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ON A NONLOCAL BOUNDARY VALUE PROBLEM FOR SECOND ORDER NONLINEAR EQUATIONS

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Below we will use the following notation:

R is the set of real numbers.

$L([a,b])$ is the set of the functions $p:]a,b[\to R$ which are Lebesgue integrable on $[a,b]$.

$L_{loc}(]a,b[)$ is the set of the functions $p:]a,b[\to R$ which are Lebesgue integrable on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

$K_0(]a,b[\times R^2)$ is the set of the functions $q:]a,b[\times R^2 \to R$ for which the mapping $t \longmapsto g(t, x_1(t), x_2(t))$ is measurable for any continuous functions $x_i:]a,b[\to R$ $(i=1,2)$.

$\sigma: L_{loc}(]a,b[) \to L_{loc}(]a,b[)$ is an operator defined by the equality

$$\sigma(p)(t) = \exp\Big[\int_{\frac{a+b}{2}}^{t} p(s)ds\Big].$$

If $\sigma(p) \in L([a,b])$, $\alpha \in [a,b]$ and $\beta \in]\alpha, b]$, then

$$\sigma_\alpha(p)(t) = \frac{1}{\sigma(p)(t)}\Big|\int_\alpha^t \sigma(p)(s)ds\Big|,$$

$$\sigma_{\alpha\beta}(p)(t) = \frac{1}{\sigma(p)(t)}\int_\alpha^t \sigma(p)(s)ds \cdot \int_t^\beta \sigma(p)(s)ds.$$

$u(s+)$ and $u(s-)$ are the limits of the function u at the point s from the right and from the left, respectively.

If $\mu: [a,b] \to R$ is a function of bounded variation, then by $\mu^*(t)$ we denote the full variation of the function μ on the segment $[a,t]$.

Under solution of the equation

$$u'' = f(t, u, u'), \tag{1}$$

where $f:]a,b[\times R^2 \to R$ satisfies the Carathéodory conditions on every compactum contained in $]a,b[\times R^2$, we understand a function $u:]a,b[\to R$ which is absolutely continuous along with its first derivative on every segment from $]a,b[$, and satisfies (1) a.e.

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In the present paper, we consider the problem of existence and uniqueness of solution of the equation (1) satisfying the boundary conditions

$$u(a+) = 0, \quad u(b-) = \int_a^b u(s)d\mu(s), \tag{2}$$

where $\mu : [a,b] \to R$ is a function of bounded variation.

Some criteria for unique solvability of the problem in a linear case are contained in [1, 2]. In the nonlinear case, a problem of the type (1), (2) has been considered in [3–5]. However, in those works μ is assumed to be a piecewise constant function ($\mu(t) = 0$ for $a \le t \le t_0$ and $\mu(t) = 1$ for $t_0 < t \le b$).

Theorems of existence and uniqueness of solution of the problem (1), (2) given in the present paper cover the case where μ is not, generaly speaking, piecewise constant, and f is not integrable in the first argument on the segment $[a,b]$, having singularities at the points $t = a$ and $t = b$.

Before passing to the formulation of basic results, let us introduce the following definitions.

Definition 1. We say that a vector-function $(p_1, p_2) :]a,b[\to R^2$ belongs to the class $U_\mu(]a,b[)$ if

$$\sigma(p_2), \ \sigma_{ab}(p_2)p_1 \in L(]a,b[)$$

and the solution u_1 of the singular Cauchy problem

$$u'' = p_1(t)u + p_2(t)u'; \quad u(a+) = 0, \quad \lim_{t \to a+} \frac{u'(t)}{\sigma(p_2)(t)} = 1$$

satisfies the conditions

$$u_1(t) > 0 \quad \text{for} \quad a < t < b, \quad u_1(b-) > \int_a^b u_1(s)d\mu^*(s).$$

Definition 2. We say that a vector-function $(p_1, p_{12}, p_{22}) :]a,b[\to R^3$ belongs to the class $V_\mu(]a,b[)$ if

$$p_{12}(t) \le p_{22} \quad \text{for} \quad a < t < b, \tag{3}$$

$$p_{i2}, \ p_1 \in L_{loc}(]a,b[), \quad i = 1,2, \tag{4}$$

$$\sigma(p_{i2}) \in L([a,b]), \quad \sigma_{ab}(p_{i2})p_1 \in L([a,b]), \quad i = 1,2, \tag{5}$$

and $(p_1, p_2) \in U_\mu(]a,b[)$ for any measurable function $p_2 :]a,b[\to R$ satisfying

$$p_{12}(t) \le p_2(t) \le p_{22}(t) \quad \text{for} \quad a < t < b.$$

Theorem 1. *On the set* $]a,b[\times R^2$*, let the inequalities*

$$\Big[f(t,x,y) - p_1(t)x - p_2(t,x,y)y\Big] \operatorname{sgn} x \ge -p(t),$$
$$p_{12}(t) \le p_2(t,x,y) \le p_{22}(t)$$

be fulfilled, where $p_2 \in K_0(]a,b[\times R^2)$ *and* $(p_1, p_2, p_{22}) \in V_\mu(]a,b[)$*. Furthermore, let* $\sigma_{ab}(p_{i2})p \in L([a,b])$ $(i = 1,2)$*, and for some point* $t_1 \in]a,b[$ *let*

$$\Big|f(t,x,y) - p_1(t)x - p_2(t,x,y)y\Big| \le p(t) \quad \text{for} \quad t_1 < t < b, \quad x \in R, \quad y \in R. \tag{6}$$

Then the problem (1), (2) *has at least one solution.*

Remark 1. Let μ be nondecreasing, the conditions (3)–(5) be fulfilled, and let $(p_1, p_{12}, p_{22}) \notin V_\mu(]a,b[)$. Then there exists a function f satisfying the conditions of Theorem 1 for which the problem (1), (2) has no solution.

Remark 2. The condition (6) can be replaced by the condition

$$\Big|f(t,x,y) - p_1(t)x - \widetilde{p}_2(t,x,y)y\Big| \le p(t) \quad \text{for} \quad t_1 < t < b, \;\; x \in R, \;\; y \in R,$$

where $\widetilde{p}_2 \in K_0(]a,b[\times R^2)$ and $p_{12}(t) \le \widetilde{p}_2(t) \le p_{22}(t,x,y)$ for $(t,x,y) \in]t_1,b[\times R^2$.

As an example, let us consider the problem (1), (2) in the case where μ increases, $\mu(b) - \mu(a) < 1$, and

$$f(t,x,y) = p_0(t) + p_1(t)x + p_2(t)y + p_3(t)x^{2n+1}|y|^k, \quad p_i \in L_{loc}(]a,b[) \;\; i = \overline{0,3},$$

where n and k are positive integers. Assume that $\lambda > 0$, $0 \le \delta < 1$, $p_1(t) \ge 0$, $p_3(t) \ge 0$, $|p_2(t)| \le \lambda + \frac{\delta}{t-a} + \frac{\delta}{b-t}$ for $a < t < b$,

$$\int\limits_a^b (s-a)(b-s)|p_i(s)|ds < +\infty, \quad i = 0,1,$$

and $p_3(t) \equiv 0$ in a neighborhood of the point b. Taking into account Theorem 1.2 in [2], we obtain from Theorem 1 that in this case the problem (1), (2) has at least one solution. As it is seen from the example, the function f may have nonintegrable singularities for $t = a$ and $t = b$.

Corollary 1. *On the set* $]a,b[\times R^2$, *let the inequality*

$$f(t,x,y)\operatorname{sgn} x \ge p_1(t)|x| - p_2(t)|y| - p(t) \tag{7}$$

be fulfilled, where $(p_1, -p_2, p_2) \in V_\mu(]a,b[)$ *and*

$$\sigma_{ab}((-1)^i p_2)p \in L([a,b]), \quad i = 1,2.$$

Furthermore, let for some point $t_1 \in]a,b[$

$$\Big|f(t,x,y) - p_1(t)x - p_2(t)y\Big| \le p(t) \quad \textit{for} \quad t_1 < t < b, \;\; x \in R, \;\; y \in R. \tag{8}$$

Then the problem (1), (2) *has at least one solution.*

Corolary 2. *Let there exist numbers* $\lambda_i \in [0,1[$, $l_i \in [0,+\infty[$, $\gamma_i \in [0,+\infty[$, $i = 1,2$, $c \in]a,b[$ *and the function* $p :]a,b[\to]0,+\infty[$ *such that*

$$\int\limits_0^{+\infty} \frac{ds}{l_1 + l_2 s + s^2} \ge \frac{(c-a)^{1-\lambda_1}}{1-\lambda_1}, \quad \int\limits_{\gamma_1}^{\gamma_2} \frac{ds}{l_1 + l_2 s + s^2} \ge \frac{(b-c)^{1-\lambda_2}}{1-\lambda_2},$$

$$\int\limits_{\gamma_1}^{\gamma_2} \frac{sds}{l_1 + l_2 s + s^2} < -\ln(\mu^*(b) - \mu^*(a)),$$

let the function $t \longmapsto (t-a)(b-t)p(t)$ *be summable on* $[a,b]$, *and on the set* $]a,b[\times R^2$ *let the inequality* (7) *be fulfilled, where*

$$p_1(t) = \begin{cases} -l_1(t-a)^{-2\lambda_1} & \textit{for} \;\; a < t \le c \\ -l_1(b-t)^{-2\lambda_2} & \textit{for} \;\; c < t < b \end{cases},$$

$$p_2(t) = \begin{cases} l_2(t-a)^{-\lambda_1} + \lambda_1(t-a)^{-1} & \textit{for} \;\; a < t \le c \\ l_2(b-t)^{-\lambda_2} + \lambda_2(b-t)^{-1} & \textit{for} \;\; c < t < b \end{cases}.$$

Moreover, let (8) *be fulfilled for some point* $t_1 \in]a,b[$ *. Then the problem* (1), (2) *has at least one solution.*

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