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ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS, I

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In the present note, we consider the question of solvability of the boundary value problem

$$u''(t) = F(u)(t), \tag{1}$$

$$u(a) = 0, \quad u(b) = 0, \tag{2}$$

where the continuous operator $F : C'([a,b]) \to L([a,b])$ satisfies the Carathéodory conditions.

Before we proceed to formulate the basic results, let us introduce the following notation:

$R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$;

$C([a,b])$ is the space of continuous functions $f : [a,b] \to R$ with the norm $\|f\|_C = \max\{|f(t)| : a \le t \le b\}$;

$C'([a,b])$ is the space of continuously differentiable functions $f : [a,b] \to R$ with the norm $\|f\|_{C'} = \|f\|_C + \|f'\|_C$; $C'_0([a,b]) = \Big\{ f \in C'([a,b]) : \; f(a) = 0, \;\; f(b) = 0 \Big\}$;

$\widetilde{C}'([a,b])$ is the set of absolutely continuous, with its first derivative, functions $f : [a,b] \to R$;

$L([a,b])$ is the space of summable on $[a,b]$ functions $f :]a,b[\to R$ with the norm $\|f\|_L = \int_a^b |f(s)|\,ds$.

$M(A,B)$ is the set of measurable functions $F : A \to B$;

$K_0([a,b])$ is the set of operators $p : C'([a,b]) \to M([a,b],R)$;

$\mathcal{L}([a,b])$ is the set of linear continuous operators $l : C([a,b]) \to L([a,b])$ such that for any $r > 0$ there exists $g_r \in L([a,b])$ satisfying

$$|l(u)(t)| \le g_r(t) \quad \text{for} \quad a < t < b, \quad \|u\|_C \le r;$$

$K([a,b])$ is the set of continuous operators $F : C'([a,b]) \to L([a,b])$ such that for any $r > 0$ there exists $g_r \in L([a,b])$ satisfying

$$|F(u)(t)| \le g_r(t) \quad \text{for} \quad a < t < b, \quad \|u\|_{C'} \le r;$$

$K_1([a,b] \times R, R_+)$ is the set of functions $q :]a,b[\times R \to R_+$ satisfying the Carathéodory condition;

$\sigma : L([a,b]) \to L([a,b])$ is an operator defined by

$$\sigma(p)(t) = \exp\Bigg[\int\limits_{\frac{a+b}{2}}^{t} p(s)\,ds \Bigg].$$

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$\sigma_\tau : L([a,b]) \to L([a,b])$ is an operator defined by

$$\sigma_\tau(p)(t) = \frac{1}{\sigma(p)(t)} \left| \int_\tau^t \sigma(p)(s)\,ds \right|,$$

$[p(t)]_+ = \frac{1}{2}\left(|p(t)| + p(t)\right)$, $[p(t)]_- = \frac{1}{2}\left(|p(t)| - p(t)\right)$.

An operator $l \in \mathcal{L}([a,b])$ is said to be positive (negative) if for any nonnegative function $u \in C([a,b])$ the function $l(u)$ is nonnegative (nonpositive).

In what follows, we assume $F \in K([a,b])$. Under solution of the equation (1) it is understood a function $u \in \widetilde{C}'([a,b])$ which almost everywhere satisfies it.

Definition. Let $l \in \mathcal{L}([a,b])$. We say that a vector function $(p, g_1, g_2) :]a,b[\to R^3$ belongs to the set $V(]a,b[\,;l)$ if p, g_1, $g_2 \in L([a,b])$ and for any function $g \in M([a,b],R)$ satisfying

$$g_1(t) \le g(t) \le g_2(t) \quad \text{for} \quad a < t < b,$$

there exists a positive function $w \in \widetilde{C}'([a,b])$ such that

$$w''(t) \le p(t)w(t) + g(t)w'(t) + l(w)(t) \quad \text{for} \quad a < t < b.$$

Remark. Let $l \in \mathcal{L}([a,b])$ be a negative operator and $p(t) + l(1)(t) \ge 0$ for $a < t < b$. Then for any g_1, $g_2 \in L([a,b])$ satisfying $g_1(t) \le g_2(t)$ for $a < t < b$, we have $(p, g_1, g_2) \in V(]a,b[\,;l)$.

Theorem 1. *Let on the set $C_0'([a,b])$ the inequalities*

$$\begin{gathered}\Big[F(v)(t) - p_1(t)v(t) - p_2(v)(t)v'(t) - l(v)(t)\Big] \operatorname{sgn} v(t) \ge -q\Big(t, \|v\|_{C'}\Big), \\ g_1(t) \le p_2(v)(t) \le g_2(t)\end{gathered} \tag{3}$$

be fulfilled, where $l \in \mathcal{L}([a,b])$ is a negative operator, $p_2 \in K_0([a,b])$, $q \in K_1([a,b] \times R, R_+)$ is nondecreasing in the second argument and

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,ds = 0. \tag{4}$$

Let, moreover,

$$(p_1, g_1, g_2) \in V\Big(]a,b[\,,l\Big).$$

Then the problem (1), (2) *has at least one solution.*

Mention two corollaries of Theorem 1 for the equation

$$u''(t) = h(t)u(\tau(t)) + G(u)(t), \tag{5}$$

where $G \in K([a,b])$, $\tau \in M([a,b],[a,b])$, and $h \in L([a,b])$ is a nonpositive function.

Corollary 1. *Let on the set $C_0'([a,b])$ the inequality*

$$G(v)(t) \operatorname{sgn} v(t) \ge -q\Big(t, \|v\|_{C'}\Big) \tag{6}$$

be fulfilled, where $q \in K_1([a,b] \times R, R_+)$ *is nondecreasing in the second argument and satisfies* (4). *Moreover, let*

$$\big(b-\tau(t)\big)\int\limits_a^{\tau(t)} (s-a)|h(s)|\,ds +$$

$$+\big(\tau(t)-a\big)\int\limits_{\tau(t)}^{b} (b-s)|h(s)|\,ds < b-a \quad \textit{for} \quad a<t<b.$$

Then the problem (5), (2) *has at least one solution.*

Corollary 2. *Let on the set* $C_0'([a,b])$ *the inequality* (6) *be fulfilled, where* $q \in K_1([a,b]\times R, R_+)$ *is nondecreasing in the second argument and satisfies* (4). *Let, moreover, there exist* $c \in [a,b]$ *such that*

$$\int\limits_a^c \sigma_a(p)(s)|h(s)|\,ds < 1, \quad \int\limits_c^b \sigma_b(p)(s)|h(s)|\,ds < 1,$$

$$\big(t-\tau(t)\big)\sigma(p)(t)\int\limits_t^c \frac{|h(s)|}{\sigma(p)(s)}\,d \le 1 \quad \textit{for} \quad a<t<b,$$

where $p(t) = h(t)(\tau(t)-t)$ *for* $a<t<b$. *Then the problem* (5), (2) *has at least one solution.*

Finally, we give a corollary of Theorem 1 for the equation

$$u''(t) = p_1(t)u(t) + p_2(u)(t)u'(t) + h(t)u(\tau(t)) + G(u)(t), \tag{7}$$

where p_2, $G \in K([a,b])$, $\tau \in M([a,b],[a,b])$, p_1, $h \in L([a,b])$ and h is positive.

Corollary 3. *Let on the set* $C_0'([a,b])$ *the inequalities* (3) *and* (6) *be fulfilled, where* g_1, $g_2 \in L([a,b])$, $q \in K_1([a,b]\times R, R_+)$ *is nondecreasing in the second argument and satisfies* (4). *Let, moreover, there exist* $\lambda_i \in [0,1[$, $\alpha_{ij} \in [0,+\infty[$, $i,j=1,2$, *and* $c \in [a,b]$ *such that*

$$\int\limits_0^{+\infty} \frac{ds}{\alpha_{11}+\alpha_{12}s+s^2} > \frac{(c-a)^{1-\lambda_1}}{1-\lambda_1}, \quad \int\limits_0^{+\infty} \frac{ds}{\alpha_{21}+\alpha_{22}s+s^2} > \frac{(b-c)^{1-\lambda_2}}{1-\lambda_2}$$

and

$$(t-a)^{2\lambda_1}\big[p_1(t)+h(t)\big] \ge -\alpha_{11}, \quad (t-a)^{\lambda_1}\Big[g_1(t)+\frac{\lambda_1}{t-a}+\big(\tau(t)-t\big)h(t)\Big] \ge -\alpha_{12} \quad \textit{for} \ a<t<c,$$

$$(b-t)^{2\lambda_2}\big[p_1(t)+h(t)\big] \ge -\alpha_{21}, \quad (b-t)^{\lambda_2}\Big[g_2(t)-\frac{\lambda_2}{b-t}+\big(\tau(t)-t\big)h(t)\Big] \le \alpha_{22} \quad \textit{for} \ c<t<b.$$

Then the problem (7), (2) *has at least one solution.*

References

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