Mem. Differential Equations Math. Phys. 15(1998), 157–159

S. A. Mazanik

LAPPO-DANILEVSKIĬ SYSTEMS AND THEIR PLACE AMONG LINEAR SYSTEMS

(Reported on June, 29 1998)

Consider the linear system

$$Dx = A(t)x, \quad x \in \mathbb{R}^n \quad t \geq 0, \quad D = d/dt, \tag{1_A}$$

where $A(t)$ is an $n \times n$ matrix of real-valued continuous and bounded functions of the real variable t on the non-negative half-line. We say that

i) $A(t)$ is a right Lappo-Danilevskiĭ matrix ($A \in LD_r(s)$) if there exists s, $s \geq 0$, such that for all $t \geq s$

$$A(t)\int_s^t A(u)du = \int_s^t A(u)du A(t); \tag{2}$$

ii) $A(t)$ is a left Lappo-Danilevskiĭ matrix ($A \in LD_l(s)$) if there exists s, $s > 0$, such that (2) is fulfilled for all $0 \leq t \leq s$;

iii) $A(t)$ is a bilateral Lappo-Danilevskiĭ matrix ($A \in LD_b(s)$) if there exists s, $s \geq 0$, such that (2) is fulfilled for all $t \geq 0$.

The corresponding systems (1_A) are called right, left or bilateral Lappo-Danilevskiĭ systems (cf. [1, p. 117]). In this paper we present some results on the distribution of the Lappo-Danilevskiĭ systems among linear systems.

Let $\rho(A, B) = \sup_{t \geq 0} \|A(t) - B(t)\|$, where $\|.\|$ be an arbitrary matrix norm, and let $LD_r = \bigcup\limits_{s \geq 0} LD_r(s)$, $LD_l = \bigcup\limits_{s > 0} LD_l(s)$, $LD_b = \bigcup\limits_{s \geq 0} LD_b(s)$. Let, for simplicity, $n = 2$.

Theorem 1. *Among linear differential systems there is a linear system* (1_A) *such that for some* $\varepsilon > 0$ *the system* (1_{A+Q}) *is neither a bilateral nor a right Lappo-Danilevskiĭ system for any matrix* Q *such that* $\rho(A, A+Q) \leq \varepsilon$.

Theorem 2. *Among linear differential systems there is a linear system* (1_A) *such that for any* $s > 0$ *there exists* $\varepsilon > 0$ *such that the matrix* $A+Q \notin LD_l(s)$ *for any matrix* Q *such that* $\rho(A, A+Q) \leq \varepsilon$.

To prove these theorems it is sufficient to consider the matrix $A(t) = (a_{ij}(t))$, $i, j = 1, 2$, where $a_{11}(t) = \sin \ln (t+1)$, $a_{12}(t) = 1$, $a_{21}(t) = \exp(-t)$, $a_{22}(t) = \cos \ln (t+1)$. (Let the symbol $[.,.]$ be used to indicate the Lie brackets, and let $[.,.]_{ij}$ be (i,j)-element of the matrix $[\cdot]$.) We have

$$[A(t)+Q(t), \int_s^t (A(u)+Q(u))du] = [A(t), \int_s^t A(u)du] + [A(t), \int_s^t Q(u)du] +$$
$$+[Q(t), \int_s^t A(u)du] + [Q(t), \int_s^t Q(u)du].$$

1991 *Mathematics Subject Classification.* 34A30.

Key words and phrases. Linear differential systems, Lappo–Danilevskiĭ systems.

It is easy to verify that if $\rho(A, A+Q) \leq \varepsilon$, then for all $t \geq 0$, $s \geq 0$, and for all sufficiently small ε we have:

$$|[A(t), \int_s^t Q(u)du]_{11}| \leq 4\varepsilon|t-s|, \quad |[A(t), \int_s^t Q(u)du]_{12}| \leq 4\varepsilon|t-s|,$$
$$|[Q(t), \int_s^t A(u)du]_{11}| \leq 4\varepsilon|t-s|, \quad |[Q(t), \int_s^t A(u)du]_{12}| \leq 4\varepsilon|t-s|,$$
$$|[Q(t), \int_s^t Q(u)du]_{11}| \leq 4\varepsilon|t-s|, \quad |[Q(t), \int_s^t Q(u)du]_{12}| \leq 4\varepsilon|t-s|.$$

Therefore, $\forall t \geq 0, s \geq 0$ we have

$$F_{12}(t,s) = |[A(t)+Q(t), \int_s^t (A(u)+Q(u))du]_{12}| \geq |[A(t), \int_s^t A(u)du]_{12}| - 12\varepsilon|t-s|.$$

Set $t_k = \exp(\pi/2 + 2k\pi) - 1$, $k \in \mathbb{N}$. It follows that $F_{12}(t_k, s) \geq |t_k - s - (s+1)\cos\ln(s+1)| - 12\varepsilon|t_k - s|$. It is easy to see that for sufficiently large k we have $F_{12}(t_k, s) > 0$, so $F_{12}(t,s) \not\equiv 0$. Thus $A+Q \notin LD_r$ and $A+Q \notin LD_b$.

Similarly one can show that

$$F_{11}(t,s) = |[A(t)+Q(t), \int_s^t (A(u)+Q(u))du]_{11}| \geq$$
$$\geq |\exp(-s) - \exp(-t) - (t-s)\exp(-t)| - 12\varepsilon|t-s|.$$

Since $F_{11}(0,s) \geq |\exp(-s) - 1 + s| - 12\varepsilon s$ and $|\exp(-s) - 1 + s| > 0$ for all $s > 0$, we see that $F_{11}(t,s) \not\equiv 0$, i.e., $A+Q \notin LD_l(s)$.

Theorem 3. *For any Lappo-Danilevskiĭ system* (1_A)*and for any* ε *there exists a system* (1_B) *such that* $\rho(A,B) \leq \varepsilon$ *but* (1_B) *is not a Lappo-Danilevskiĭ system.*

Indeed, if a_{12} and a_{21} are constant, then we can set $b_{12}(t) = a_{12}(t) + \alpha a_{12}(t) + \varphi(t)$, $b_{21}(t) = a_{21}(t) + \beta + \varphi(t)$, where φ is a continuous function, $0 \leq \alpha \leq \varepsilon$, $0 \leq \beta \leq \varepsilon$, and $b_{11}(t) = a_{11}(t)$, $b_{22}(t) = a_{22}(t)$. If we choose α and β such that $a_{12} - a_{21} + \alpha a_{12} - \beta \neq 0$ (the existence of such α and β is obvious), then one can show that $B \notin \{LD_b \bigcup LD_r \bigcup LD_l\}$. If $a_{21} \not\equiv const$ or $a_{12} \not\equiv const$, then we set $b_{12}(t) = \alpha + a_{12}(t)$, where $0 \leq \alpha \leq \varepsilon$, and $b_{ij}(t) = a_{ij}(t)$ for all $i,j = 1,2$, $(i,j) \neq (1,2)$, or $b_{21}(t) = \beta + a_{21}(t)$, where $0 \leq \beta \leq \varepsilon$, and $b_{ij}(t) = a_{ij}(t)$ for all $i,j = 1,2$, $(i,j) \neq (2,1)$, respectively. For both these cases one can show that $B \notin \{LD_b \bigcup LD_r \bigcup LD_l\}$.

Theorem 4. *Let* $A_i \in LD_\alpha(s_i)$, $i \in \mathbb{N}$, $\alpha \in \{b, r\}$, *and* $\rho(A, A_i) \to 0$ *as* $i \to +\infty$. *If there exists* M *such that* $s_i \leq M < +\infty$ *for all* $i \in \mathbb{N}$, *then* A *is a bilateral or right Lappo-Danilevskiĭ matrix.*

Indeed, since the sequence (s_i) is bounded, there exists a subsequence (s_{i_k}) such that $s_{i_k} \to s \geq 0$ as $i_k \to +\infty$. Without loss of generality, $s_i \to s$ as $i \to +\infty$. So for the corresponding values of t we have $[A_i(t), \int_s^t A_i(u)du] = [A_i(t), \int_{s_i}^t A_i(u)du] + [A_i(t), \int_s^{s_i} A_i(u)du] = [A_i(t), \int_s^{s_i} A_i(u)du]$. Since A_i is uniformly bounded on $[0, +\infty[$, we have $[A_i(t), \int_s^{s_i} A_i(u)du] \to 0$ as $i \to +\infty$. On the other hand, the sequence A_i is uniformly convergent on the non-negative half-line Therefore $[A_i(t), \int_s^t A_i(u)du] \to [A(t), \int_s^t A(u)du]$ as $i \to +\infty$. So for the corresponding values of t we have $[A(t), \int_s^t A(u)du] \equiv 0$, i.e., A is a bilateral or right Lappo-Danilevskiĭ matrix.

Similarly one can prove

Theorem 5. *Let* $A_i \in LD_l(s_i)$, $i \in \mathbb{N}$, *and* $\rho(A, A_i) \to 0$ *as* $i \to +\infty$. *If there exist* m, M *such that* $0 < m \leq s_i \leq M < +\infty$ *for all* $i \in \mathbb{N}$, *then* A *is a left Lappo-Danilevskiĭ matrix.*

Theorem 6. *There exists a sequence* A_i, $A_i \in LD_r(s_i)$, $i \in \mathbb{N}$, $\rho(A, A_{s_i}) \to 0$ *and* $s_i \to +\infty$ *as* $i \to +\infty$, *such that* $A \notin LD_r$.

To prove this statement, it is sufficient to consider a sequence $A_k(t) = a_{ijk}(t)$, $i, j = 1, 2$, such that $a_{11k}(t) = a_{22k}(t) = g(t) with$ g continuous and bounded, and $a_{21k}(t) = \exp(-t)$, $a_{12k} = f_k(t)$, where

$$f_k = \begin{cases} (1 - \exp(-t)) \exp(-t), & 0 \le t \le k, \\ (1 - \exp(-k)) \exp(-t), & t > k. \end{cases}$$

Theorem 7. *There exists a sequence* A_i, $A_i \in LD_l(s_i)$, $i \in \mathbb{N}$, $\rho(A, A_{s_i}) \to 0$ *and* $s_i \to +0$ *as* $i \to +\infty$, *such that* $A \notin LD_l$.

To prove this statement, it is sufficient to consider a sequence $A_k(t) = a_{ijk}(t)$, $i, j = 1, 2$, such that $a_{11k}(t) = a_{22k}(t) = g(t)$ with g continuous and bounded, $a_{21k}(t) = \exp(-t)$, $a_{12k} = f_k(t)$, where

$$f_k = \begin{cases} \exp(-k^{-1} - t)), & 0 \le t \le k^{-1}, \\ \exp(-2t), & t > k^{-1}. \end{cases}$$

Theorem 8. *Let* $A_i \in LD_b(s_i)$, $i \in \mathbb{N}$. *If* $\rho(A, A_{s_i}) \to 0$ *as* $i \to +\infty$, *then* A *is a bilateral Lappo-Danilevskiĭ matrix.*

Theorem 9. *Let* $A_i \in LD_l(s_i)$, $i \in \mathbb{N}$. *If there exists* m *such that* $0 < m \le s_i$ *for all* $i \in \mathbb{N}$, *then* A *is a left Lappo-Danilevskiĭ matrix.*

The proofs of Theorem 8 and Theorem 9 are based on the following lemmas.

Lemma 1. *Let continuous scalar functions* f *and* g *satisfy* $f(t) \int_s^t g(u)du = g(t) \times \int_s^t f(u)du$ *for some* $s \ge 0$ *and for all* t, $t \in]b, c[\subset [0, +\infty[$. *If* $\int_s^t g(u)du \ne 0$ *for all* t, $t \in]b, c[$, *then there exists a number* λ *such that* $\int_s^t f(u)du = \lambda \int_s^t g(u)du$ *and* $f(t) = \lambda g(t)$ $\forall\, t \in [b, c]$.

Let $Z(g; s) = \{t \ge 0 \mid \int_s^t g(u)du = 0\}$, $N(g; s) = \{t \in Z(g; s) \mid g(t) \ne 0\}$. Denote by $R(g; s)$ the subset of $Z(g; s) \backslash N(g; s)$ with the following property: $\forall t_0 \in R(g; s)$ $\forall \delta > 0$ $\exists t_\delta$, $t_0 < t_\delta \le t_0 + \delta$, $t_\delta \notin Z(g; s)$. Denote by $L(g; s)$ the subset of $Z(g; s) \backslash N(g; s)$ with the following property: $\forall t_0 \in L(g; s)$ $\forall \delta$, $0 < \delta \le t_0$, $\exists t_\delta$, $t_0 - \delta \le t_\delta < t_0$, $t_\delta \notin Z(g; s)$.

Lemma 2. *Let continuous scalar functions* f *and* g *satisfy* $f(t) \int_s^t g(u)du = g(t) \times \int_s^t f(u)du$, *for some* $s \ge 0$ *and for all* $t \ge 0$. *Then* $N(g; s) \bigcup R(g; s) \bigcup L(g; s) \subset Z(f; s)$.

Lemma 3. *Let a sequence of continuous scalar functions* (g_i) *uniformly over* $[0, +\infty[$ *converge to a function* g. *Then for any* $\sigma \in]0, +\infty[$, $g(\sigma) \ne 0$, *there exist positive* ε *and* ν *such that for all* $i \ge \nu$ *the inequalities* $g_i(t) \ne 0$, $g(t) \ne 0$ *hold for all* $t \in [\sigma - \varepsilon, \sigma + \varepsilon]$.

Lemma 4. *Let sequences of continuous scalar functions* (g_i) (f_i) *uniformly over* $[0, +\infty[$ *converge to functions* g *and* f *respectively. If for any* $i \in \mathbb{N}$ *there is* s_i *such that for all* $t \ge 0$ *we have* $f_i(t) \int_{s_i}^t g_i(u)du = g_i(t) \int_{s_i}^t f_i(u)du$, *then there exists* $s \ge 0$ *such that the equality* $f(t) \int_s^t g(u)du = g(t) \int_s^t f(u)du$ *holds for all* $t \ge 0$.

References

1. B. P. Demidovich, Lectures on mathematical stability theory. (Russian) *Nauka, Moscow,* 1967.

Author's address:
Department of Higher Mathematics
Belarussian State University
4, F.Skorina Ave., Minsk 220050
Belarus