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ON A SUCCESIVE APPROXIMATIONS METHOD OF SOLVING THE CAUCHY PROBLEM FOR THE SYSTEM OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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In the present note, we consider a succesive approximations method of construction of the solution of the Cauchy problem

$$dx(t) = dA(t)\cdot f(t,x(t)), \tag{1}$$

$$x(t_0) = c_0, \tag{2}$$

where $t_0 \in [a,b]$, $c_0 \in R^n$, $A = (a_{ik})_{i,k=1}^n : [a,b] \to R^{n\times n}$ is a matrix-function whose components have bounded variation and are continuous from the right on $[a,t_0[$ and continuous from the left on $]t_0,b]$, $f = (f_k)_{k=1}^n : [a,b]\times R^n \to R^n$ is a vector-function belonging to the Carathéodory class corresponding to A.

The following notation and definitions will be used: $R =]-\infty,+\infty[$, $[a,b]$ $(a,b\in R)$ is a closed segment, R^n is the space of all real column n-vectors $x = (x_i)_{i=1}^n$ with the norm $\|x\| = \sum\limits_{i=1}^n |x_i|$, $R^{n\times n}$ is the set of all real $n\times n$ matrices.

$\mathrm{BV}([a,b],R^n)$ is the set of all vector-functions $x = (x_i) : [a,b]\to R^n$ with components of bounded variation on $[a,b]$, $x(t-) = (x_i(t-))_{i=1}^n$ and $x(t+) = (x_i(t+))_{i=1}^n$ are the left and the right limits of x at the point $t\in[a,b]$ $(x(a-) = x(a),\ x(b+) = x(b))$, $d_1x(t) = x(t) - x(t-)$, $d_2x(t) = x(t+) - x(t)$.

If $g : [a,b]\to R$ is a nondecreasing function, $x : [a,b]\to R$ and $a\le s<t\le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,dg(\tau) + x(t)d_1g(t) + x(s)d_2g(s),$$

where $\int\limits_{]s,t[} x(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure μ_g corresponding to the finction g (if $s=t$, then $\int\limits_s^t x(\tau)dg(\tau) = 0$).

If $G_j = (g_{jik})_{i,k=1}^n : [a,b]\to R^{n\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G = G_1 - G_2$ and $x = (x_k)_{k=1}^n : [a,b]\to R^n$, then

$$\int_s^t dG(\tau)\cdot x(\tau) = \Big(\sum_{k=1}^n\Big(\int_s^t x_k(\tau)\,dg_{1ik}(\tau) - \int_s^t x_k(\tau)dg_{2ik}(\tau)\Big)\Big)_{i=1}^n$$
$$\text{for}\quad a\le s\le t\le b;$$

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$K([a,b]\times R^n, R^n; G_j)$ is the Carathéodery class corresponding to G_j, i.e., the set of all vector-functions $\varphi=(\varphi_k)_{k=1}^n : [a,b]\times R^n \to R^n$ such that for each $i,k=\{1,\dots,n\}$: (a) the function $\varphi_k(\cdot,x) : [a,b]\to R^n$ is $\mu_{g_{jik}}$-measurable for every $x\in R^n$; (b) the function $\varphi_k(t,\cdot) : R^n \to R^n$ is continuous for $\mu_{g_{jik}}$-almost every $t\in[a,b]$, and the function $\sup\{\varphi_k(\cdot,x) : \|x\|\le r\}$ is $\mu_{g_{jik}}$-integrable on $[a,b]$ for every positive number r,

$$K([a,b]\times R^n, R^n; G)=\bigcap_{j=1}^{2} K([a,b]\times R^n, R^n; G_j).$$

A vector-function $x\in \mathrm{BV}([a,b],R^n)$ is said to be a solution of the problem (1),(2) if it satisfies the condition (2) and

$$x(t)=x(s)+\int_s^t dA(\tau)\cdot f(\tau,x(\tau)) \quad \text{for} \quad a\le s\le t\le b.$$

Theorem. *Let* $f\in K([a,b]\times R^n, R^n; A)$ *and*

$$\|f(x,t)-f(t,y)\|\le L\|x-y\| \quad \textit{for} \quad (t,x,y)\in[a,b]\times R^{2n},$$

where $L=const$. *Then the problem* (1),(2) *has a unique solution* x, *and*

$$\lim_{k\to+\infty} x_k(t)=x(t) \quad \textit{uniformly on} \;\; [a,b],$$

where

$$x_0(t)\equiv c_0,$$
$$x_k(t)\equiv c_0+\int_{t_0}^t dA(\tau)\cdot f(\tau,x_{k-1}(\tau)) \quad (k=1,2,\dots).$$

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