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CRITICAL CASE OF MULTIPLE PAIRS OF PURE IMAGINARY ROOTS OF A NONAUTONOMOUS ESSENTIALLY NONLINEAR n-th ORDER EQUATION

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We investigate the asymptotic stability in the Lyapunov sense as $t \uparrow \omega$ of the zero solution of a differential equation of the form

$$y^{(n)} + \sum_{k=1}^{n-1} p_k(t) \cdot y^{(n-k)} + p_n(t) \cdot y = F\Big(t, y, y', \dots, y^{(n-1)}\Big), \tag{1}$$

where $t \in \Delta \equiv [a_0, \omega[$, $-\infty < a_0 < \omega \le +\infty$, $p_s : \Delta \to R$, $R \equiv]-\infty, +\infty[$, $p_s \equiv \pi^s \cdot a_s$, $\pi : \Delta \to R_+$, $R_+ =]0, +\infty[$, $a_s \equiv a_{s0} + o_s(1)$, $a_s^{(1)} = o_{sl}(1)$ as $t \uparrow \omega$, $a_{s0} \in R$, $l \in \{\overline{1, h}\}$, $h \in N$, $N \equiv \{1, 2, \dots\}$, $s = \overline{1, n}$, and the following conditions are fulfilled:

(1) the equation $P_n(\lambda) \equiv \lambda^n + \sum_{s=1}^n a_{s0} \cdot \lambda^{n-s} = 0$ possesses $2 \cdot n_0$, $1 \le n_0 \le \left[\frac{1}{2} \cdot n\right]$ roots λ_0 satisfying $\operatorname{Re} \lambda_0 = 0$; the remaining roots λ possess the property $\operatorname{Re} \lambda < 0$;

(2) $F(t, X) \equiv \sum_{|Q|=2}^m F_Q(t) \cdot X^Q + R_m(t, X)$, $X \equiv (x_1, \dots, x_n)$, $F : \Delta \times S(X, r) \to R$, $S(X, r) \equiv \{X, X^T : \|X\| \le r\}$, $r \in R_+$, $Q = (q_1, \dots, q_n)$, $q_k \in \{0, N\}$, $k = \overline{1, n}$, $\|Q\| = \sum_{k=1}^n q_k$, $X^Q \equiv \prod_{k=1}^n x_k^{q_k}$, $F_Q \in C_\Delta^k$, $h \in N$, $\|Q\| = \overline{2, m}$, $m \in N \setminus \{1\}$, $|R_m| \le L \cdot \left(\sum_{k=1}^n |x_k|\right)^{m+a}$, $L : \Delta \to [0, +\infty[$, $L \in C_\Delta$, $a \in R_+$.

Below the use is made of the following definitions and notation:

Definition 1. The differential equation (1) possesses the property St as $t \uparrow \omega$ if for any arbitrarily small $\varepsilon \in R_+$ there exist $\delta_\varepsilon \in]0, \varepsilon]$ and $T_\varepsilon \in \Delta$ such that any solution $y = y(t)$ of (1) satisfying $|y(T_\varepsilon)| < \delta_\varepsilon \cdot \pi(T_\varepsilon)$, $|y^{(s-1)}(T_\varepsilon)| < \delta_\varepsilon \cdot \pi^s(T_\varepsilon)$, $s = \overline{2, n}$, possesses the property $|y(t)| < \varepsilon_\varepsilon \cdot \pi$, $|y^{(s-1)}(t)| < \varepsilon \cdot \pi^s$ for all $t \in [T_\varepsilon, \omega[$, $s = \overline{2, n}$.

Definition 2. The differential equation (2) possesses the property $AsSt$ as $t \uparrow \omega$ if Definition 1 is fulfilled, and $\pi^{-1} \cdot y(t) = o(1)$ and $\pi^{-s} \cdot y^{(s-1)}(t) = o(1)$ as $t \uparrow \omega$, $s = \overline{2, n}$.

Definition 1′. The differential system

$$Y' = f(t, Y), \;\; Y \equiv \operatorname{col}(y_1, \dots, y_n), \;\; f(t, \overline{0}) \equiv \overline{0}, \;\; \overline{0} \equiv \operatorname{col}(0, \dots, 0), \tag{2}$$

possesses the property St as $t \uparrow \omega$ if for any arbitrarily small $\varepsilon \in R$ there exist $\delta_\varepsilon \in]0, \varepsilon]$ and $T_\varepsilon \in \Delta$ such that any solution $Y = Y(t)$ of the differential system (2) with the condition $\|Y(T_\varepsilon)\| < \delta_\varepsilon$ possesses the property $\|Y(t)\| < \varepsilon$ for all $t \in [T_\varepsilon, \omega[$.

For $\omega < +\infty$, the property St of the differential system (2) is defined by a rephrasing of this property for $\omega = +\infty$ [2, p. 168].

Definition 2′. The differential system (2) possesses the property $AsSt$ as $t \uparrow \omega$ if Definition 1′ is fulfilled, and $\|Y(t)\| = o(1)$ as $t \uparrow \omega$.

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E_k, H_k are respectively the unit and the displacement matrices of the dimension $k\times k$; Y_k is a column vector of the dimension k;

$$Y^{-1}\equiv \operatorname{col}(y_1^{-1},\dots,y_n^{-1}),\quad Z=\operatorname{col}(z_1,\dots,z_n)=\operatorname{col}(Z_{n_1},\dots,Z_{n_k}),$$

$$Y\cdot Z\equiv \operatorname{col}(y_1\cdot z_1,\dots,y_n\cdot z_n),\quad <Y,Z>\equiv \sum_{k=1}^{n} y_k z_k,$$

$$\|Y\|^2\equiv \sum_{k=1}^{n}|y_k|^2,\quad \operatorname{grad} V(t,Y)\equiv \operatorname{col}\Big(\frac{\partial V}{\partial y_1},\dots,\frac{\partial V}{\partial y_n}\Big),$$

$$E_k^T\equiv(0,\dots,0,\underset{k}{1},0,\dots,0);\quad L_\Delta\equiv\Big\{f:\Delta\to R,\int^{\omega}|f|\cdot dt<+\infty\Big\};$$

$$\Lambda\equiv \max i\big\{f_s:\Delta\to R; s=\overline{1,n}\big\},\quad \text{for }\ \Lambda:\Delta\to R_+,$$

$$\Lambda^{-1}\cdot f_s=c_s+o_s(1)\ \text{ as }\ t\uparrow\omega,\quad c_s\in R,\quad \sum_{s=1}^{n}|c_s|>0.$$

The results of this paper are effectively applied to the differential equation (1) whose coefficients are slowly varying functions, i.e., the functions whose derivatives are small as $t\uparrow\omega$ in comparison with the functions themselves. For example,

$$p_k\equiv t^{k\cdot\beta}\cdot\Big[a_{k0}+b_k\cdot t^{-a_k}\cdot(\ln t)^{\beta_k}\cdot\sin t^{\gamma_k}\Big],$$
$$F_Q\equiv t^{\gamma\cdot\|Q\|}\cdot\Big[F_{Q0}+g_Q\cdot t^{-a_Q}\cdot(\ln t)^{\beta_Q}\cdot\sin t^{\gamma_Q}\Big],$$
$$k,\beta,a_{k0},b_{k0},\beta_k,\gamma,F_{Q0},g_Q,\beta_Q\in R,\quad a_k,a_Q\in\{0,R_+\},$$
$$\gamma_k,\gamma_Q\in]0,1],\quad k=\overline{1,n},\quad \|Q\|=\overline{2,m}.$$

Lemma. *If $\pi'\cdot\pi^{-2}=o(1)$ as $t\uparrow\omega$, then the transformation $y=\pi\cdot y_1$, $y^{(s)}=\pi^{s+1}\cdot y_{s+1}$, $s=\overline{1,n-1}$, reduces the differential equation (1) to that of the kind*

$$Y'=\pi\cdot P\cdot Y+G, \tag{3}$$

where $P=\|p_{sk}\|$, $s,k=\overline{1,n}$, $p_{ss}\equiv -s\cdot\pi'\cdot\pi^{-2}$, $s=\overline{1,n-1}$, $p_{nn}\equiv -a_1-n\cdot\pi'\cdot\pi^{-2}$, $p_{s,s+1}\equiv 1$, $s=\overline{1,n-1}$, $p_{sk}\equiv 0$,

$$s=\overline{1,n-2},\quad k=\overline{s+2,n},\quad p_{sk}\equiv 0,\quad s=\overline{2,n-1},\quad k=\overline{s-1,n-2},$$
$$p_{nk}\equiv -a_{n-k+1},\quad k=\overline{1,n-1},\quad G\equiv\operatorname{col}(\overline{0},G_n).$$
$$G_n\equiv\sum_{|Q|=2}^{m} f_Q\cdot\pi^{-n+\sum_{s=1}^{n}s\cdot q_s}\cdot Y^Q+R_m^*,$$
$$|R_m^*|\le\Big(\sum_{k=1}^{n}\pi^k\Big)^{m+a}\cdot\pi^{-n}\cdot L\cdot\Big(\sum_{k=1}^{n}|y_k|\Big)^{m+a},$$
$$\det[P(\omega)-\lambda\cdot E_n]\equiv P_n(\lambda).$$

The proof of the lemma is obvious.

Assume first that using the generalized "shearing" [3], "frozen" [4] and K.P. Persidsky's methods of transformations, we can construct a nondegenerate substitution

$y_s = h_s(t, Z)$, $h_s(t, \overline{0}) \equiv 0$, $s = \overline{1,n}$, which reduces the differential system (3) to that of the special form

$$\begin{cases} Z'_{n_s} = \pi_s \cdot (i \cdot \mu_s \cdot E_{n_s} + \Omega_{n_s}) \cdot Z_{n_s} + \Phi_{n_s}, \\ i^2 = -1, \quad s = \overline{1,k_0}, \quad \sum\limits_{s=1}^{k_0} n_s = n_0 \\ Z'_{n-2\cdot n_0} = \pi \cdot P_{n-2\cdot n_0} \cdot Z_{n-2\cdot n_0} + \Phi_{n-2\cdot n_0}, \end{cases} \tag{4}$$

where $\pi_s : \Delta \to R_+$, $\mu_s \in R_+$, $s = \overline{1,k_0}$ are known numbers, $\|\Omega_{n_s}\| = o(1)$ as $t \uparrow \omega$, $s = \overline{1,k_0}$, the roots μ of the equation $\det(P_{n-2\cdot n_0} - \mu \cdot E_{n-2\cdot n_0}) = 0$ possess the property $\operatorname{Re}\mu : \Delta \to]-\infty, -\gamma]$, $y \in R_+$; $s = \overline{1,k_0}$, $\Phi_{n-2\cdot n_0}$ are small in a sense.

Theorem 1. *Let the differential equation* (1) *be such that*

(1) $\pi' \cdot \pi^{-2} = o(1)$ *as* $t \uparrow \omega$, *and the transformation* $y = \pi \cdot h_1(t, Z)$, $y^{(s)} = \pi^{s+1} \times h_{s+1}(t, Z)$, $s = \overline{1, n-1}$, *reduces the differential equation* (1) *to the differential system* (4) *with* $\pi_s \cdot \|\operatorname{Re}\Omega_{n_s}\| \in L_\Delta$, $s = \overline{1,k_0}$;

(2) *for all* $Z \in S(Z, r)$, *it holds* $\|\Phi_{n_s}\|$, $\|\Phi_{n-2\cdot n_0}\| \in L_D$, $s = \overline{1,k_0}$, $h_k(t, Z) = o(1)$ *as* $t \uparrow \omega$, $k = \overline{1,n}$.

Then the differential equation (1) *possesses the property* $AsSt$ *as* $t \uparrow \omega$.

Proof. Consider the differential system (4) in terms of the quasi-linear differential system and apply the results of [6]. □

Assume now that by the methods of "shearing" [3] and "frozen" [4] transformations we can construct a nondegenerate change of variables

$$y_k = f_s(t, Z) \equiv \sum_{|Q|=2}^{m} f_{sQ} \cdot Z^Q, \quad s = \overline{1,n},$$

reducing the differential system (3) to that of the special form

$$\begin{cases} Z'_{n_s} = \pi_s \cdot (i \cdot \mu_s \cdot E_{n_s} + H_{n_s}) \cdot Z_{n_s} + \\ \quad + \sum\limits_{\|Q_{n_s} + L_{n_s}\|=2}^{m} g_{n_s, Q_{n_s}, L_{n_s}} \cdot Z_{n_s}^{Q_{n_s}} \cdot \overline{Z}_{n_s}^{l_{n_s}} + \Theta_{n_s}, \\ i^2 = -1, \ \sum\limits_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot (\|Q_{n_k} - L_{n_k}\|) + \mu_s \cdot \pi_s = 0, \quad s = \overline{1,k_0}, \quad \sum\limits_{s=1}^{k_0} n_s = n_0, \\ Z'_{n-2\cdot n_0} = \pi \cdot P_{n-2\cdot n_0} \cdot Z_{n-2\cdot n_0} + \\ + Z_{n-2\cdot n_0} \cdot \sum\limits_{\|Q_{n_s}\|=1}^{m-1} g_{n-2\cdot n_0} \cdot Z_{n_s}^{Q_{n_s}} \overline{Z}_{n_s}^{Q_{n_s}} + \Theta_{n-2\cdot n_0}, \end{cases} \tag{5}$$

where $\pi_s : \Delta \to R_+$, $\mu_s \in R_+$, $g_{n_s, Q_{n_s}, L_{n_s}}$, $g_{n-2\cdot n_0, Q_{n_s}}$ are known values, the equation $\det(P_{n-2\cdot n_0} - \mu \cdot E_{n-2\cdot n_0}) = 0$ possesses only the roots μ with the property $\operatorname{Re}\mu : \Delta \to]-\infty, \gamma]$, Θ_{n_s}, $\Theta_{n-2\cdot n_0}$ are small in a sense.

Select from the differential system (5) that of the form

$$\begin{cases} Z'_{n_s} = \pi_s \cdot (i \cdot \mu_s \cdot E_{n_s} + H_{n_s}) \cdot Z_{n_s} + \sum\limits_{\|Q_{n_s} + L_{n_s}\|=2}^{m} g_{n_s, Q_{n_s}, L_{n_s}} \cdot Z_{n_s}^{Q_{n_s}} \cdot \overline{Z}_{n_s}^{l_{n_s}}, \\ \sum\limits_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot (\|Q_{n_k}\| - \|L_{n_k}\|) + \mu_s \cdot \pi_s = 0, \quad s = \overline{1,k_0}, \quad \sum\limits_{s=1}^{k_0} n_s = n_0. \end{cases} \tag{6}$$

Assume that the differential system (6) can be substituted by an equivalent $2 \cdot n_0$-th order differential equation with respect to one of the components of the vector $\operatorname{col}(Z_{n_s}, \dots, Z_{n_{k_0}})$. Then, using a method presented in [7], we can obtain asymptotic representation of all proper solutions of the above obtained differential equation. Denote by $\Psi_{n_s} = \Psi_{n_s}(t)$, $s = \overline{1, k_0}$, an asymptotic representation of one of the proper solutions of the differential system (6).

Theorem 2. *Let the differential equation* (1) *be such that*

(1) $\pi' \cdot \pi^{-2} = o(1)$ *as* $t \uparrow \omega$, *and the transformation* $y = \pi \cdot f_1(t, Z)$, $y^{(s)} = \pi^{s+1} \cdot f_{s+1}(t, Z)$, $s = \overline{1, n-1}$, *reduces the differential equation* (1) *to* (5) *with* $\|P'_{n-2\cdot n_0}\| \cdot \pi^{-1} = o(1)$ *as* $t \uparrow \omega$;

(2) *there exists an asymptotic representation of one of the proper solutions of the differential system* (6), $\Psi_{n_s} = \Psi_{n_s}(t)$, *such that* $\|\Psi_{n_s}\| = o(1)$ *and* $\|\Psi'_{n_s} \cdot \Psi_{n_s}^{-1}\| \cdot \pi_s^{-1} = o(1)$ *as* $t \uparrow \omega$, $s = \overline{1, k_0}$;

(3) *there exist positive definite Lyapunov functions* $V = V_{n_s}(Z_{n_s})$ *such that for all* $t \in \Delta$ *and all* $\operatorname{col}(Z_{n_s}, \dots, Z_{n_s}, \overline{O}) \in S(Z, r)$

$$\operatorname{Re} < \operatorname{grad} V_{n_s}(Z_{n_s}),\ \ \pi_s \cdot \Big[(i \cdot \mu_s - \Psi'_{n_s} \cdot \Psi_{n_s}^{-1} \cdot \pi_s^{-1}) \cdot E_{n_s} + H_{n_s}\Big] \cdot Z_{n_s} +$$
$$+ \sum_{\|Q_{n_s} + L_{n_s}\| = 2}^{m} g_{n_s, Q_{n_s}, L_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}, -E_{n_s}^T} \cdot \overline{\Psi}_{n_s}^{L_{n_s}} \cdot Z_{n_s}^{Q_{n_s}} \cdot \overline{Z}_{n_s}^{L_{n_s}} > \equiv$$
$$\equiv \Lambda_s \cdot \Big[W_{0s}(Z_{n_s}) + W_{1s}(t, Z_{n_s})\Big],$$
$$\Lambda_s \equiv \max\Big\{ \Big\|g_{n_s, Q_{n_s}, L_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}, -E_{n_s}^T} \cdot \overline{\Psi}_{n_s}^{L_{n_s}}\Big\|, \sum_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot (\|Q_{n_k}\| - \|L_{n_k}\|) +$$
$$+ \mu_s \cdot \pi_s = 0,\ \|Q_{n_s} + L_{n_s}\| = \overline{2, m}\Big\},$$
$$W_{0s}(Z_{n_s}) < 0,\ \ Z_{n_s} \neq \overline{0},\ \ W_{0s}(\overline{0}) = 0,$$
$$W_{1s}(t, Z_{n_s}) = o(1),\ \ \Lambda_s^{-1} \cdot \pi_s = o(1),$$
$$\Lambda_s^{-1} \cdot g_{n-2\cdot n_0, Q_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}, -E_{n_s}^T} \cdot \overline{\Psi}_{n_s}^{L_{n_s}} = o(1) \ \ as\ \ t \uparrow \omega,\ \ s = \overline{1, k_0};$$

(4) *there exists* $\nu : \Delta \to R_+$, $\nu \in C_{\Delta}^{-1}$, *such that* $\nu = o(1)$, $\nu' \cdot \nu^{-1} \cdot \pi^{-1} = o(1)$, $t \uparrow \omega$ *and for all* $Z \in S(Z, r) \setminus \overline{O}$

$$\Big[\sum_{s=1}^{k_0} \Big\|\Theta_{n_s}(t, \Psi_{n_1} \cdot Z_{n_1}, \dots, \Psi_{n_{k_0}} \cdot Z_{n_{k_0}}, \nu \cdot Z_{n-2\cdot n_0}) \cdot \Psi_{n_s}^{-1}\Big\| +$$
$$+ \Big\|\Theta_{n-2\cdot n_0}(t, \Psi_{n_1} \cdot Z_{n_1}, \dots, \Psi_{n_{k_0}} \cdot Z_{n_{k_0}}, \nu \cdot Z_{n-2\cdot n_0}) \cdot v^{-1}\Big\|\Big] \times$$
$$\times \Big[\sum_{s=1}^{k_0} \Lambda_s \cdot W_{0s}(Z_{n_s}) - \pi \cdot \|Z_{n-2\cdot n_0}\|^2\Big]^{-1} = o(1),$$
$$f_s(t, \Psi_{n_1} \cdot Z_{n_1}, \dots, \Psi_{n_{k_0}} \cdot Z_{n_{k_0}}, v \cdot Z_{n-2\cdot n_0}) = o(1) \ \ as\ \ t \uparrow \omega,\ \ s = \overline{1, k_0}.$$

Then the differential equation (1) *possesses the property AsSt as* $t \uparrow \omega$.

Proof. In the differential system (5), we make the substitution $Z_{n_s} = \Psi_{n_s} \cdot Y_{n_s}$, $s = \overline{1, k_0}$, $Z_{n-2\cdot n_0} = v \cdot Y_{n-2\cdot n_0}$ and apply to the differential system with respect to Y_{n_s}, $s = \overline{1, k_0}$, $Y_{n-2\cdot n_0}$ the analogue of the lemma [4] on the stability in a ring-shaped domain involving the origin. □

Remark. If the coefficients of the differential equation (1) are slowly varying functions, then applying several times the method of "frozen' transformations, one can attain for fixed Z that the functions Φ_{n_s}, $s=\overline{1,k_0}$, $\Phi_{n-2\cdot n_0}$ and Θ_{n_s}, $s=\overline{1,k_0}$, $\Theta_{n-2\cdot n_0}$ in the differential systems (4) and (5), respectively, would tend rapidly enough to zero as $t\uparrow\omega$.

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