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# CRITICAL CASE OF MULTIPLE PAIRS OF PURE IMAGINARY ROOTS OF A NONAUTONOMOUS ESSENTIALLY NONLINEAR DIFFERENTIAL SYSTEM

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In the present note, we suggest a criterion of the asymptotic stability (in the Lyapunov sense) as $t \uparrow \omega$ of the trivial solution of a differential system of the kind

$$X' = F(t, X), \tag{1}$$

where $X = \operatorname{col}(x_1, \dots, x_n)$, $t \in \Delta \equiv [a_0, \omega[$, $-\infty < a_0 < \omega \le +\infty$, $F : \Delta \times S(X, r) \to R^n$, $R^n$ is the $n$-dimensional real Euclidean space, $S(X, r) \equiv \{X, X^T : \|X\| \le r; r \in R_+\}$, $R_+ \equiv ]0, +\infty[$,

$$F(t, X) \equiv \pi_1 \cdot P_1 \cdot X + \sum_{\|Q\|=2}^{m} F_Q \cdot X^Q + R_m, \;\; \pi_1 : \Delta \to R_+, \;\; P_1 = \|p_{sk}\|,$$

$$s, k = \overline{1, n}, \quad \|P_1\| : \Delta \to ]0, M], \quad M \in R_+,$$

$$F_Q \equiv \operatorname{col}(F_{1Q}, \dots, F_{nQ}), \quad F_{kQ} : \Delta \to R, \quad k = \overline{1, n}, \quad Q = (q_1, \dots, q_n),$$

$$q_k \in \{0, 1, 2, \dots\}, \quad \|Q\| = \sum_{k=1}^{n} q_k, \quad X^Q \equiv \prod_{k=1}^{n} x_k^{q_k},$$

and the following conditions are fulfilled:

(1) $\pi_1, p_{sk}, F_{kQ} \in C_\Delta^h$, $p_{sk}^{(1)} = o(1)$ as $t \uparrow \omega, s, k = \overline{1, n}$, $l \in \{\overline{1, h}\}$, $h \in N$, $\|Q\| = \overline{2, m}$;

(2) the equation $\det(P_0 - \lambda \cdot E) = 0$, $P_0 = \lim_{t \uparrow \omega} P_1$ has $2 \cdot n_0$, $1 \le n_0 \le \left[\frac{1}{2} \cdot n\right]$ roots $\lambda_0$ satisfying $\operatorname{Re} \lambda_0 = 0$, while the rest of roots $\lambda$ of the same equation has negative real parts;

(3) $\|R_m\| \le L\Big(\sum_{k=1}^{n} |x_k|\Big)^{m+a}$, $L \in C_\Delta$, $L : \Delta \to [0, +\infty[$, $a \in R_+$.

The results of this paper are effectively applied to differential systems whose coefficients are slowly varying functions, i.e., the functions whose derivatives are small as $t \uparrow \omega$ in comparison with the functions themselves. For example, $t^a$, $(\ln t)^b$, $\sin t^c$, $a, b \in R$, $c \in ]0, 1[$, $\omega = +\infty$, etc.

Below we use the following definitions and notation:

**Definition 1.** The differential system (1) possesses the property $St$ as $t \uparrow \omega$ if for every arbitrarily small $\varepsilon \in R$ there exist $\delta_k \in ]0, \varepsilon]$, $T_\varepsilon \in \Delta$ such that any solution $X = X(t)$ under the condition $\|X(T_\varepsilon)\| < \delta_\varepsilon$ possesses the property $\|X(t)\| < \varepsilon$ for all $t \in [T_\varepsilon, \omega[$.

For $\omega < +\infty$, the property $St$ of the differential system (1) is defined by a rephrasing of this property for $\omega = +\infty$ [2, p. 168].

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**Definition 2.** The differential system (1) possesses the property $AsSt$ as $t \uparrow \omega$ if Definition 1 is fulfilled, and $\|X(t)\| = o(1)$ as $t \uparrow \omega$.

$E_k, H_k$ are respectively the unit and the displacement matrices of the dimension $r \times k; Y_k$ is a vector column of the dimension $k$;

$$Y = \operatorname{col}(y_1, \ldots, y_n) = \operatorname{col}(Y_{n_1}, \ldots, Y_{n_{k_0}}, \overline{Y}_{n_1}, \ldots, \overline{Y}_{n_{k_0}}, Y_{n-2\cdot n_0}),$$

$$X^{-1} \equiv \operatorname{col}(x_1^{-1}, \ldots, x_n^{-1}), < X, Y > \equiv \sum_{k=1}^{n} x_k \cdot y_k,$$

$$\|X\|^2 \not\equiv \sum_{k=1}^{n} |x_k|^2, \quad X \cdot Y \equiv \operatorname{col}(x_1 \cdot y_1, \ldots, x_n \cdot y_n),$$

$$\operatorname{grad} V(t, X) \equiv \operatorname{col}\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right), \quad \overline{O} \equiv \operatorname{col}(0, \ldots, 0),$$

$$\Lambda \equiv \max i\{g_s : \Delta \to R; s = \overline{1, n}\}, \quad \text{for } \Lambda : \Delta \Rightarrow R_+,$$

$$\Lambda^{-1} \cdot g_s = c_s + o_s(1), \quad t \uparrow \omega, \quad c_s \in R, \quad s = \overline{1, n}, \quad \sum_{s=1}^{n} |c_s| > 0;$$

$$E_k^T \equiv (0, \ldots, 0, 1, 0, \ldots, 0).$$

Assume that by using the methods of generalized "shearing" [3] and "frozen" [4] transformations we can construct a nondegenerate substitution $X = G(t, Y)$ with $G(t, Y)$ an $m$-th degree polynomial in $Y$, $G(t, \overline{O}) \equiv \overline{O}$, which reduces the differential system (1) to that of the special kind

$$\begin{cases} Y'_{n_s} = \pi_s \cdot (i \cdot \mu_s \cdot E_{n_s} + H_{n_s}) \cdot Y_{n_s} + \\ \quad + \displaystyle\sum_{\|Q_{n_s} + L_{n_s}\| = 2}^{m} f_{n_s, Q_{n_s}, L_{n_s}} \cdot Y_{n_s}^{Q_{n_s}} \cdot \overline{Y}_{n_s}^{L_{n_s}} + \Phi_{n_s}, \\ i^2 = -1, \displaystyle\sum_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot \Big(\|Q_{n_k}\| - \|L_{n_k}\|\Big) + \mu_s \cdot \pi_s = 0, \\ \qquad s = \overline{1, k_0}, \quad \displaystyle\sum_{s=1}^{k_0} n_s = n_0, \\ Y'_{n-2\cdot n_0} = \pi_1 \cdot P_{n-2\cdot n_0} \cdot Y_{n-2\cdot n_0} + \\ \quad + Y_{n-2\cdot n_0} \cdot \displaystyle\sum_{\|Q_{n_s}\| = 1}^{m-1} g_{n-2\cdot n_0, Q_{n_s}} \cdot Y_{n_s}^{Q_{n_s}} \cdot \overline{Y}_{n_s}^{Q_{n_s}} + \Phi_{n-2\cdot n_0}, \end{cases} \tag{2}$$

where $\pi_s : \Delta \to R_+$, $\mu_s \in R_+$, $f_{n_s, Q_{n_s}, L_{n_s}}$, $\|Q_{n_s} + L_{n_s}\| = \overline{2, m}$, $g_{n-2\cdot n_0, Q_{n_s}}$, $\|Q_{n_s}\| = \overline{1, m-1}$, $s = \overline{1, k_0}$, are known values; $\|P_{n-2\cdot n_0}\| : \Delta \to ]0, M]$, the roots of the equation $\det(P_{n-2\cdot n_0} - \lambda \cdot E_{n-2\cdot n_0}) = 0$ possess the property $\operatorname{Re} \lambda : \Delta \to ]0, -\gamma]$, $y \in R_+$; $\Phi_{n_s}$, $s = \overline{1, k_0}$, $\Phi_{n-2\cdot n_0}$ are small in a sense.

For autonomous differential systems, an analogous critical case for two simple pairs of pure imaginary roots has been investigated by G.V. Kamenkov [5] and I.G. Malkin [6].

**Lemma.** *Let for a differential system of the kind*

$$X' = U(t, X), \quad t \in \Delta, \quad X \in S(X, r), \quad U(t, \overline{O}) \equiv \overline{O}, \tag{3}$$

*there exist a positively definite Lyapunov function $V = V(t, X)$ admitting an infinitely small higher limit, such that*

(1) *for all* $t \in \Delta$ *and all* $X \in S(X,r)$

$$< \mathrm{grad}\, V(t,X),\ \ U(t,X) > \equiv G_0(t,X)\cdot[1+G_1(t,X)],$$
$$G_0(t,\overline{O}) \equiv 0,\ \ G_0(t,X) < 0,\ \ X \neq \overline{O};$$

(2) *there exists* $c_0 \in R$ *such that for all* $t \in \Delta$, *it holds* $S(t,X) \equiv \{X : V(t,X) = c_0\} \in S(X,r)$*;*

(3) *for all* $X \in S(X,r) \setminus \overline{O}$, $\frac{\partial V(t,X)}{\partial t} \cdot G_0^{-1}(t,X) = o(1)$ *and* $G_1(t,X) = o(1)$ *as* $t \uparrow \omega$. *Then there exists* $T_0 \in \Delta$ *such that any solution* $X = X(t)$ *of the differential system* (3) *with the initial condition* $\|X(T_0)\| \leq \inf_{t\in\Delta,\ X\in S(t,X)} \|X\|$ *possesses the property* $\|X(t)\| \leq \sup_{t\in\Delta,\ X\in S(t,X)} \|X\|$ *for all* $t \in [T_0, \omega[$.

The proof can be performed by reductio ad absurdum.

Select from the differential system (2) that of the kind

$$\begin{cases} Y'_{n_s} = \pi_s \cdot (i\cdot\mu_s \cdot E_{n_s} + H_{n_s}) \cdot Y_{n_s} + \displaystyle\sum_{\|Q_{n_s}+L_{n_s}\|=2}^{m} f_{n_s,Q_{n_s},L_{n_s}} \cdot Y_{n_s}^{Q_{n_s}} \cdot \overline{Y}_{n_s}^{L_{n_s}}, \\ \displaystyle\sum_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot \big(\|Q_{n_s}\| - \|L_{n_s}\|\big) + \mu_s \cdot \pi_s = 0, \\ \quad s = \overline{1,k_0},\ \ n_1 + \cdots + n_{k_0} = n_0. \end{cases} \tag{4}$$

Suppose that the differential system (4) can be substituted by an equivalent $2\cdot n_0$-th order differential equation with respect to one of the components of the vector $\mathrm{col}(Y_{n_1}, ..., Y_{n_{k_0}})$. Then, using the method presented in [7], one can obtain asymptotic representations of all proper solutions of the above-obtained differential equation.

Let $\Psi_{n_s} = \Psi_{n_s}(t)$, $s = \overline{1,k_0}$, be an asymptotic representation of one of the proper solutions of the differential system (4).

**Theorem.** *Let the differential system* (1) *be such that*

(1) *the transformation* $X = G(t,Y)$ *reduces the differential system* (1) *to* (2) *in which* $\|P'_{n-2\cdot n_0}\| \cdot \pi_1^{-1} = o(1)$ *as* $t \uparrow \omega$*;*

(2) *there exists an asymptotic representation of one of the proper solutions of the differential system* (4), $\Psi_{n_s} = \Psi_{n_s}(t)$, *such that* $\|\Psi_{n_s}\| = o(1)$ *and* $\|\Psi'_{n_s} \cdot \Psi_{n_s}^{-1}\| \cdot \pi_s^{-1} = o(1)$ *as* $t \uparrow \omega$, $s = \overline{1,k_0}$*;*

(3) *there exist positive definite Lyapunov functions* $V = V_s(Y_{n_s})$ *such that for all* $t \in \Delta$ *and all* $(Y_{n_s}, \ldots, Y_{n_{k_0}}, \overline{O}) \in S(Y,r)$, *we have*

$$\mathrm{Re} < \mathrm{grad}\, V_s(Y_{n_s}),\ \pi_s \cdot \Big[(i\cdot\mu_s - \Psi'_{n_s}\cdot\Psi_{n_s}^{-1}\cdot\pi_s^{-1})\cdot E_{n_s} + H_{n_s}\Big]\cdot Y_{n_s} +$$
$$+ \sum_{\|Q_{n_s}+L_{n_s}\|=2}^{m} f_{n_s,Q_{n_s},L_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}-E_{n_s}^T} \cdot \overline{\Psi}_{n_s}^{L_{n_s}} \cdot Y_{n_s}^{Q_{n_s}} \cdot \overline{Y}_{n_s}^{L_{n_s}} > \equiv$$
$$\equiv \Lambda_s \cdot \Big[W_{0s}(Y_{n_s}) + W_{1s}(t, Y_{n_s})\Big],$$
$$\Lambda_s \equiv \max i \Big\{ \Big\| f_{n_s,Q_{n_s},L_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}-E_{n_s}^T} \cdot \overline{\Psi}_{n_s}^{L_{n_s}} \Big\|;$$
$$\sum_{k=1}^{k_0} \mu_k \cdot \pi_k \cdot \big(\|Q_{n_s}\| - \|L_{n_s}\|\big) + \mu_s\cdot\pi_s,\ \|Q_{n_s}+L_{n_s}\| = \overline{2,m} \Big\},$$
$$W_{0s}(Y_{n_s}) < 0,\ \ Y_{n_s} \neq \overline{O},\ \ W_{0s}(\overline{O}) = 0,$$
$$W_{1s}(t, Y_{n_s}) = o(1),\ \ \Lambda_s^{-1}\cdot\pi_s = o(1),$$
$$\Lambda_s^{-1} \cdot g_{n-2\cdot n_0, Q_{n_s}} \cdot \Psi_{n_s}^{Q_{n_s}} \cdot \overline{\Psi}_{n_s}^{Q_{n_s}} = o(1)\ \ as\ \ t\uparrow\omega,\ \ s = \overline{1,k_0};$$

(4) *there exists* $v : \Delta \to R_+$, $v \in C^1_\Delta$, *such that* $v = o(1)$, $v' \cdot v^{-1} \cdot \pi_1^{-1} = o(1)$ *as* $t \uparrow \omega$, *and for all* $Y \in S(Y, r) \setminus \overline{O}$, *it holds*

$$\Big[\sum_{s=1}^{k_0} \Big\|\Phi_{n_s}(t, \Psi_{n_1} \cdot Y_{n_1}, \dots, \Psi_{n_{k_0}} \cdot Y_{n_{k_0}}, v \cdot Y_{n-2\cdot n_0}) \cdot \Psi_{n_s}^{-1}\Big\| +$$
$$+ \Big\|\Phi_{n-2\cdot n_0}(t, \Psi_{n_1} \cdot Y_{n_1}, \dots, \Psi_{n_{k_0}} \cdot Y_{n_{k_0}}, v \cdot Y_{n-2\cdot n_0}) \cdot v^{-1}\Big\|\Big] \times$$
$$\times \Big[\sum_{s=1}^{k_0} \Lambda_s \cdot W_{0s}(Y_{n_s}) - \pi_1 \cdot \|Y_{n-2\cdot n_0}\|^2\Big]^{-1} = o(1),$$
$$\Big\|G(t, \Psi_{n_1} \cdot Y_{n_1}, \dots, \Psi_{n_{k_0}} \cdot Y_{n_{k_0}}, v \cdot Y_{n-2\cdot n_0})\Big\| = o(1), \quad as \ \ t \uparrow \omega.$$

*Then the differential system* (1) *possesses the property* $AsSt$ *as* $t \uparrow \omega$.

*Proof.* In the differential system (2), we make the substitution $Y_{n_s} = \psi_{n_k} \cdot X_{n_k}$, $s = \overline{1, k_0}$, $Y_{n-2\cdot n_0} = v \cdot X_{n-2\cdot n_0}$ and use the lemma for the differential system with respect to $X_{n_k}$, $s = \overline{1, k_0}$, $X_{n-2\cdot n_0}$. □

*Remark* 1. If the coefficients of the differential system (1) are slowly varying functions, then using several times the method of "frozen" $t$, one can attain that for a fixed $Y$, the functions $\Phi_{n_k}$, $s = \overline{1, k_0}$, $\Phi_{n-2\cdot n_0}$ in the differential system (2) would tend rapidly to zero as $t \uparrow \omega$.

*Remark* 2. When the differential system (2) possesses only simple pairs of pure imaginary roots, then the number of equations of the differential system (4) which determines the stability of the differential system (1), can be reduced exactly by half. This facilitates finding of asymptotic representations of proper solutions. In this case, $n_s = 1$, $s = \overline{1, k_0}$, $k_o = n_0$, and the differential system (4) takes the form

$$\begin{gathered} x_s' = i \cdot \mu_s \cdot \pi_s \cdot x_s + \sum_{k=1}^{m_0} f_{s, 2\cdot k+1} \cdot x_s^{k+1} \cdot \overline{x}_s^k, \\ s = \overline{1, k_0}, \quad m_0 = \frac{1}{4} \cdot \big[2 \cdot m - 3 + (-1)^m\big]. \end{gathered} \tag{5}$$

The substitution $x_s = \rho_s \cdot \exp(i \cdot \theta_k)$, $s = \overline{1, k_0}$, reduces the differential system (5) to that of the kind

$$\rho_s' = \sum_{k=1}^{m_0} \operatorname{Re} f_{s, 2\cdot k+1} \cdot \rho_s^{2\cdot k+1},$$
$$\theta_s' = \mu_s \cdot \pi_s + \sum_{k=1}^{m_0} \operatorname{Im} f_{s, 2\cdot k+1} \cdot \rho_s^{2k}, \quad s = \overline{1, k_0}.$$

It follows from the substitution that the variables $x_s$ and $\rho_s, s = \overline{1, k_0}$ are equivalent in terms of stability. Therefore one can neglect the differential equation with respect to $\theta_s, s = \overline{1, k_0}$. Then the necessary asymptotic representations of proper solutions of the differential system (5) are to be found among the functions $\big[- \operatorname{Re} f_{s, 2\cdot k+1} \cdot \operatorname{Re}^{-1} f_{s, 2l+1}\big]^{\frac{1}{2(k-l)}}$, $k \neq l$, $\Big[- 2k \int_\tau^l \operatorname{Re} f_{s, 2k+1} \cdot d\tau\Big]^{-\frac{1}{2k}}$, $s = \overline{1, k_0}$, $k, l = \overline{1, m_0}$.

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