alpha}\Lambda(T_0;a)-k^{-\alpha}\varepsilon\le\Lambda(T_0;a^0)\le k^{\alpha}\Lambda(T_0;a^0)+k^{\alpha}\varepsilon.$$

Estimating the difference

$$k^{\alpha}\Lambda(T_0;a^0)+k^{\alpha}\varepsilon-k^{-\alpha}\Lambda(T_0;a^0)+k^{-\alpha}\varepsilon=\Big(k^{\alpha}-\frac{1}{k^{\alpha}}\Big)\Lambda(T_0;a^0)+\Big(k^{\alpha}+\frac{1}{k^{\alpha}}\Big)\varepsilon,$$

we can conclude that it may become arbitrarily small as $k\to 1$ and ε is sufficiently small. ■

Theorem 8. *Let M be a compactum from K_0 and let $\mathcal{F}=\{T\in\mathcal{A}(\mathbb{R}^n;\rho):\mu(T)=V\}$. Then there exists $(T_0;a^0)\in\mathcal{F}\times M$ such that*

$$\Lambda(T_0;a^0)=\inf_{(T,a)\in\mathcal{F}\times M}\Lambda(T;a).$$

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