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ON THE QUESTION OF SOLVABILITY OF THE PERIODIC BOUNDARY VALUE PROBLEM FOR A SYSTEM OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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In the present note sufficient conditions are given for the solvability of the ω-periodic boundary value problem

$$dx(t) = dA(t) \cdot f(t, x(t)), \tag{1}$$

$$x(0) = x(\omega), \tag{2}$$

where ω is a positive number, $A(t) = (a_{ik}(t))_{i,k=1}^{n}$, $a_{ik}(t) \equiv a_{ik}^{(1)}(t) - a_{ik}^{(2)}(t)$, $a_{ik}^{(\sigma)} : R \to R$ $(\sigma = 1, 2)$ are functions nondecreasing on $[0, \omega]$, $a_{ik}^{(\sigma)}(t+\omega) = a_{ik}^{(\sigma)}(t) + a_{ik}^{(\sigma)}(\omega)$ for $t \in R$; $f = (f_k)_{k=1}^{n} : R \times R^n \to R^n$ is an ω-periodic with respect to the first variable vector-function such that its restriction on $[0, \omega] \times R^n$ belongs to the Carathéodory classes corresponding to the matrix-functions $A^{(1)}$ and $A^{(2)}$, $A^{(\sigma)}(t) \equiv (a_{ik}^{(\sigma)}(t))_{i,k=1}^{n}$ $(\sigma = 1, 2)$.

The following notation and definitions will be used: $R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$, $[a, b]$ $(a, b \in R)$ is a closed segment, $R^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ik})_{i,k=1}^{n,m}$ with the norm $\|X\| = \max_{k=1,\dots,m} \sum_{i=1}^{n} |x_{ik}|$; $|X| = (|x_{ik}|)_{i,k=1}^{n,m}$; if $X \in R^{n\times n}$, then $\det(X)$ is the determinant of X, I_n is the identity $n \times n$-matrix; $R^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$.

$\mathrm{BV}([a, b], R^{n\times m})$ is the set of all matrix-functions $X = (x_{ik})_{i,k=1}^{n,m} : [a, b] \to R^{n\times m}$ such that every its component x_{ik} has a bounded total variation on $[a, b]$.

$s_k : \mathrm{BV}([a, b], R) \to \mathrm{BV}([a, b], R)$ $(k = 0, 1, 2)$ are the operators defined by $s_1(x)(a) = s_2(x)(a) = 0$,

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad \text{for} \quad t \in]a, b],$$

$$s_0(x)(t) \equiv x(t) - s_1(x)(t) - s_2(x)(t).$$

$\mathrm{BV}_{\omega}^{n\times m}$ is the set of all matrix-functions $X : R \to R^{n\times m}$ such that $X(t+\omega) = X(t) + X(\omega)$ for $t \in R$, and its restriction on $[0, \omega]$ belongs to $\mathrm{BV}([0, \omega], R^{n\times m})$; $X(t-)$ and $X(t+)$ are the left and the right limits of X at the point $t \in R$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

If $g : R \to R$ is nondecreasing on the interval $I \subset R$, $x : R \to R$ and $s < t$ $(s, t \in I)$, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_{]s,t[} x(\tau)\, dg(\tau) + x(t)d_1g(t) + x(s)d_2g(s),$$

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where $\int\limits_{]s,t[} x(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure μ_g corresponding to the function g (if $s=t$, then $\int\limits_s^t x(\tau)dg(\tau) = 0$); $L([a,b],R;g)$ is the set of all μ_g-measurable functions $x : [0,\omega] \to R$ such that $\int\limits_a^b |x(t)|dg(t) < +\infty$.

A matrix-function is said to be nondecreasing if every of its components is such.

If $G = (g_{ik})_{i,k=1}^{l,n} : R \to R^{l\times n}$ is a matrix-function nondeacreasing on the interval $I \subset R$ and $X = (x_{kj})_{k,j=1}^{n,m} : R \to R^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau) = \Big(\sum_{k=1}^n \int\limits_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\Big)_{i,j=1}^{l,m} \quad \text{for} \quad s \le t \ \ (s,t \in I);$$

$L([a,b],R^{n\times m};G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to R^{n\times m}$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i=1,\dots,l;\ k=1,\dots,n;\ j=1,\dots,m)$;

$K([a,b]\times D_1,D_2;G)$ $(D_1 \subset R^n,\ D_2 \subset R^{n\times m})$ is the Carathéodory class corresponding to G, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$: a) the function $f_{kj}(\cdot,x) : [a,b] \to R$ is $\mu_{g_{ik}}$-measurable for every $x \in D_1$; b) the function $f_{kj}(t,\cdot) : D_1 \to R$ is continuous for $\mu_{g_{ik}}$-almost every $t \in [a,b]$, and $\sup\{|f_{kj}(\cdot,x)| : x \in D_0\} \in L([a,b],R;g_{ik})$ for every compact $D_0 \subset D_1$.

If $G^{(\sigma)} : R \to R^{l\times n}$ $(\sigma = 1,2)$ are matrix-functions nondecreasing on the interval $I \subset R$, $G = G^{(1)} - G^{(2)}$ and $X : R \to R^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau) = \int\limits_s^t dG^{(1)}(\tau)\cdot X(\tau) - \int\limits_s^t dG^{(2)}(\tau)\cdot X(\tau) \quad \text{for} \quad s \le t \ \ (s,t \in I).$$

An ω-periodic vector-function $x : R \to R^n$ is said to be a solution of the problem (1),(2) if its restriction on $[s,t]$ belongs to $\mathrm{BV}([s,t],R^n)$ and

$$x(t) = x(s) + \int\limits_s^t dA(\tau)\cdot f(\tau,x(\tau)) \quad \text{for} \quad s < t \ \ (s,t \in R).$$

Let $B = (b_{ik})_{i,k=1}^n \in \mathrm{BV}_\omega^{n\times n}$ and let natural numbers m, $r_1,\dots,r_m$ and $n_1,\dots,n_m$ $(0 = n_0 < n_1 < \dots < n_m = n)$, nondecreasing functions $g_{lj} : [0,\omega] \to R$ $(l = 1,\dots,r_j;$ $j = 1,\dots,m)$, functions $\alpha_{lj} \in L([0,\omega],R;g_{lj})$ $(l = 1,\dots,r_j;\ j = 1,\dots,m)$ and matrix-functions $\mathcal{P}_{lj} = (p_{ljik})_{i,k=1}^n$ $(l = 1,\dots,r_j;\ j = 1,\dots,m)$, $p_{ljik} \in L([0,\omega],R;g_{lj})$ $(i,k = n_{j-1}+1,\dots,n_j)$ be such that $b_{ik}(t) \equiv 0$ $(i = n_{j-1}+1,\dots,n_j;\ k = n_j+1,\dots,n;$ $j = 1,\dots,m-1)$,

$$\sigma_j \sum_{i,k=n_{j-1}+1}^{n_j} p_{ljik}(t)x_ix_k \ge \alpha_{lj}(t) \sum_{i=n_{j-1}+1}^{n_j} x_i^2$$

for $\mu_{g_{lj}}$-almost everywhere $t \in [0,\omega]$, $(x_i)_{i=1}^n \in R^n$ $(l = 1,\dots,r_j;\ \ j = 1,\dots,m)$,

$$\sigma_i\Big(b_{jii}(t) - b_{jii}(s) - \sum_{l=1}^{r_j} \int\limits_s^t p_{ljii}(\tau)\,dg_{lj}(\tau)\Big) \ge 0$$

$$\text{for} \quad 0 \le s \le t \le \omega \quad (i = n_{j-1}+1, \dots, n_j; \quad j = 1, \dots, m)$$

and

$$b_{jik}(t) = \sum_{l=1}^{r_j} \int_0^t p_{ljik}(\tau)\, dg_{lj}(\tau) \quad \text{for} \quad t \in [0,\omega]$$

$$(i \ne k;\;\; i,k = n_{j-1}+1, \dots, n_j;\;\; j = 1, \dots, m),$$

where $\sigma_j \in \{-1,1\}$ $(j = 1, \dots, m)$ and

$$b_{jik}(t) \equiv b_{ik}(t) - \frac{1}{2}\Big(\sum_{0<\tau\le t} \sum_{\sigma=n_{j-1}+1}^{n_j} d_1 b_{\sigma i}(\tau) \cdot d_1 b_{\sigma k}(\tau) -$$

$$- \sum_{0\le\tau<t} \sum_{\sigma=n_{j-1}+1}^{n_j} d_2 b_{\sigma i}(\tau) \cdot d_2 b_{\sigma k}(\tau)\Big) \quad (i,k = n_{j-1}+1, \dots, n_j; \quad j = 1, \dots, m).$$

Then we will say that

$$B \in Q_\omega^{n\times n}\Big(m, \Big(r_j, n_j, (g_{lj}, \alpha_{lj}, \mathcal{P}_{lj})_{l=1}^{r_j}\Big)_{j=1}^{m}\Big). \tag{3}$$

Theorem 1. *Let the conditions* (3),

$$|f(t,x) - \mathcal{P}(t)x| \le q(t, \|x\|),$$

$$\det\Big(I_n + (-1)^\sigma d_\sigma A(t) \cdot \mathcal{P}(t)\Big) \ne 0 \quad (\sigma = 1,2),$$

$$\lim_{\rho\to+\infty} \frac{1}{\rho} \int_0^\omega d\Big(A^{(1)}(t) + A^{(2)}(t)\Big) \cdot q(t,\rho) = 0, \tag{4}$$

$$(1+\sigma_j) d_1 g_j(t) + (1-\sigma_j) d_2 g_j(t) < 2, \tag{5}$$

$$(1-\sigma_j) d_1 g_j(t) + (1+\sigma_j) d_2 g_j(t) \ne -2 \tag{6}$$

and

$$\exp\Big(s_0(g_j)(\omega)\Big) > \frac{1}{2}\Big[(1+\sigma_j) \prod_{0<\tau\le\omega} (1 - d_1 g_j(\tau)) \prod_{0\le\tau<\omega} (1 + d_2 g_j(\tau))^{-1} +$$

$$+ (1-\sigma_j) \prod_{0<\tau\le\omega} (1 + d_1 g_j(\tau))^{-1} \prod_{0\le\tau<\omega} (1 - d_2 g_j(\tau))\Big] \tag{7}$$

be fulfilled on $[0,\omega] \times R^n$ *for every* $j \in \{1, \dots, m\}$, *where* $\sigma_j \in \{-1,1\}$; $m, r_1, \dots, r_m$ *and* $n_1, \dots, n_m$ $(0 = n_0 < n_1 < \dots < n_m = n)$ *are natural numbers;* $\alpha_{lj} \in L([0,\omega], R; g_{lj})$, $\mathcal{P}_{lj} = (p_{ljik})_{i,k=1}^n$, $p_{ljik} \in L([0,\omega], R; g_{lj})$, $g_{lj} : [0,\omega] \to R$ *is nondecreasing* $(i,k = n_{j-1}+1, \dots, n_j)$; $\mathcal{P} \in \bigcap_{\sigma=1}^{2} L([0,\omega], R^{n\times n}; A^{(\sigma)})$; $B(t) = \int_0^t dA(\tau) \cdot \mathcal{P}(\tau)$ *for* $t \in [0,\omega]$, $B(t+\omega) = B(t) + B(\omega)$; $g_j(t) \equiv 2 \sum_{l=1}^{r_j} \int_0^t \alpha_{lj}(\tau) dg_{lj}(\tau)$; $q \in \bigcap_{\sigma=1}^{2} K([0,\omega] \times R_+, R_+^n; A^{(\sigma)})$ *is a vector-function nondecreasing in the second variable. Then the problem* (1),(2) *is solvable.*

Theorem 2. *Let the conditions* (3)–(7) *be fulfilled. Let, moreover,*

$$|f(t,x) - \mathcal{P}(t,x)x| \le q(t, \|x\|),$$

$$\sigma_j \sum_{i,k=n_{j-1}+1}^{n_j} \Big(\sum_{\nu=1}^{n} a_{li\nu}(\tau) p_{\nu k}(\tau, y) x_i x_k \geq \alpha_{lj}(\tau) \sum_{i=n_{j-1}+1}^{n_j} x_i^2$$

$$\text{for } \mu_{g_{lj}}\text{-almost every } \tau \in [0,\omega] \quad (l = 1, \dots, r_j),$$

$$(-1)^\sigma \sigma_j \sum_{i,k,l=n_{j-1}+1}^{n_j} \Big(\sum_{\nu,\mu=1}^{n} p_{\nu i}(t,y) p_{\mu k}(t,y) d_\sigma a_{l\nu})t) \cdot d_\sigma a_{l\mu}(t) \Big) x_i x_k \geq$$

$$\geq d_\sigma \beta_j(t) \sum_{i=n_{j-1}+1}^{n_j} x_i^2 \quad (\sigma = 1, 2),$$

$$|p_{ik}(t,y)| \leq \varphi_{ik}(t) \quad (i, k = 1, \dots, n)$$

and

$$\sum_{j=1}^{m} \sum_{i,k=n_{j-1}+1}^{n_j} \sum_{l=1}^{n} \varphi_{lk}(t) d_\sigma \left(a_{il}^{(1)}(t) + a_{il}^{(2)}(t) \right) < 1 \quad (\sigma = 1, 2)$$

be fulfilled for $(t,x,y) \in [0,\omega] \times R^{2n}$ $(j = 1, \dots, m)$ *where* $\sigma_j \in \{-1, 1\}$*;* $m, r_1, \dots, r_m$ *and* $n_1, \dots, n_m$ $(0 = n_0 < n_1 < \dots < n_m = n)$ *are natural numbers;* $\beta_j \in \mathrm{BV}([0,\omega], R)$*;* a_{lik} *and* α_{lj} *belong to* $L([0,\omega], R; g_{lj})$*,* $g_{lj} : [0,\omega] \to R$ *are nondecreasing* $(i, k = n_{j-1} + 1, \dots, n_j)$*;* $\mathcal{P} = (p_{ik})_{i,k=1}^n \in \bigcap_{\sigma=1}^{2} K([0,\omega] \times R^n, R^{n\times n}; A^{(\sigma)})$*,* $(\varphi_{ik})_{i,k=1}^n \in \bigcap_{\sigma=1}^{2} L([0,\omega], R_+^{n\times n}; A^{(\sigma)})$*;* $a_{ik}(t) = \sum_{l=1}^{n} \int_0^t a_{lik}(\tau) dg_{lj}(\tau)$ *and* $\sum_{l=1}^{n} \int_0^t p_{lk}(\tau, y_1, \dots, y_n) da_{il}(\tau) = 0$ *for* $t \in [0,\omega]$ $(i = n_{j-1} + 1, \dots, n_j;\ k = n_j + 1, \dots, n;\ j = 1, \dots, m-1)$*;* $g_j(t) \equiv 2 \sum_{l=1}^{r_j} \int_0^t \alpha_{lj}(\tau) dg_{lj}(\tau) + \sum_{\sigma=1}^{2} s_\sigma(\beta_j)(t)$*; and* $q \in \bigcap_{\sigma=1}^{2} K([0,\omega] \times R_+, R_+^n; A^{(\sigma)})$ *is a vector-function nondecreasing in the second variable. Then the problem* (1),(2) *is solvable.*

The analogous question has been considered in [1] for a system of ordinary differential equations.

References

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