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PERMANENCE AND UNIFORM ASYMPTOTIC STABILITY OF POSITIVE SOLUTIONS OF SAIQH MODELS ON TIME SCALES

Abstract. We introduce a Susceptible-Asymptomatic-Infectious-Quarantined-Hospitalized (SAIQH) compartmental model formulated on time scales and construct an appropriate Lyapunov functional. Our main results establish the existence of solutions and the permanence of the system. We derive sufficient conditions guaranteeing a unique almost-periodic solution that is uniformly asymptotically stable. A numerical example is provided to illustrate and support the theoretical findings.

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1 Introduction

In 2019, the COVID-19 pandemic has appeared for the first time in Wuhan, China, attracting many researchers to investigate outbreaks and the spread of viruses [1,2]. Some of the studies provided new mathematical compartmental models, clearly illustrating the important contributions of mathematics in combating communicable diseases. Such models have been widely used to analyze the corresponding dynamics and to supply useful tools in disease epidemiology [3, 11, 16].

Kim et al. studied a SAIQH (Susceptible-Asymptomatic-Infectious-Quarantined-Hospitalized) mathematical model to analyze the transmission dynamics of MERS and to estimate transmission rates [10]. Lemos-Paião et al. developed a SAIQH type model for COVID-19, which was important to describe and understand the pandemic in Portugal [11]. Similar to [11], here we also consider the H_{IC} class of hospitalized individuals in intensive care units.

Let the total living population under study, denoted by $N(t)$, $t \geq 0$, be divided into six classes:

(i) the susceptible individuals $S(t)$;

(ii) the infected individuals without (or with mild) symptoms $A(t)$ (asymptomatic);

(iii) infected individuals $I(t)$ with visible symptoms;

(iv) quarantined individuals $Q(t)$ in home isolation; (v) hospitalized individuals $H(t)$;

(vi) hospitalized individuals $H_{IC}(t)$ in intensive care units.

The mathematical model introduced and studied in [11] reads as follows:

$$\begin{cases} \dot{S}(t) = \Lambda + \omega n Q(t) - \big[\lambda(t)(1-p) + \phi p + \gamma\big] S(t), \\ \dot{A}(t) = \lambda(t)(1-p)S(t) - [q\nu + \gamma]A(t), \\ \dot{I}(t) = q\nu A(t) - [\delta_1 + \gamma]I(t), \\ \dot{Q}(t) = \phi p S(t) + \delta_1 f_1 I(t) + \delta_2(1 - f_2 - f_3)H(t) - [\omega m + \gamma]Q(t), \\ \dot{H}(t) = \delta_1(1-f_1)I(t) + \eta(1-k)H_{IC}(t) - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] H(t), \\ \dot{H_{IC}}(t) = \delta_2 f_2 H(t) - [\eta(1-k) + \alpha_2 k + \gamma]H_{IC}(t), \end{cases} \tag{1.1}$$

where, for all time $t \geq 0$,

$$\lambda(t) = \frac{\beta(l_A A(t) + I(t) + l_H H(t))}{N(t)}$$

is a bounded positive function with $N(t) = S(t) + A(t) + I(t) + Q(t) + H(t) + H_{IC}(t)$, $\beta, l_A, l_H > 0$, and all the other parameters in the model are nonnegative. In addition, $p, 1-p, k, 1-k, q, f_1, 1-f_1, f_2, f_3, 1-f_2-f_3 \in [0,1]$. For more details on the mathematical model (1.1), we refer the reader to [11].

The time-scale approach unifies continuous and discrete dynamics into a single mathematical framework, allowing the results to hold simultaneously for differential, difference, and hybrid models. For an epidemiological model like SAIQH, this offers clear advantages:

1. It captures hybrid processes where infection evolves continuously, but interventions (screening, quarantines, vaccination campaigns) occur at discrete times.
2. It accommodates data sampled at irregular intervals and models dynamics with variable step lengths (seasonal changes, shifting intervention frequencies) without switching formalisms.
3. It aligns theory with numerical practice: discretization schemes and simulations can be treated within the same framework, preserving qualitative properties.
4. It increases the applicability and robustness of theoretical results (existence, permanence, almost-periodicity, uniform asymptotic stability) by proving them for a broader class of temporal structures.

Thus, inspired by Prasad and Khuddush (2019) [14], we extend (1.1) to a time scale $\mathbb{T}$, making the SAIQH model more general and better suited to real-world, mixed-time epidemic scenarios. Precisely, here we extend (1.1) to a time scale $\mathbb{T}$, studying the permanence and uniform asymptotic stability of the unique positive almost periodic solution of the following SAIQH type model on time scales:

$$\begin{cases} S^{\Delta}(t) = \Lambda + \omega n Q(t) - \big[\lambda(t)(1-p) + \phi p + \gamma\big] S^{\sigma}(t), \\ A^{\Delta}(t) = \lambda(t)(1-p)S(t) - [q\nu + \gamma]A^{\sigma}(t), \\ I^{\Delta}(t) = q\nu A(t) - [\delta_1 + \gamma]I^{\sigma}(t), \\ Q^{\Delta}(t) = \phi p S(t) + \delta_1 f_1 I(t) + \delta_2(1-f_2-f_3)H(t) - [\omega m + \gamma]Q^{\sigma}(t), \\ H^{\Delta}(t) = \delta_1(1-f_1)I(t) + \eta(1-k)H_{IC}(t) - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big]H^{\sigma}(t), \\ H_{IC}^{\Delta}(t) = \delta_2 f_2 H(t) - [\eta(1-k) + \alpha_2 k + \gamma]H_{IC}^{\sigma}(t), \end{cases} \tag{1.2}$$

where all parameters are real nonnegative numbers and $\lambda(t)$ is bounded positive, as mentioned before. In the particular case $\mathbb{T} = \mathbb{R}^+$, system (1.2) reduces to (1.1). For a more detailed explanation of the topic of time scales, we refer the reader to [4–7, 9, 14, 15, 17, 18].

The article is organized as follows. In Section 2, we recall the main necessary notions and results needed in the sequel. The original results are then formulated and proved in Section 3: we show that system (1.2) is permanent (Theorem 3.1); we give a sufficient condition for the existence of a solution to system (1.2) (Theorem 3.2); and we obtain conditions implying the dynamic system (1.2) to have a unique almost periodic solution that is uniformly asymptotically stable (Theorem 3.3).

For convenience, in the sequel, we put $x_1 = S$, $x_2 = A$, $x_3 = I$, $x_4 = Q$, $x_5 = H$, and $x_6 = H_{IC}$.

2 Preliminaries

For a function $f(t)$ defined on $t \in \mathbb{T}^+$, we set $f^L := \inf\{f(t) :\ t \in \mathbb{T}^+\}$, $f^U := \sup\{f(t) :\ t \in \mathbb{T}^+\}$.

We begin by recalling some basic but fundamental concepts of time scale calculus [5]. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$; on $\mathbb{T}$, we consider the topology inherited from the real numbers with the standard topology. The jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess function $\mu : \mathbb{T} \to mathbbR^+$ are defined, respectively, by $\sigma(t) = \inf\{\tau \in \mathbb{T} :\ \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} :\ \tau < t\}$ and $\mu(t) = \sigma(t) - t$, supplemented by $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, or right-scattered when $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, and $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} := \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}^{\kappa} := \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$, then $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is given by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 2.1 (see [5]). A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$; a function $g : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} :\ 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Lemma 2.1 (see [8]). *Assume that $\alpha > 0$, $b > 0$, and $-\alpha \in \mathcal{R}^+$. If*

$$y^{\Delta}(t) \geq\ (\leq)\, b - \alpha y^{\sigma}(t), \ \ y(t) > 0, \ \ t \in [t_0, \infty)_{\mathbb{T}},$$

then

$$y(t) \geq\ (\leq)\, \frac{b}{\alpha}\left[1 + \left(\frac{\alpha y(t_0)}{b} - 1\right)e_{(-\alpha)}(t, t_0)\right], \ \ t \in [t_0, \infty)_{\mathbb{T}},$$

where $e_{\cdot}(\cdot, \cdot)$ is the standard exponential function of the time-scale calculus [5].

Now, we give the definition of the Dini derivative. Let $s \in \mathbb{T}$ approach t from the right ($s > t$).

Case A: t is right-dense ($\sigma(t) = t$). In this case, the definitions coincide with the classical ones in real analysis, restricted to points in $\mathbb{T}$:

$$D_{\Delta}^{+} f(t) = \limsup_{\substack{s \to t^+ \\ s \in \mathbb{T}}} \frac{f(s) - f(t)}{s - t}\,, \quad D_{\Delta}^{-} f(t) = \liminf_{\substack{s \to t^+ \\ s \in \mathbb{T}}} \frac{f(s) - f(t)}{s - t}\,.$$

Case B: t is right-scattered ($\sigma(t) > t$). In this situation, limits are trivial, since the closest point to t on the right is $\sigma(t)$. Hence, the upper and lower Dini derivatives coincide and reduce to the slope between t and its forward jump:

$$D^+_\Delta f(t) = D^-_\Delta f(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(*Note:* Similar definitions can be introduced for the left Dini derivatives using the backward jump operator $\rho(t)$. However, the delta calculus focuses mainly on the forward direction.)

Unified notation. For $t \in \mathbb{T}^\kappa$, the upper Dini delta derivative of f can be defined in a unified way by

$$D^+_\Delta f(t) = \inf_{\delta>0} \sup \left\{ \frac{f(s) - f(t)}{s - t} : \ s \in (t, t+\delta) \cap \mathbb{T} \right\}.$$

If t is right-scattered, the set $(t, t+\delta) \cap \mathbb{T}$ reduces eventually to $\{\sigma(t)\}$ as $\delta \to 0^+$, and the above definition collapses to the simple difference quotient.

Lemma 2.2 (see [13])**.** *Let $f \in C(\mathbb{T}, \mathbb{R})$ be Δ-differentiable at $t \in \mathbb{T}$. Then $D^+_\Delta f(t) \leq \operatorname{sign}(f^\sigma(t)) f^\Delta(t)$, where $f^\sigma(t) = f(\sigma(t))$.*

Definition 2.2 (see [12])**.** A time scale $\mathbb{T}$ is called an almost periodic time scale if $\Pi = \{\tau \in \mathbb{R} : t + \tau \in \mathbb{T} \text{ for all } t \in \mathbb{T}\} \neq \{0\}$.

Definition 2.3 (see [12])**.** Let $\mathbb{T}$ be an almost periodic time scale. A function $x \in C(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of x, that is, $E\{\varepsilon, x\} = \{\tau \in \Pi : \ |x(t+\tau) - x(t)| < \varepsilon \text{ for all } t \in \mathbb{T}\}$ is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau \in E\{\varepsilon, x\}$ for which $|x(t+\tau) - x(t)| < \varepsilon$ for all $t \in \mathbb{T}$. The value τ is known as the ε-translation number of x, and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, x\}$.

Definition 2.4 (see [12])**.** Let $\mathbb{D}$ be an open set in $\mathbb{R}^n$ and let $\mathbb{T}$ be a positive almost periodic time scale. A function $f \in C(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in \mathbb{D}$ if the ε-translation set of f,

$$E\{\varepsilon, f, \mathbb{S}\} = \big\{\tau \in \Pi : \ \ |f(t+\tau) - f(t)| < \varepsilon \ \text{ for all } \ (t, x) \in \mathbb{T} \times \mathbb{S}\big\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and, for each compact subset $\mathbb{S}$ of $\mathbb{D}$, that is, for any given $\varepsilon > 0$ and each compact subset $\mathbb{S}$ of $\mathbb{D}$, there exists a constant $l(\varepsilon, \mathbb{S}) > 0$ such that each interval of length $l(\varepsilon, \mathbb{S})$ contains $\tau(\varepsilon, \mathbb{S}) \in E\{\varepsilon, f, \mathbb{S}\}$ for which $|f(t+\tau, x) - f(t, x)| < \varepsilon$ for all $(t, x) \in \mathbb{T} \times \mathbb{S}$.

Consider a system

$$x^\Delta(t) = h(t, x), \tag{2.1}$$

where $h : \mathbb{T}^+ \times \mathbb{S}_B \to \mathbb{R}^n$, $\mathbb{S}_B = \{x \in \mathbb{R}^n : \ \|x\| < B\}$ and $h(t, x)$ is almost periodic in t uniformly for $x \in \mathbb{S}_B$ and is continuous in x. In [18], the question of existence of a unique almost periodic solution $f(t) \in \mathbb{S}$ of (2.1), which is uniformly asymptotically stable, is investigated. For our model, we obtain from [18] the following result.

Lemma 2.3 (see [18])**.** *Suppose that there exists a Lyapunov function $V(t, x, z)$ defined on $\mathbb{T}^+ \times \mathbb{S}_B \times \mathbb{S}_B$ satisfying the following conditions:*

(i) $a(\|x - z\|) \leq V(t, x, z) \leq b(\|x - z\|)$, *where $a, b \in \mathbb{K}$ with $\mathbb{K} = \{\alpha \in C(\mathbb{R}^+, \mathbb{R}^+) : \ \alpha(0) = 0$ and α is increasing$\}$;*

(ii) $V(t, x, z) - V(t, x_1, z_1)| \leq L(\|x - x_1\| + \|z - z_1\|)$, *where $L > 0$ is a constant;*

(iii) $D^+_\Delta V(t, x, z) \leq -cV(t, x, z)$, *where $c > 0$, $-c \in \mathcal{R}^+$,* *and $D^+_\Delta V$ is the Dini derivative of V. Furthermore, if there exists a solution $x(t) \in \mathbb{S}$ of the system*

$$\begin{cases} x_1^\Delta(t) = \Lambda + \omega n x_4(t) - [\lambda(t)(1-p) + \phi p + \gamma] x_1^\sigma(t), \\ x_2^\Delta(t) = \lambda(t)(1-p)x_1(t) - [q\nu + \gamma] x_2^\sigma(t), \\ x_3^\Delta(t) = q\nu x_2(t) - [\delta_1 + \gamma] x_3^\sigma(t), \\ x_4^\Delta(t) = \phi p x_1(t) + \delta_1 f_1 x_3(t) + \delta_2(1-f_2-f_3)x_5(t) - [\omega n + \gamma] x_4^\sigma(t), \\ x_5^\Delta(t) = \delta_1(1-f_1)x_3(t) + \eta(1-k)x_6(t) - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] x_5^\sigma(t), \\ x_6^\Delta(t) = \delta_2 f_2 x_5(t) - \big[\eta(1-k) + \alpha_2 k + \gamma\big] x_6^\sigma(t), \end{cases} \tag{2.2}$$

for $t \in \mathbb{T}^+$, where $\mathbb{S} \cup \mathbb{S}_B$ is a compact set, then there exists a unique almost periodic solution $f(t) \in \mathbb{S}$ of system (2.2), which is uniformly asymptotically stable.

3 Main Results

Let $t_0 \in \mathbb{T}$ be a fixed positive initial time. Our main results are: the proof that system (2.2) is permanent (Section 3.1); a sufficient condition for the existence of a solution to system (2.2) (Section 3.2); and conditions that imply the dynamic system (2.2) to have a unique almost periodic solution that is uniformly asymptotically stable (Section 3.3).

3.1 Permanence of solutions

We begin by introducing the notion of the permanence of solutions.

Definition 3.1. System (2.2) is said to be permanent if there exist positive constants m and M such that $m \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le M$, $i = 1, 2, \ldots, 6$, for any solution $(x_1(t), \ldots, x_6(t))$ of (2.2).

Theorem 3.1. *System (2.2) is permanent.*

Proof. Let $Z(t) = (x_1(t), \ldots, x_6(t))$ be a positive solution of system (2.2). Then

$$\begin{cases} x_1^\Delta(t) \ge \Lambda - \big[\lambda^U(1-p) + \phi p + \gamma\big] x_1^\sigma(t), \\ x_2^\Delta(t) \ge \lambda^L(1-p)m_1 - (q\nu + \gamma) x_2^\sigma(t), \\ x_3^\Delta(t) \ge q\nu m_2 - (\delta_1 + \gamma) x_3^\sigma(t), \\ x_4^\Delta(t) \ge \phi p m_1 + \delta_1 f_1 m_3 - (\omega n + \gamma) x_4^\sigma(t), \\ x_5^\Delta(t) \ge \delta_1(1-f_1)m_3 - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] x_5^\sigma(t), \\ x_6^\Delta(t) \ge \delta_2 f_2 m_5 - \big[\eta(1-k) + \alpha_2 k + \gamma\big] x_6^\sigma(t). \end{cases}$$

From Lemma 2.1, it follows that

$$x_1(t) \ge \frac{\Lambda}{\lambda^u(1-p) + \phi p + \gamma}\Big[1 + \Big(\frac{\lambda^U(1-p) + \phi p + \gamma}{\Lambda} x_1(t_0) - 1\Big) e_{-(\lambda^U(1-p)+\phi p+\gamma)}(t, t_0)\Big],$$

and we have $e_{-(\lambda^U(1-p)+\phi p+\gamma)}(t, t_0) \to 0$ as $t \to \infty$. Thus,

$$\begin{cases} x_1(t) \ge m_1 := \dfrac{\Lambda}{\lambda^U(1-p) + \phi p + \gamma} & \text{for } t \ge T_1, \\ x_2(t) \ge m_2 := \dfrac{\lambda^L(1-p)m_1}{q\nu + \gamma} & \text{for } t \ge T_2, \\ x_3(t) \ge m_3 := \dfrac{q\nu m_2}{\delta_1 + \gamma} & \text{for } t \ge T_3, \\ x_4(t) \ge m_4 := \dfrac{\phi p m_1 + \delta_1 f_1 m_3}{\omega n + \gamma} & \text{for } t \ge T_4, \\ x_5(t) \ge m_5 := \dfrac{\delta_1(1-f_1)m_3}{\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma} & \text{for } t \ge T_5, \\ x_6(t) \ge m_6 := \dfrac{\delta_2 f_2 m_5}{\eta(1-k) + \alpha_2 k + \gamma} & \text{for } t \ge T_6. \end{cases}$$

Let $m = \min_{1\leqslant i\leqslant 6} m_i$ and $T = \max_{1\leqslant i\leqslant 6} T_i$. We can then write that $x_i(t) \geq m$ for all $t > T$. $\square$

For system (2.2), we introduce the following assumption:

(H_1) $\lambda(t)$ is a bounded almost periodic function satisfying $0 < \lambda^L \leq \lambda(t) \leq \lambda^U$.

To complete the proof of our Theorem 3.2, we first present a technical lemma.

Lemma 3.1. *If $(H1)$ holds, then, for any positive solution $Z(t) = (x_1(t), \dots, x_6(t))$ of system (2.2), there exist positive constants M and T such that $x_i(t) < M$, $i = 1, \dots, 6$, for all $t > T$.*

Proof. Let $Z(t) = (x_1(t), \dots, x_6(t))$ be a positive solution of system (2.2). Then

$$\begin{cases} x_1^\Delta(t) \leqslant \Lambda + \omega n \dfrac{\Lambda}{\gamma} - \big[\lambda^L(1-p) + \phi p + \gamma\big] x_1^\sigma(t), \\ x_2^\Delta(t) \leqslant \lambda^U(1-p)M_1 - (q\nu + \gamma)x_2^\sigma(t), \\ x_3^\Delta(t) \leqslant q\nu M_2 - (\delta_1 + \gamma)x_3^\sigma(t), \\ x_4^\Delta(t) \leqslant \phi p M_1 + \delta_1 f_1 M_3 + \delta_2(1 - f_2 - f_3)\dfrac{\Lambda}{\gamma} - (\omega n + \gamma)x_4^\sigma(t), \\ x_5^\Delta(t) \leqslant \delta_1(1 - f_1)M_3 + \eta(1-k)\frac{\Lambda}{\gamma} - \big[\delta_2(1 - f_2 - f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] x_5^\sigma(t), \\ x_6^\Delta(t) \leqslant \delta_2 f_2 M_5 - \big[\eta(1-k) + \alpha_2 k + \gamma\big] x_6^\sigma(t). \end{cases}$$

From Lemma 2.1, it follows that

$$x_1(t) \leqslant \frac{\Lambda + \omega n \frac{\Lambda}{\gamma}}{\lambda^L(1-p) + \phi p + \gamma}\left[1 + \left(\frac{\lambda^L(1-p) + \phi p + \gamma}{\Lambda + \omega n \frac{\Lambda}{\gamma}} x_1(t_0) - 1\right) e_{-(\lambda^L(1-p)+\phi p+\gamma)}(t, t_0)\right]$$

and we have $e_{-(\lambda^L(1-p)+\phi p+\gamma)}(t, t_0) \to 0$ as $t \to \infty$. Then

$$x_1(t) \leqslant M_1 := \frac{\Lambda + \omega n \frac{\Lambda}{\gamma}}{\lambda^L(1-p) + \phi p + \gamma},$$

$$\begin{cases} x_1(t) \leqslant M_1 := \dfrac{\Lambda + \omega n \frac{\Lambda}{\gamma}}{\lambda^L(1-p) + \phi p + \gamma} & \text{for } t \geq T_1, \\ x_2(t) \leqslant M_2 := \dfrac{\lambda^U(1-p)M_1}{q\nu + \gamma} & \text{for } t \geq T_2, \\ x_3(t) \leqslant M_3 := \dfrac{q\nu M_2}{\delta_1 + \gamma} & \text{for } t \geq T_3, \\ x_4(t) \leqslant M_4 := \dfrac{\phi p M_1 + \delta_1 f_1 M_3 + \delta_2(1 - f_2 - f_3)\frac{\Lambda}{\gamma}}{\omega n + \gamma} & \text{for } t \geq T_4, \\ x_5(t) \leqslant M_5 := \dfrac{\delta_1(1 - f_1)M_3 + \eta(1-k)\frac{\Lambda}{\gamma}}{\delta_2(1 - f_2 - f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma} & \text{for } t \geq T_5, \\ x_6(t) \leqslant M_6 := \dfrac{\delta_2 f_2 M_5}{\eta(1-k) + \alpha_2 k + \gamma} & \text{for } t \geq T_6. \end{cases}$$

Let $M = \max_{1\leqslant i\leqslant 6} M_i$ and $T = \max_{1\leqslant i\leqslant 6} T_i$. Then $x_i(t) \leqslant M$ for all $t > T$. $\square$

3.2 Existence solutions of system (2.2)

Let us define the following set:

$$\Omega = \big\{(x_1(t), \dots, x_6(t)) : \ (x_1(t), \dots, x_6(t)) \text{ is a solution of } (2.2) \text{ and } 0 < m \leq x_i \leq M, \ i = 1, \dots, 6\big\}.$$

Theorem 3.2. *If $(H1)$ holds, then the set Ω is nonempty.*

Proof. The almost periodicity of $\lambda(t)$ implies that there is a sequence $\{\xi_l\} \subseteq \mathbb{T}^+$ with $l \to \infty$ such that $\lambda(t+\xi_l) \to \lambda(t)$ as $l \to \infty$. From Theorem 3.1 and Lemma 3.1, for each sufficiently small $\varepsilon > 0$, there exists $t_1 \in \mathbb{T}^+$ such that $m - \varepsilon \leq x_i(t) \leq M + \varepsilon$ for all $t \geq t_1$, $i = 1, \ldots, 6$. Set $x_{il}(t) = x_i(t+\xi_l)$ for $t \geq t_1 - \xi_l$, $l = 1, 2, \ldots$. For any positive integer k, we find that there exists a sequence $\{x_{il}(t) : \ l \geq k\}$ such that the sequence $\{x_{il}(t)\}$ has a subsequence, denoted by $\{x^*_{il}(t)\}$ $(x^*_{il}(t) = x_{il}(t+\xi^*_l))$, converging on any finite interval of $\mathbb{T}^+$ as $l \to \infty$. So, we have a sequence $\{y_i(t)\}$ such that, for $t \in \mathbb{T}^+$,

$$x^*_{il}(t) \to y_i(t) \ \text{ as } \ l \to \infty, \ \ i = 1, \ldots, 6. \tag{3.1}$$

It is easy to see that the above sequence $\{\xi^*_l\} \subseteq \mathbb{T}^+$ with $\xi^*_l \to \tau$ for $l \to \infty$ is such that $\lambda(t+\xi^*_l) \to \lambda(t)$ as $l \to \infty$, which, together with (3.1) and

$$\begin{cases} x^{*\Delta}_{1l}(t) = \Lambda + \omega n x^*_{4l}(t) - \big[\lambda(t+\xi^*_l)(1-p) + \phi p + \gamma\big] x^{*\sigma}_{1l}(t), \\ x^{*\Delta}_{2l}(t) = \lambda(t+\xi^*_l))(1-p) x^*_{1l}(t) - [q\nu + \gamma] x^{*\sigma}_{2l}(t), \\ x^{*\Delta}_{3l}(t) = q\nu x^*_{2l}(t) - [\delta_1 + \gamma] x^{*\sigma}_{3l}(t), \\ x^{*\Delta}_{4l}(t) = \phi p x^*_{1l}(t) + \delta_1 f_1 x^*_{3l}(t) + \delta_2(1 - f_2 - f_3) x^*_{5l}(t) - [\omega n + \gamma] x^{*\sigma}_{4l}(t), \\ x^{*\Delta}_{5l}(t) = \delta_1(1-f_1) x^*_{3l}(t) + \eta(1-k) x^*_{6l}(t) - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] x^{*\sigma}_{5l}(t), \\ x^{*\Delta}_{6l}(t) = \delta_2 f_2 x^*_{5l}(t) - \big[\eta(1-k) + \alpha_2 k + \gamma\big] x^{*\sigma}_{6l}(t), \end{cases}$$

yields

$$\begin{cases} y^{\Delta}_1(t) = \Lambda + \omega n y_4(t) - \big[\lambda(t)(1-p) + \phi p + \gamma\big] y^{\sigma}_1(t), \\ y^{\Delta}_2(t) = \lambda(t)(1-p) y_1(t) - [q\nu + \gamma] y^{\sigma}_2(t), \\ y^{\Delta}_3(t) = q\nu y_2(t) - [\delta_1 + \gamma] y^{\sigma}_3(t), \\ y^{\Delta}_4(t) = \phi p y_1(t) + \delta_1 f_1 y_3(t) + \delta_2(1 - f_2 - f_3) y_5(t) - [\omega n + \gamma] y^{\sigma}_4(t), \\ y^{\Delta}_5(t) = \delta_1(1-f_1) y_3(t) + \eta(1-k) y_6(t) - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] y^{\sigma}_5(t), \\ y^{\Delta}_6(t) = \delta_2 f_2 y_5(t) - \big[\eta(1-k) + \alpha_2 k + \gamma\big] y^{\sigma}_6(t). \end{cases}$$

It is clear that $(y_1(t), \ldots, y_6(t))$ is a solution of system (2.2) and $m - \varepsilon \leq y_i(t) \leq M + \varepsilon$ for all $t \in \mathbb{T}^+$, $i = 1, \ldots, 6$. Since ε is arbitrary, it follows that $m \leq y_i(t) \leq M$ for $t \in \mathbb{T}^+$, $i = 1, \ldots, 6$. □

3.3 Uniform asymptotic stability

Now, we establish sufficient conditions for the existence of a unique positive almost periodic solution to system (2.2) that is uniform asymptotically stable. We introduce some more notations. Let

$$A_1 := \lambda^L(1-p) + \phi p + \gamma, \quad A_2 := q\nu + \gamma, \quad A_3 := \delta_1 + \gamma,$$
$$A_4 := \omega n + \gamma, \quad A_5 := \delta_2(1-f_3) + \alpha_1 f_3 + \gamma, \quad A_6 := \eta(1-k) + \alpha_2 k + \gamma,$$
$$B_1 := \lambda^U(1-p) + \phi p, \quad B_2 := q\nu + 2\,\frac{\gamma\beta l_A(1-p)M}{\Lambda}, \quad B_3 := \delta_1 + 2\,\frac{\gamma\beta(1-p)M}{\Lambda},$$
$$B_4 := \omega n, \quad B_5 := \delta_2(1-f_3) + 2\,\frac{\gamma\beta l_H(1-p)M}{\Lambda}, \quad B_6 := \eta(1-k).$$

Moreover, let $A := \min\limits_{1\leq i\leq 6} A_i$ and $B := \max\limits_{1\leq i\leq 6} B_i$.

In our next result (Theorem 3.3), we assume the following additional hypothesis:

(H2) $B < A$.

Theorem 3.3. *If* $(H1)$ *and* $(H2)$ *hold, then the dynamic system* (2.2) *has a unique almost periodic solution* $Z(t) = (x_1(t), \ldots, x_6(t)) \in \Omega$ *that is uniformly asymptotically stable.*

Proof. According to Theorem 3.1, every solution $Z(t) = (x_1(t), \dots, x_6(t))$ of system (2.2) satisfies $x_i^L \le x_i \le x_i^U$. Hence, $|x_i(t)| \le K_i$, $i = 1, \dots, 6$. Suppose that $Z(t) = (x_1(t), \dots, x_6(t))$ and $\widehat{Z}(t) = (\widehat{x}_1(t), \dots, \widehat{x}_6(t))$ are two positive solutions of system (2.2). We have

$$\begin{cases} x_1^\Delta(t) = \Lambda + \omega n x_4(t) - \big[\lambda(t)(1-p) + \phi p + \gamma\big] x_1^\sigma(t), \\ x_2^\Delta(t) = \lambda(t)(1-p)x_1(t) - [q\nu + \gamma] x_2^\sigma(t), \\ x_3^\Delta(t) = q\nu x_2(t) - [\delta_1 + \gamma] x_3^\sigma(t), \\ x_4^\Delta(t) = \phi p x_1(t) + \delta_1 f_1 x_3(t) + \delta_2(1 - f_2 - f_3)x_5(t) - [\omega n + \gamma] x_4^\sigma(t), \\ x_5^\Delta(t) = \delta_1(1-f_1)x_3(t) + \eta(1-k)x_6(t) - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] x_5^\sigma(t), \\ x_6^\Delta(t) = \delta_2 f_2 x_5(t) - \big[\eta(1-k) + \alpha_2 k + \gamma\big] x_6^\sigma(t), \end{cases}$$

$$\begin{cases} \widehat{x}_1^\Delta(t) = \Lambda + \omega n \widehat{x}_4(t) - \big[\widehat{\lambda}(t)(1-p) + \phi p + \gamma\big] \widehat{x}_1^\sigma(t), \\ \widehat{x}_2^\Delta(t) = \widehat{\lambda}(t)(1-p)\widehat{x}_1(t) - [q\nu + \gamma] \widehat{x}_2^\sigma(t), \\ \widehat{x}_3^\Delta(t) = q\nu \widehat{x}_2(t) - [\delta_1 + \gamma] \widehat{x}_3^\sigma(t), \\ \widehat{x}_4^\Delta(t) = \phi p \widehat{x}_1(t) + \delta_1 f_1 \widehat{x}_3(t) + \delta_2(1 - f_2 - f_3)\widehat{x}_5(t) - [\omega n + \gamma] \widehat{x}_4^\sigma(t), \\ \widehat{x}_5^\Delta(t) = \delta_1(1-f_1)\widehat{x}_3(t) + \eta(1-k)\widehat{x}_6(t) - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] \widehat{x}_5^\sigma(t), \\ \widehat{x}_6^\Delta(t) = \delta_2 f_2 \widehat{x}_5(t) - \big[\eta(1-k) + \alpha_2 k + \gamma\big] \widehat{x}_6^\sigma(t). \end{cases}$$

Denote

$$\|Z\| = \|(x_1(t), \dots, x_6(t))\| = \sup_{t \in \mathbb{T}^+} \sum_{i=1}^{6} |x_i(t)|.$$

Then $\|Z\| \le K$ and $\|\widehat{Z}\| \le K$, where $K = \sum_{i=1}^{6} K_i$. Define the Lyapunov function $V(t, Z, \widehat{Z})$ on $\mathbb{T}^+ \times \Omega \times \Omega$ as

$$V(t, Z, \widehat{Z}) = \sum_{i=1}^{6} |x_i(t) - \widehat{x}_i(t)| = \sum_{i=1}^{6} V_i(t),$$

where $V_i(t) = |x_i(t) - \widehat{x}_i(t)|$. Then the two norms

$$\|Z - \widehat{Z}\| = \sup_{T \in \mathbb{T}^+} \sum_{i=1}^{6} |x_i(t) - \widehat{x}_i(t)|, \quad \|Z - \widehat{Z}\|_* = \sup_{t \in \mathbb{R}_+} \Big[\sum_{i=1}^{6} (x_i(t) - \widehat{x}_i(t))^2 \Big]^{\frac{1}{2}}$$

are equivalent, that is, there exist two constants $\eta_1, \eta_2 > 0$ such that $\eta_1 \|Z - \widehat{Z}\|_* \le \|Z - \widehat{Z}\| \le \eta_2 \|Z - \widehat{Z}\|_*$. Hence, $\eta_1 \|Z - \widehat{Z}\|_* \le V(t, Z, \widehat{Z}) \le \eta_2 \|Z - \widehat{Z}\|_*$. Let a, $b \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(x) = \eta_1 x$, and $b(x) = \eta_2 x$. Then assumption (i) of Lemma 2.3 is satisfied. On the other hand, we have

$$\begin{aligned} \big|V(t, Z(t), \widehat{Z}(t)) - V(t, Z^*(t), \widehat{Z}^*(t))\big| &= \Big| \sum_{i=1}^{6} |x_i(t) - \widehat{x}_i(t)| - \sum_{i=1}^{6} |x_i^*(t) - \widehat{x}_i^*(t)| \Big| \\ &\le \sum_{i=1}^{6} \Big| |x_i(t) - \widehat{x}_i(t)| - |x_i^*(t) - \widehat{x}_i^*(t)| \Big| \le \sum_{i=1}^{6} \big| (x_i(t) - \widehat{x}_i(t)) - (x_i^*(t) - \widehat{x}_i^*(t)) \big| \\ = \sum_{i=1}^{6} \big| (x_i(t) - x_i^*(t)) - (\widehat{x}_i(t) - \widehat{x}_i^*(t)) \big| &\le \sum_{i=1}^{6} |x_i(t) - x_i^*(t)| + \sum_{i=1}^{6} |\widehat{x}_i(t) - \widehat{x}_i^*(t)| \le \|Z - Z^*\| + \|\widehat{Z} - \widehat{Z}^*\|, \end{aligned}$$

where $L = 1$, so condition (ii) of Lemma 2.3 is also satisfied.

Now, using Lemma 2.2, it follows that

$$D_\Delta^+ V_i(t) \le \operatorname{sign}(x_i^\sigma(t) - \widehat{x}_i^\sigma(t))(x_i^\Delta(t) - \widehat{x}_i^\Delta(t)), \quad i = 1, \dots, 6,$$

where $D_\Delta^+ V_i$ is the Dini derivative of V_i. For $i=1$,

$$\begin{aligned}
D_\Delta^+ V_1(t) &\leq \operatorname{sign}(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))(x_1^\Delta(t)-\widehat{x}_1^\Delta(t))\\
&= \operatorname{sign}(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))\Big[\omega n(x_4(t)-\widehat{x}_4(t))-\lambda(t)(1-p)x_1^\sigma(t)\\
&\qquad -[\phi p+\gamma](x_1^\sigma(t)-\widehat{x}_1^\sigma(t))+\widehat{\lambda}(t)(1-p)\widehat{x}_1^\sigma(t)\Big]\\
&= \operatorname{sign}(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))\Big[\omega n(x_4(t)-\widehat{x}_4(t))-\lambda(t)(1-p)x_1^\sigma(t)-[\phi p+\gamma](x_1^\sigma(t)\\
&\qquad -\widehat{x}_1^\sigma(t))+\widehat{\lambda}(t)(1-p)\widehat{x}_1^\sigma(t)+\lambda(t)(1-p)\widehat{x}_1^\sigma(t)-\lambda(t)(1-p)\widehat{x}_1^\sigma(t)\Big]\\
&= \operatorname{sign}(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))\Big[\omega n(x_4(t)-\widehat{x}_4(t))-\big[\lambda(t)(1-p)+\phi p+\gamma\big](x_1^\sigma(t)-\widehat{x}_1^\sigma(t))\\
&\qquad +(\widehat{\lambda}(t)-\lambda(t))(1-p)\widehat{x}_1^\sigma(t)\Big]\\
&= \operatorname{sign}(x_1^\sigma(t)-\widehat{x}_1^\sigma(t))\Big[\omega n(x_4(t)-\widehat{x}_4(t))-\big[\lambda(t)(1-p)+\phi p+\gamma\big](x_1^\sigma(t)-\widehat{x}_1^\sigma(t))\\
&\qquad +(\widehat{\lambda}(t)-\lambda(t))(1-p)\widehat{x}_1^\sigma(t)\Big]\\
&\leq \omega n|x_4(t)-\widehat{x}_4(t)|-\left[\lambda^L(1-p)+\phi p+\gamma\right]|x_1^\sigma(t)-\widehat{x}_1^\sigma(t)|+\frac{\gamma\beta l_A(1-p)M}{\Lambda}|x_2(t)-\widehat{x}_2(t)|\\
&\qquad +\frac{\gamma\beta(1-p)M}{\Lambda}|x_3(t)-\widehat{x}_3(t)|+\frac{\gamma\beta l_H(1-p)M}{\Lambda}|x_5(t)-\widehat{x}_5(t)|.
\end{aligned}$$

For $i=2$,

$$\begin{aligned}
D_\Delta^+ V_2(t) &\leq \operatorname{sign}(x_2^\sigma(t)-\widehat{x}_2^\sigma(t))(x_2^\Delta(t)-\widehat{x}_2^\Delta(t))\\
&= \operatorname{sign}(x_2^\sigma(t)-\widehat{x}_2^\sigma(t))\Big[\lambda(t)(1-p)x_1(t)-[q\nu+\gamma](x_2^\sigma(t)-\widehat{x}_2^\sigma(t))-\widehat{\lambda}(t)(1-p)\widehat{x}_1(t)\Big]\\
&= \operatorname{sign}(x_2^\sigma(t)-\widehat{x}_2^\sigma(t))\Big[\lambda(t)(1-p)x_1(t)-[q\nu+\gamma](x_2^\sigma(t)-\widehat{x}_2^\sigma(t))-\widehat{\lambda}(t)(1-p)\widehat{x}_1(t)\\
&\qquad +\lambda(t)(1-p)\widehat{x}_1(t)-\lambda(t)(1-p)\widehat{x}_1(t)\Big]\\
&= \operatorname{sign}(x_2^\sigma(t)-\widehat{x}_2^\sigma(t))\Big[-[q\nu+\gamma](x_2^\sigma(t)-\widehat{x}_2^\sigma(t))+\lambda(t)(1-p)(x_1(t)-\widehat{x}_1(t))\\
&\qquad -(\widehat{\lambda}(t)-\lambda(t))(1-p)\widehat{x}_1(t)\Big]\\
&\leq \lambda^U(1-p)|x_1(t)-\widehat{x}_1(t)|-[q\nu+\gamma]|x_2^\sigma(t)-\widehat{x}_2^\sigma(t)|+(1-p)M|\lambda(t)-\widehat{\lambda}(t)|\\
&\leq \lambda^U(1-p)|x_1(t)-\widehat{x}_1(t)|-[q\nu+\gamma]|x_2^\sigma(t)-\widehat{x}_2^\sigma(t)|+\frac{\gamma\beta l_A(1-p)M}{\Lambda}|x_2(t)-\widehat{x}_2(t)|\\
&\qquad +\frac{\gamma\beta(1-p)M}{\Lambda}|x_3(t)-\widehat{x}_3(t)|+\frac{\gamma\beta l_H(1-p)M}{\Lambda}|x_5(t)-\widehat{x}_5(t)|.
\end{aligned}$$

For $i=3$,

$$\begin{aligned}
D_\Delta^+ V_3(t) &\leq \operatorname{sign}(x_3^\sigma(t)-\widehat{x}_3^\sigma(t))(x_3^\Delta(t)-\widehat{x}_3^\Delta(t))\\
&= \operatorname{sign}(x_3^\sigma(t)-\widehat{x}_3^\sigma(t))\big[q\nu(x_2(t)-\widehat{x}_2(t))-[\delta_1+\gamma](x_3^\sigma(t)-\widehat{x}_3^\sigma(t))\big]\\
&\leq q\nu|x_2(t)-\widehat{x}_2(t)|-[\delta_1+\gamma]|x_3^\sigma(t)-\widehat{x}_3^\sigma(t)|.
\end{aligned}$$

For $i=4$,

$$\begin{aligned}
D_\Delta^+ V_4(t) &\leq \operatorname{sign}(x_4^\sigma(t)-\widehat{x}_4^\sigma(t))(x_4^\Delta(t)-\widehat{x}_4^\Delta(t))\\
&= \operatorname{sign}(x_4^\sigma(t)-\widehat{x}_4^\sigma(t))\Big[\phi p(x_1(t)-\widehat{x}_1(t))+\delta_1 f_1(x_3(t)-\widehat{x}_3(t))\\
&\qquad +\delta_2(1-f_2-f_3)(x_5(t)-\widehat{x}_5(t))-[\omega n+\gamma](x_4^\sigma(t)-\widehat{x}_4^\sigma(t))\Big]\\
&\leq \phi p|x_1(t)-\widehat{x}_1(t)|+\delta_1 f_1|x_3(t)-\widehat{x}_3(t)|+\delta_2(1-f_2-f_3)|x_5(t)-\widehat{x}_5(t)|-[\omega n+\gamma]|x_4^\sigma(t)-\widehat{x}_4^\sigma(t)|.
\end{aligned}$$

For $i = 5$,

$$
\begin{aligned}
D_\Delta^+ V_5(t) &\le \operatorname{sign}(x_5^\sigma(t) - \widehat{x}_5^\sigma(t))(x_5^\Delta(t) - \widehat{x}_5^\Delta(t)) \\
&= \operatorname{sign}(x_5^\sigma(t) - \widehat{x}_5^\sigma(t))\Big[\delta_1(1-f_1)(x_3(t) - \widehat{x}_3(t)) + \eta(1-k)(x_6(t) - \widehat{x}_6(t)) \\
&\qquad - \big[\delta_2(1-f_2-f_3) + \delta_1 f_1 + \alpha_1 f_3 + \gamma\big](x_5^\sigma(t) - \widehat{x}_5^\sigma(t))\Big] \\
&\le \delta_1(1-f_1)|x_3(t) - \widehat{x}_3(t)| + \eta(1-k)|x_6(t) - \widehat{x}_6(t)| \\
&\qquad - \big[\delta_2(1-f_2-f_3) + \delta_1 f_1 + \alpha_1 f_3 + \gamma\big]\, |x_5^\sigma(t) - \widehat{x}_5^\sigma(t)|.
\end{aligned}
$$

Finally, for $i = 6$,

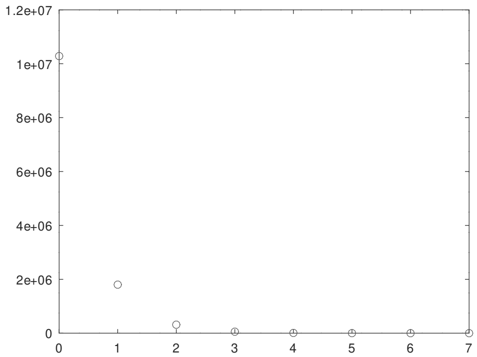
$$
\begin{aligned}
D_\Delta^+ V_6(t) &\le \operatorname{sign}(x_6^\sigma(t) - \widehat{x}_6^\sigma(t))(x_6^\Delta(t) - \widehat{x}_6^\Delta(t)) \\
&= \operatorname{sign}(x_6^\sigma(t) - \widehat{x}_6^\sigma(t))\Big[\delta_2 f_2(x_5(t) - \widehat{x}_5(t)) - \big[\eta(1-k) + \alpha_2 k + \gamma\big](x_6^\sigma(t) - \widehat{x}_6^\sigma(t))\Big] \\
&\le \delta_2 f_2|x_5(t) - \widehat{x}_5(t)| - \big[\eta(1-k) + \alpha_2 k + \gamma\big]\, |x_6^\sigma(t) - \widehat{x}_6^\sigma(t)|.
\end{aligned}
$$

Based on all the previous inequalities, we obtain that

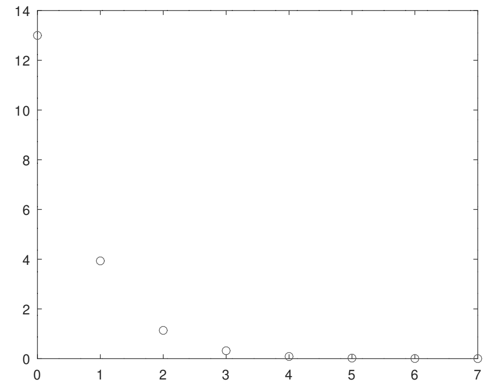
$$
\begin{aligned}
D_\Delta^+ V(t) &\le \omega n|x_4(t) - \widehat{x}_4(t)| - \big[\lambda^L(1-p) + \phi p + \gamma\big]\, |x_1^\sigma(t) - \widehat{x}_1^\sigma(t)| + \frac{\gamma\beta l_A(1-p)M}{\Lambda}\, |x_2(t) - \widehat{x}_2(t)| \\
&\quad + \frac{\gamma\beta(1-p)M}{\Lambda}\, |x_3(t) - \widehat{x}_3(t)| + \frac{\gamma\beta l_H(1-p)M}{\Lambda}\, |x_5(t) - \widehat{x}_5(t)| \\
&\quad + \lambda^U(1-p)|x_1(t) - \widehat{x}_1(t)| - [q\nu + \gamma]\, |x_2^\sigma(t) - \widehat{x}_2^\sigma(t)| \\
&\quad + \frac{\gamma\beta l_A(1-p)M}{\Lambda}\, |x_2(t) - \widehat{x}_2(t)| + \frac{\gamma\beta(1-p)M}{\Lambda}|x_3(t) - \widehat{x}_3(t)| \\
&\quad + \frac{\gamma\beta l_H(1-p)M}{\Lambda}\, |x_5(t) - \widehat{x}_5(t)| + q\nu|x_2(t) - \widehat{x}_2(t)| - [\delta_1 + \gamma]\, |x_3^\sigma(t) - \widehat{x}_3^\sigma(t)| \\
&\quad + \phi p|x_1(t) - \widehat{x}_1(t)| + \delta_1 f_1|x_3(t) - \widehat{x}_3(t)| + \delta_2(1-f_2-f_3)|x_5(t) - \widehat{x}_5(t)| \\
&\quad - [\omega n + \gamma]\, |x_4^\sigma(t) - \widehat{x}_4^\sigma(t)| + \delta_1(1-f_1)|x_3(t) - \widehat{x}_3(t)| + \eta(1-k)|x_6(t) - \widehat{x}_6(t)| \\
&\quad - \big[\delta_2(1-f_2-f_3) + \delta_1 f_1 + \alpha_1 f_3 + \gamma\big]\, |x_5^\sigma(t) - \widehat{x}_5^\sigma(t)| + \delta_2 f_2|x_5(t) - \widehat{x}_5(t)| \\
&\quad - \big[\eta(1-k) + \alpha_2 k + \gamma\big]\, |x_6^\sigma(t) - \widehat{x}_6^\sigma(t)| \\
&= -\big[\lambda(t)(1-p) + \phi p + \gamma\big]\, |x_1^\sigma(t) - \widehat{x}_1^\sigma(t)| - [q\nu + \gamma]\, |x_2^\sigma(t) - \widehat{x}_2^\sigma(t)| - [\delta_1 + \gamma]\, |x_3^\sigma(t) - \widehat{x}_3^\sigma(t)| \\
&\quad - [\omega n + \gamma]\, |x_4^\sigma(t) - \widehat{x}_4^\sigma(t)| - \big[\delta_2(1-f_2-f_3) + \delta_1 f_1 + \alpha_1 f_3 + \gamma\big]\, |x_5^\sigma(t) - \widehat{x}_5^\sigma(t)| \\
&\quad - \big[\eta(1-k) + \alpha_2 k + \gamma\big]\, |x_6^\sigma(t) - \widehat{x}_6^\sigma(t)| + \{\lambda^U(1-p) + \phi p\}|x_1(t) - \widehat{x}_1(t)| \\
&\quad + \Big\{q\nu + \frac{\gamma\beta l_A(1-p)M}{\Lambda} + \frac{\gamma\beta l_A(1-p)M}{\Lambda}\Big\}|x_2(t) - \widehat{x}_2(t)| \\
&\quad + \Big\{\delta_1 f_1 + \delta_1(1-f_1) + \frac{\gamma\beta(1-p)M}{\Lambda} + \frac{\gamma\beta(1-p)M}{\Lambda}\Big\}|x_3(t) - \widehat{x}_3(t)| + \omega n|x_4(t) - \widehat{x}_4(t)| \\
&\quad + \Big\{\delta_2(1-f_2-f_3) + \delta_2 f_2 + \frac{\gamma\beta l_H(1-p)M}{\Lambda} + \frac{\gamma\beta l_H(1-p)M}{\Lambda}\Big\}|x_5(t) - \widehat{x}_5(t)| \\
&\qquad + \eta(1-k)|x_6(t) - \widehat{x}_6(t)| \\
&= -A_1|x_1^\sigma(t) - \widehat{x}_1^\sigma(t)| - A_2|x_2^\sigma(t) - \widehat{x}_2^\sigma(t)| - A_3|x_3^\sigma(t) - \widehat{x}_3^\sigma(t)| - A_4|x_4^\sigma(t) - \widehat{x}_4^\sigma(t)| \\
&\quad - A_5|x_5^\sigma(t) - \widehat{x}_5^\sigma(t)| - A_6|x_6^\sigma(t) - \widehat{x}_6^\sigma(t)| + B_1|x_1(t) - \widehat{x}_1(t)| + B_2|x_2(t) - \widehat{x}_2(t)| \\
&\quad + B_3|x_3(t) - \widehat{x}_3(t)| + B_4|x_4(t) - \widehat{x}_4(t)| + B_5|x_5(t) - \widehat{x}_5(t)| + B_6|x_6(t) - \widehat{x}_6(t)| \\
&= -AV(\sigma(t)) + BV(t) = (B - A)V(t) - A\mu(t)D^+V^\Delta(t),
\end{aligned}
$$

and $D_\Delta^+ V(t) \le \frac{B-A}{1+A\mu(t)}\, V(t) \le -\psi(t)V(t)$ with $\psi = \frac{A-B}{1+A\mu^U}$. By $(H2)$, we have $\psi(t) = \frac{A-B}{1+A\mu^U} > 0$ and $1 - \psi\mu(t) = 1 + A(\mu^U - \mu(t)) + \mu(t)B > 0$. Hence, $-\psi \in \mathcal{R}^+$. Thus, assumption (iii) of Lemma 2.3 is satisfied, and it follows from Lemma 2.3 that there exists a unique almost periodic solution $Z(t) = (x_1(t), \dots, x_6(t))$ of the dynamic system (2.2) that is uniformly asymptotically stable with $Z(t) \in \Omega$. □

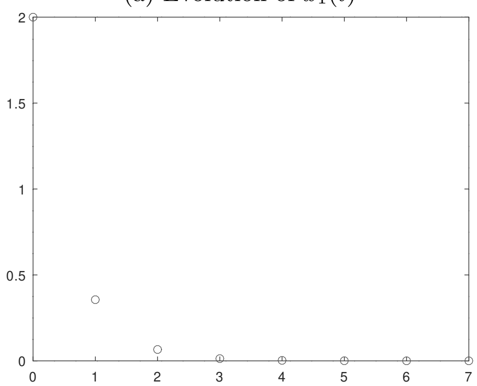
We illustrate our results with an example.



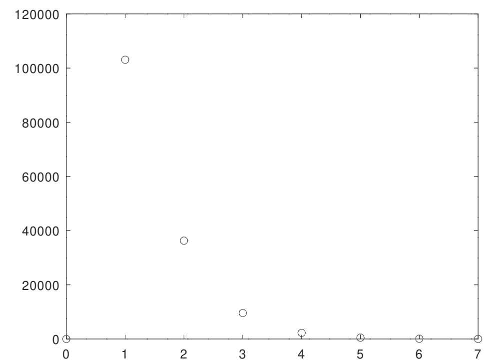
(a) Evolution of $x_1(t)$



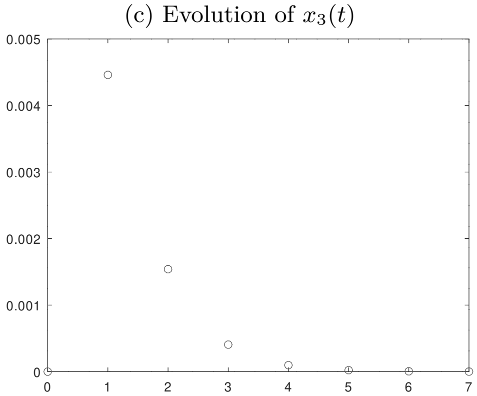
(b) Evolution of $x_2(t)$



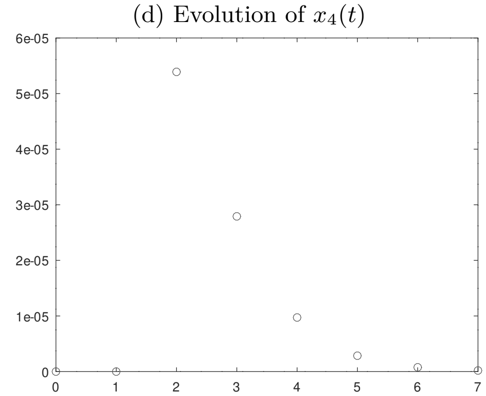
(c) Evolution of $x_3(t)$



(d) Evolution of $x_4(t)$



(e) Evolution of $x_5(t)$



(f) Evolution of $x_6(t)$

Figure 1: solution of (3.2) during 7 days.

Example 3.1. Based on [11], let us consider the following system on the time scale $\mathbb{T} = \mathbb{Z}_0^+$:

$$\begin{cases} x_1^{\Delta}(t) = \Lambda + \omega n x_4(t) - \big[\lambda(t)(1-p) + \phi p + \gamma\big] x_1^{\sigma}(t), \\ x_2^{\Delta}(t) = \lambda(t)(1-p)x_1(t) - [q\nu + \gamma] x_2^{\sigma}(t), \\ x_3^{\Delta}(t) = q\nu x_2(t) - [\delta_1 + \gamma] x_3^{\sigma}(t), \\ x_4^{\Delta}(t) = \phi p x_1(t) + \delta_1 f_1 x_3(t) + \delta_2(1 - f_2 - f_3)x_5(t) - [\omega n + \gamma] x_4^{\sigma}(t), \\ x_5^{\Delta}(t) = \delta_1(1-f_1)x_3(t) + \eta(1-k)x_6(t) - \big[\delta_2(1-f_2-f_3) + \delta_2 f_2 + \alpha_1 f_3 + \gamma\big] x_5^{\sigma}(t), \\ x_6^{\Delta}(t) = \delta_2 f_2 x_5(t) - \big[\eta(1-k) + \alpha_2 k + \gamma\big] x_6^{\sigma}(t), \end{cases} \tag{3.2}$$

subject to $x_1(0) = 10283785$, $x_2(0) = 13$, $x_3(0) = 2$, $x_4(0) = 0$, $x_5(0) = 0$, $x_6(0) = 0$, where $\Lambda = \frac{22614}{53}$, $\omega = 1/31$, $n = 0.075$, $\lambda(t) = \frac{\beta(l_A x_2(t)+x_3(t)+l_H x_5(t))}{N(t)}$ with $\beta = 1.93$, $l_A = 1$, $l_H = 0.1$, and $N(t) = \sum_{i=1}^{6} x_i(t)$, $p = 0.68$, $\phi = 1/12$, $\gamma = \frac{47833615}{N_0}$ with $N_0 = N(0)$, $q = 0.15$, $\nu = 1/15$, $\delta_1 = 1/3$, $\delta_2 = 1/3$, $f_1 = 0.96$, $f_2 = 0.21$, $f_3 = 0.03$, $\eta = 1/7$, $k = 0.03$, $\alpha_1 = 1/7$, and $\alpha_2 = 1/15$.

System (3.2) is permanent with $\lambda^L = 1.876738171 \times 10^{-7}$, $\lambda^U = 1.93$, $M_1 = 65.83271997$, $M_2 = 8.722416333$, $M_3 = 0.017498412509$, $M_4 = 4.428264471$, $M_5 = 59.126$, $M_6 = 0.02788991356$, $M = \max_{i=1,\dots,6}(M_i) = M_1$, $m_1 = 58.16800031$, $m_2 = 7.7068897$, $m_3 = 0.0154611$, $m_4 = 0.7093454$, $m_5 = 0.0000414$, $m_6 = 6.0482208 \times 10^{-7}$, and $m = \min_{i=1,\dots,6}(m_i) = m_6$. In addition, the conditions of Theorem 3.3 are verified and we have $4.653775371 = A > B = 4.148857053$, $\psi = 0.0505839$, $1 - \psi\mu(t) = 0.94941603 > 0$. We conclude that system (3.2) has a unique positive almost periodic solution, which is uniformly asymptotic stable. This is illustrated in Figure 1.

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