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ON SYSTEMS OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL AND INTEGRAL INEQUALITIES

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In the present note, we consider the questions of estimates of the solutions of the system of differential inequlities

$$dx(t)\cdot \operatorname{sign}(t-t_0)\le dC(t)\cdot x(t)+dq(t)\quad \text{for}\quad t\in[a,b]\setminus\{t_0\},\tag{1}$$

satisfying the condition

$$x(t_0)+(-1)^j d_j x(t_0)\le c_0+d_jC(t_0)\cdot c_0+d_jq(t_0)\ \ (j=1,2),\tag{2}$$

and of the solutions of the system of integral inequalities

$$x(t)\le c_0+\Big(\int\limits_{t_0}^{t} dC(\tau)\cdot x(\tau)+q(t)-q(t_0)\Big)\operatorname{sign}(t-t_0)\quad \text{for}\quad t\in[a,b],\tag{3}$$

satisfying the condition (2), where $t_0\in[a,b]$, $c_0\in R^n$, $q\in \mathrm{BV}([a,b],R^n)$ and $C=(c_{ik})_{i,k=1}^n\in \mathrm{BV}([a,b],R^{n\times n})$.

The following notation and definitions will be used: $R=]-\infty,+\infty[$, $[a,b]$ $(a,b\in R)$ is a closed segment, $R^{n\times m}$ is the set of all real $n\times m$-matrices $X=(x_{ik})_{i,k=1}^{n,m}$; if $X\in R^{n\times n}$, then $\det(X)$ is the determinant of X, I_n is the identity $n\times n$-matrix; $R^n=R^{n\times 1}$ is the set of all real column n-vectors $x=(x_i)_{i=1}^n$.

$\mathrm{BV}([a,b],R^{n\times m})$ is the set of all matrix-functions $X=(x_{ik})_{i,k=1}^{n,m}:[a,b]\to R^{n\times m}$ such that every its component x_{ik} has bounded total variation on $[a,b]$. If $I\subset R$ is an interval, then $\mathrm{BV}(I,R^{n\times m})$ is the set of all matrix-functions $X:I\to R^{n\times m}$ such that $X\in \mathrm{BV}([c,d],R^{n\times m})$ for every $c,d\in I$. $X(t-)=(x_{ik}(t-))_{i,k=1}^{n,m}$ and $X(t+)=(x_{ik}(t+))_{i,k=1}^{n,m}$ are the left and the right limits of X at the point $t\in[a,b]$ ($X(a-)=X(a)$, $X(b+)=X(b)$), $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

If $g:[a,b]\to R$ is a nondecreasing function, $x:[a,b]\to R$ and $a\le s<t\le b$, then

$$\int\limits_s^t x(\tau)\,dg(\tau)=\int\limits_{]s,t[} x(\tau)\,dg(\tau)+x(t)d_1g(t)+x(s)d_2g(s),$$

where $\int\limits_{]s,t[} x(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure μ_g corresponding to the function g (if $s=t$, then $\int\limits_s^t x(\tau)dg(\tau)=0$).

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If $g_j : [a,b] \to R$ $(j = 1,2)$ are nondeacreasing functions, $g = g_1 - g_2$ and $x : [a,b] \to R$, then

$$\int_s^t x(\tau)dg(\tau) = \int_s^t x(\tau)\, dg_1(\tau) - \int_s^t x(\tau)dg_2(\tau) \quad \text{for} \quad a \le s \le t \le b.$$

If $G = (g_{ik})_{i,k=1}^n \in \mathrm{BV}([a,b], R^{n\times n})$, $x = (x_k)_{k=1}^n \in \mathrm{BV}([a,b], R^n)$, then

$$\int_s^t dG(\tau)\cdot x(\tau) = \Big(\sum_{k=1}^n \int_s^t x_k(\tau)dg_{ik}(\tau)\Big)_{i=1}^n \quad \text{for} \quad a \le s \le t \le b.$$

Let $I \subset [a,b]$ be an interval and $A \in \mathrm{BV}(I, R^{n\times n})$. A vector-function is said to be a solution of the system of the linear generalized ordinary differential equations $dx(t) = dA(t)\cdot x(t) + dq(t)$ (inequalities $dx(t) \le dA(t)\cdot x(t) + dq(t)$) on I if

$$x(t) - x(s) - \int_s^t dA(\tau)\cdot x(\tau) - q(t) + q(s) = 0 \ \ (\le 0) \quad \text{for} \quad s \le t \ \ (s,t \in I).$$

Theorem 1. *Let c_{ik} $(i \ne k;\ i,k = 1,\dots,n)$ be functions nondecreasing on $[a,b]$, $C(t) = (c_{ik}(t))_{i,k=1}^n$,*

$$\det\Big(I_n + (-1)^j d_j C(t)\Big) \ne 0 \quad \textit{for} \quad (-1)^j(t-t_0) \ge 0 \ \ (j = 1,2), \tag{4}$$

$$1 + d_j c_{ii}(t) > 0 \quad \textit{for} \quad (-1)^j(t-t_0) \ge 0 \quad (j = 1,2;\ \ i = 1,\dots,n), \tag{5}$$

and

$$\sum_{i=1}^n d_j c_{ik}(t) < 1 \quad \textit{for} \quad (-1)^j(t-t_0) < 0 \ \ (j = 1,2;\ \ k = 1,\dots,n). \tag{6}$$

Let, moreover, $x \in \mathrm{BV}([a,t_0[, R^n) \cap \mathrm{BV}(]t_0,b], R^n)$ be a solution of the system (1) *satisfying the condition* (2). *Then*

$$x(t) \le y(t) \quad \textit{for} \quad t \in [a,b]\setminus\{t_0\}, \tag{7}$$

where $y \in \mathrm{BV}([a,b], R^n)$ is a solution of the problem

$$dy(t) = \Big[dC(t)\cdot y(t) + dq(t)\Big]\,\mathrm{sign}(t-t_0) \quad \textit{for} \quad t \in [a,b]\setminus\{t_0\}, \tag{8}$$

$$(-1)^j d_j y(t_0) = d_j C(t_0)\cdot y(t_0) + d_j q(t_0) \quad (j = 1,2), \tag{9}$$

$$y(t_0) = c_0. \tag{10}$$

Theorem 2. *Let c_{ik} $(i,k = 1,\dots,n)$ be functions nondecreasing on $[a,b]$ and* (4) *and* (6) *hold, where $C(t) = (c_{ik}(t))_{i,k=1}^n$. Then for every solution $x \in \mathrm{BV}([a,t_0[, R^n) \cap \mathrm{BV}(]t_0,b], R^n)$ of the system* (3) *satisfying the condition* (2), *the estimate* (7) *holds, where $y_0 \in \mathrm{BV}([a,b], R^n)$ is a solution of the problem* (8)–(10).

Remark. The condition

$$\max_{k=1,\dots,n} \sum_{i=1}^n |d_j c_{ik}(t)| < 1 \quad \text{for} \quad t \in [a,b] \quad (j = 1,2)$$

guarantees the conditions (4)–(6). Moreover, in view of (4) the problem (8)–(10) has a unique solution (see [1,Theorem III.1.4]).

References

1. Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations: boundary value problems and adjoints. *Academia, Praha,* 1979.

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