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ON THE SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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In the present note, basing on the results of our previous work [1] we establish sufficient conditions for the existence and uniqueness of a solution of the boundary value problem

$$\frac{dx(t)}{dt} = g\Big(t, x(\tau_1(t)), \dots, x(\tau_m(t))\Big), \tag{1}$$

$$h(x) = 0, \tag{2}$$

where $g : [a,b] \times R^{nm} \to R^n$ is a vector function satisfying the local Carathéodory conditions, $\tau_i : [a,b] \to [a,b]$ $(i = 1,\dots,m)$ are measurable functions and $h : C([a,b]; R^n) \to R^n$ is a continuous operator.

Under solution of the system (1) we understand an absolutely continuous vector function $x : [a,b] \to R^n$ which almost everywhere on $[a,b]$ satisfies it, and under solution of the problem (1), (2) we mean a solution of the system (1) which satisfies the condition (2).

The use is made of the following notation:

$I = [a,b]$, $R =]-\infty,+\infty[$, $R_+ = [0,+\infty[$;

R^n – the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with $x_i \in R$ $(i = 1,\dots,n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$R^{n\times n}$ – the space of $n \times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with $x_{ik} \in R$ $(i,k = 1,\dots,n)$ and the norm

$$\|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

$$R_+^n = \Big\{(x_i)_{i=1}^n \in R^n : \ x_i \ge 0 \ (i = 1,\dots,n)\Big\},$$

$$R_+^{n\times n} = \Big\{(x_{ik})_{i,k=1}^n \in R^{n\times n} : \ x_{ik} \ge 0 \ \ (i,k = 1,\dots,n)\Big\};$$

if $x, y \in R^n$ and $X, Y \in R^{n\times n}$, then

$$x \le y \Longleftrightarrow y - x \in R_+^n \quad \text{and} \quad X \le Y \Longleftrightarrow Y - X \in R_+^{n\times n};$$

if $x = (x_i)_{i=1}^n \in R^n$ and $X = (x_{ik})_{i,k=1}^n \in R^{n\times n}$, then

$$|x| = (|x_i|)_{i=1}^n, \quad |X| = (|x_{ik}|)_{i,k=1}^n;$$

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$C(I;R^n)$ – the space of continuous vector functions* $x : I \to R^n$ with the norm

$$\|x\|_C = \max\{\|x(t)\| : t \in I\};$$

$$C(I;R^n_+) = \Big\{x \in C(I;R^n) : \ x(t) \geq 0 \ \text{ for } \ a \leq t \leq b\Big\};$$

$L(I;R^n)$ – the space of summable vector functions $x : I \to R^n$ with the norm

$$\|x\|_L = \int_a^b \|x(t)\|\,dt.$$

Definition 1. Let $\mathcal{P} : I \times R^{n_0} \to R^{n\times n}$ be a matrix function satisfying the local Carathéodory conditions. We say that a summable matrix function $\mathcal{P}_0 : I \to R^{n\times n}$ belongs to the set $\mathcal{E}^{n_0}_{\mathcal{P}}$ if there exists a sequence $u_k \in C(I;R^{n_0})$ $(k = 1,2,\dots)$ such that

$$\lim_{k\to\infty} \int_a^t \mathcal{P}(s,u_k(s))\,ds = \int_a^t \mathcal{P}_0(s)\,ds \quad \text{uniformly on} \quad I.$$

Definition 2. Let $l : C(I;R^{n_0}) \times C(I;R^n) \to R^n$ be a continuous operator. We say that a linear operator $l_0 : C(I;R^n) \to R^n$ belongs to the set $\mathcal{E}^{n_0}_l$ if there exists a sequence $u_k \in C(I;R^{n_0})$ $(k = 1,2,\dots)$ such that

$$\lim_{k\to\infty} l(u_k,v) = l_0(v) \quad \text{for} \quad v \in C(I;R^n).$$

Definition 3. An operator $h_0 : C(I;R^n_+) \to R^n_+$ is said to be positively homogeneous if for any $\lambda \in R_+$ and $u \in C(I;R^n_+)$ we have $h_0(\lambda u) = \lambda h_0(u)$. However, if for any $u,v \in C(I;R^n_+)$ satisfying $u(t) \leq v(t)$ for $t \in I$ the inequality $h_0(u) \leq h_0(v)$ is fulfilled, then h_0 is said to be nondecreasing.

Definition 4. Let $Q_k : I \to R^{n\times n}_+$ $(k = 1,\dots,m)$ be summable matrix functions, $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,m)$ be measurable functions and $h_0 : C(I;R^n_+) \to R^n_+$ be a positively homogeneous continuous nondecreasing operator. Then the writing

$$(\mathcal{P}_1,\dots,\mathcal{P}_m;l) \in O^{n_1,n_2}_{Q_1,\dots,Q_m;\tau_1,\dots,\tau_m;h_0}$$

means that

(i) $\mathcal{P}_k : I \times R^{n_1} \to R^{n\times n}$ $(k = 1,\dots,n)$ are matrix functions satisfying the local Carathéodory conditions and $l : C(I;R^{n_2}) \times C(I;R^n) \to R^n$ is a continuous operator; moreover, $l(u,\cdot) : C(I;R^n) \to R^n$ is linear for arbitrarily fixed $u \in C(I;R^{n_2})$.

(ii) there exist a summable function $\alpha : I \to R_+$ and a positive number α_0 such that the inequalities

$$\|\mathcal{P}_k(t,x)\| \leq \alpha(t) \quad (k = 1,\dots,m) \quad \text{and} \quad \|l(u,v)\| \leq \alpha_0\|v\|_C$$

are fulfilled on $I \times R^{n_1}$ and $C(I;R^{n_2}) \times C(I;R^n)$, respectively;

(iii) for any $\mathcal{P}_{0k} \in \mathcal{E}^{n_1}_{\mathcal{P}}$ $(k = 1,\dots,m)$ and $l_0 \in \mathcal{E}^{n_2}_l$, the problem

$$\left|\frac{dv(t)}{dt} - \sum_{k=1}^m \mathcal{P}_{0k}(t)v(\tau_k(\tau))\right| \leq \sum_{k=1}^m Q_k(t)|v(\tau_k(t)|, \quad |l_0(v)| \leq h_0(|v|)$$

has only the trivial solution.

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*A vector or matrix function is said to be continuous, absolutely continuous, summable, etc., if all its components have such a property.

Theorem 1. *Let on $I \times R^{mn}$ the inequality*

$$\Big|g(t,x_1,\dots,x_m) - \sum_{k=1}^{m} \mathcal{P}_k(t,x_1,\dots,x_m)x_k\Big| \le \sum_{k=1}^{m} Q_k(t)|x_k| + \eta(t) \tag{3}$$

and on $C(I;R^n)$ the inequality

$$\Big|h(x) - l(x,x)\Big| \le h_0(|x|) + \eta_0 \tag{4}$$

be fulfilled, where $Q_k : I \to R_+^{n\times n}$ $(k = 1,\dots,m)$ are summable matrix functions, $\eta : I \to R_+^n$ is a summable vector function, $h_0 : C(I;R_+^n) \to R_+^n$ is a positively homogeneous continuous nondecreasing operator, $\eta_0 \in R_+^n$ and

$$(\mathcal{P}_1,\dots,\mathcal{P}_m;l) \in O^{mn,n}_{Q_1,\dots,Q_m;\tau_1,\dots,\tau_m;h_0}. \tag{5}$$

Then the problem (1), (2) *has at least one solution.*

Scheme of the proof. For any $x,y \in C(I;R^n)$ and $u \in C(I;R_+^n)$, put $f(x)(t) = g(t,x(\tau_1(t),\dots,x(\tau_m(t))),\ p(x,y)(t) = \sum_{k=1}^{m} \mathcal{P}_k(t,x_1(\tau_1(t),\dots,x_m(\tau_m(t)))y(\tau_k(t))$ and $q_0(u)(t) = \sum_{k=1}^{m} Q_k(t)u(\tau_k(t))$. Then the system (1) and the condition (3) take respectively the form

$$\frac{dx(t)}{dt} = f(x)(t), \tag{1'}$$

$$\Big|f(x)(t) - p(x,x)(t)\Big| \le q_0(|x|)(t) + \eta(t). \tag{3'}$$

Owing to the restrictions imposed on g and τ_k $(k = 1,\dots,n)$, the operator $f : C(I;R^n) \to L(I;R^n)$ is continuous. On the other hand, by Definition 4 and also by Definition 1.3 of [1], we can show that the condition (5) implies the condition

$$(p,l) \in O^n_{q_0,h_0}. \tag{5'}$$

By Theorem 1.1 from [1], the conditions (3′), (4) and (5′) ensure the solvability of the problem (1′), (2). ■

According to Theorem 1, we can easily prove

Theorem 2. *Let on $I \times R^{mn}$ the inequality*

$$\Big|g(t,x_1,\dots,x_m) - g(t,y_1,\dots,y_m) - \sum_{k=1}^{m} \mathcal{P}_k(t,x_1,\dots,x_m,y_1,\dots,y_m)(x_k - y_k)\Big| \le$$
$$\le \sum_{k=1}^{m} Q_k(t)|x_k - y_k|,$$

and on $C(I;R^n)$ the inequality

$$\Big|h(x) - h(y) - l(x,y,x-y)\Big| \le h_0(|x-y|)$$

be fulfilled, where $Q_k : I \to R_+^{n\times n}$ $(k = 1,\dots,m)$ are summable matrix functions, $h_0 : C(I;R_+^n) \to R_+^n$ is a positively homogeneous continuous nondecreasing operator and

$$(\mathcal{P}_1,\dots,\mathcal{P}_m;l) \in O^{2mn,2n}_{Q_1,\dots,Q_m;\tau_1,\dots,\tau_m;h_0}.$$

Then the problem (1), (2) *has a unique solution.*

In the case where the matrix functions $\mathcal{P}_k$ $(k = 1,\dots,m)$ depend only on t and $l : C(I;R^n) \to R^n$ is a linear operator, Theorems 1 and 2 will respectively take the following form.

Corollary 1. *Let on $I \times R^{mn}$ the inequality*

$$\Big|g(t,x_1,\dots,x_m) - \sum_{k=1}^{m} \mathcal{P}_k(t)x_k\Big| \le \sum_{k=1}^{m} Q_k(t)|x_k| + \eta(t)$$

and on $C(I;R^n)$ the inequality

$$\Big|h(x) - l(x)\Big| \le h_0(|x|) + \eta_0$$

be fulfilled, where $\mathcal{P}_k : I \to R^{n\times n}$, $Q_k : I \to R_+^{n\times m}$ $(k = 1,\dots,m)$ are summable matrix functions, $\eta : I \to R_+^n$ is a summable vector function, $l : C(I;R^n) \to R^n$ is a linear bounded operator and $h_0 : C(I;R_+^n) \to R_+^n$ is a positively homogeneous continuous nondecreasing operator. Let, moreover, the problem

$$\Big|\frac{dv(t)}{dt} - \sum_{k=1}^{m} \mathcal{P}_k(t)v(\tau_k(t))\Big| \le \sum_{k=1}^{m} Q_k(t)|v(\tau_k(t))|, \quad |l(v)| \le h_0(|v|) \tag{6}$$

have only the trivial solution. Then the problem (1), (2) *has at least one solution.*

Corollary 2. *Let on $I \times R^{mn}$ the inequality*

$$\Big|g(t,x_1,\dots,x_m) - g(t,y_1,\dots,y_m) - \sum_{k=1}^{m} \mathcal{P}_k(t)(x_k - y_k)\Big| \le \sum_{k=1}^{m} Q_k(t)|x_k - y_k|$$

and on $C(I;R^n)$ the inequality

$$\Big|h(x) - h(y) - l(x-y)\Big| \le h_0(|x-y|)$$

be fulfilled, where $\mathcal{P}_k : I \to R^{n\times n}$, $Q_k : I \to R_+^{n\times n}$ $(k = 1,\dots,m)$ are summable matrix functions, $l : C(I;R^n) \to R^n$ is a linear bounded operator and $h_0 : C(I;R_+^n) \to R_+^n$ is a positively homogeneous continuous nondecreasing operator such that the problem (6) *has only the trivial solution. Then the problem* (1), (2) *has a unique solution.*

Consider now the case where the boundary conditions (2) have the form

$$\varphi\Big(x(t_1),\dots,x(t_{m_0})\Big) = 0, \tag{7}$$

where $\varphi : R^{m_0 n} \to R^n$ is a continuous vector function and $t_k \in I$ $(i = 1,\dots,m_0)$.

For the problem (1), (7), we have from Theorem 2 the following

Corollary 3. *Let:* (i) *for almost all $t \in I$ there exist $\frac{\partial g(t,x_1,\dots,x_m)}{\partial x_k}$* $(k = 1,\dots,m)$ *which are continuous with respect to $x_1,\dots,x_m$ in R^{mn} and satisfy*

$$\mathcal{P}_{1k}(t) \le \frac{\partial g(t,x_1,\dots,x_m)}{\partial x_k} \le \mathcal{P}_{2k}(t) \quad (k = 1,\dots,m),$$

where $\mathcal{P}_{1k}$ and $\mathcal{P}_{2k} : I \to R^{n\times n}$ $(k = 1,\dots,m)$ are summable matrix functions;

(ii) *the vector function φ have the first order continuous partial derivatives and*

$$A_{1k} \le \frac{\partial \varphi(x_1,\dots,x_{m_0})}{\partial x_k} \le A_{2k} \quad (k = 1,\dots,m_0)$$

on $R^{m_0 n}$, where A_{1k} and $A_{2k} \in R^{n\times n}$ $(k = 1,\dots,m_0)$; (iii) *for any summable matrix functions $\mathcal{P}_k : I \to R^{n\times n}$ $(k = 1,\dots,m)$ and matrices $A_k \in R^{n\times n}$ $(k = 1,\dots,m_0)$ satisfying*

$$\mathcal{P}_{1k}(t) \le \mathcal{P}_k(t) \le \mathcal{P}_{2k}(t) \quad \text{for} \quad t \in I \quad (k = 1,\dots,m),$$
$$A_{1k} \le A_k \le A_{2k} \quad (k = 1,\dots,m_0),$$

the boundary value problem

$$\frac{dv(t)}{dt} = \sum_{k=1}^{m} \mathcal{P}_k(t)v(\tau_k(t)), \quad \sum_{k=1}^{m_0} A_k v(t_k) = 0$$

have only the trivial solution. Then the problem (1), (7) *has a unique solution.*

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