Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2025, 1–21

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MULTIVALUED NONLINEAR DIRICHLET BOUNDARY $p(u)$-LAPLACIAN PROBLEM

Abstract. We study the following nonlinear homogenous Dirichlet boundary $p(u)$-Laplacian problem

$$\beta(u) - \operatorname{div} a(x, u, \nabla u) \ni f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

The existence and partial uniqueness results of solutions for L^1-data f are established.

2020 Mathematics Subject Classification. 35J25, 35J60, 35Dxx, 76A05.

Key words and phrases. Variable exponent $p(u)$-Laplacian, Young measure, homogeneous Dirichlet boundary condition, bounded Radon diffuse measures, maximal monotone graph.

1 Introduction

We consider the following nonlinear elliptic $p(u)$-Laplacian problem with the Dirichlet boundary condition

$$P(\beta, f) \qquad \begin{cases} \beta(u) - \operatorname{div} a(x,u,\nabla u) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary and β is a maximal monotone graph on $\mathbb{R}$ such that $0 \in \beta(0)$, a is a Leray–Lions type operator and $f \in L^1(\Omega)$. $\operatorname{div} a(x,u,\nabla u)$ is called a $p(u)$-Laplacian operator, a prototype case is $\operatorname{div}(|\nabla u|^{p(\,\cdot\,,u)-2} \cdot \nabla u)$.

The problem $P(\beta, f)$ is adapted to a generalized Leray–Lions framework under the assumption that $a : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}^N$ is a Carathéodory function with

$$a(x,z,0) = 0 \ \text{ for all } \ z \in \mathbb{R} \ \text{ and a.e. } \ x \in \Omega \tag{1.1}$$

satisfying the strict monotonicity assumption

$$\big(a(x,z,\xi) - a(x,z,\eta)\big) \cdot (\xi - \eta) > 0 \ \text{ for all } \ \xi, \eta \in \mathbb{R}^N, \ \ \xi \neq \eta, \tag{1.2}$$

as well as the growth and the coercivity assumptions with variable exponent

$$|a(x,z,\xi)|^{p'(x,z)} \leq C_1\big(|\xi|^{p(x,z)} + \mathcal{M}(x)\big), \tag{1.3}$$

$$a(x,z,\xi) \cdot \xi \geq \frac{1}{C_2}\, |\xi|^{p(x,z)}. \tag{1.4}$$

Here, C_1, C_2 are positive constants and $\mathcal{M}$ is a positive function such that $\mathcal{M} \in L^1(\Omega)$.

$p : \Omega \times \mathbb{R} \to [p_-, p_+]$ is a Carathéodory function, $1 < p_- \leq p(x,z) \leq p_+ < \infty$ and $p'(x,z) = \frac{p(x,z)}{p(x,z)-1}$ is the conjugate exponent of $p(x,z)$ with

$$p_- := \operatorname*{ess\,inf}_{(x,z) \in \overline{\Omega} \times \mathbb{R}} p(x,z) \ \text{ and } \ p_+ := \operatorname*{ess\,sup}_{(x,z) \in \overline{\Omega} \times \mathbb{R}} p(x,z).$$

We assume that

$$p_- > N \ \text{ and } p \text{ is uniformly log-Hölder continuous on } \overline{\Omega} \times [-M, M] \text{ for all } M > 0. \tag{1.5}$$

Problem $P(\beta, f)$ can be seen as an extension of the following problem:

$$\begin{cases} b(u) - \operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $b : \mathbb{R} \to \mathbb{R}$ nondecreasing, normalized by $b(0) = 0$, and $f \in L^1(\Omega)$. Andreianov et *al.* (see [2]) studied problem (1.6) and established the existence results for such problems with variable exponent $p(x,u)$, issues of uniqueness and structural stability.

Since β is nonlinear, a bounded Radon diffuse measure appears in the definition of the solution to account of the boundary of the domain. Here, we use the notion of renormalized solution for the problem $P(\beta, f)$ in the context of variable exponent. The concept of renormalized solution was introduced by Diperna and Lions [8]. Note that the standard Leray–Lions elliptic problem with L^1 source terms is well posed in the framework of renormalized solutions.

We define $\mathcal{M}_b(\Omega)$ as the set of bounded Radon measures in Ω. For the variable exponent $\pi(\,\cdot\,)$, where $\pi(\,\cdot\,)$ is to be defined later, given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,\pi(\,\cdot\,)}(\Omega)$ if $\mu(A) = 0$ for every set A such that $Cap_{\pi(\,\cdot\,)}(A, \Omega) = 0$ (see [12, 13]). For $A \subset \Omega$, we denote

$$S_{\pi(\,\cdot\,)}(A) = \Big\{ u \in W_0^{1,\pi(\,\cdot\,)}(\Omega) \cap C_0(\Omega) : \ \ u = 1 \ \text{ on } A \text{ and } u \geq 0 \text{ in } \Omega \Big\}.$$

The $\pi(\,\cdot\,)$-capacity for every subset A with respect to Ω is defined by

$$Cap_{\pi(\,\cdot\,)}(A,\Omega) = \inf_{u\in S_{\pi(\,\cdot\,)}(A)} \bigg\{ \int_\Omega |\nabla u|^{\pi(\,\cdot\,)}\,dx \bigg\}.$$

The set of bounded Radon diffuse measures in variable exponent setting $\pi(\,\cdot\,)$ is denoted by $\mathcal{M}_b^{\pi(\,\cdot\,)}(\Omega)$.

Moreover, we use the Young measure associated to the weak convergence method of sequences of solution gradients to obtain some useful convergence results (cf. [1, 2, 10, 12]). We also adapt the techniques exposed in [11] for passing to the limit in the sequence $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ which will be defined later.

The interest of the study of this kind of problem is due to the fact that it can model phenomena which arise in the study of elastic mechanics (see [3]), electrorheological fluid (see [15]) or image restoration (see [7]). In particular, in the case of image restoration, several numerical examples suggest that the consideration of the exponent $p(\,\cdot\,,u)$ preserves the edges and reduces the noise of restored images u, as presented in [16, Section 8].

The remaining part of this article is organized as follows: in the next section, we introduce some preliminary results. In the third section, we study the existence and partial uniqueness results of renormalized solutions for the problem $P(\beta,f)$.

2 Preliminary results

- We will use the so-called truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s|\le k,\\ k\,\mathrm{sign}_0(s) & \text{if } |s|>k, \end{cases} \quad \text{where} \quad \mathrm{sign}_0(s) := \begin{cases} 1 & \text{if } s>0,\\ 0 & \text{if } s=0,\\ -1 & \text{if } s<0. \end{cases}$$

The truncation function possesses the following properties:

$$T_k(-s) = -T_k(s), \quad |T_k(s)| = \min\big\{|s|,k\big\},$$
$$\lim_{k\to\infty} T_k(s) = s \ \text{ and } \ \lim_{k\to 0}\frac{1}{k}T_k(s) = \mathrm{sign}_0(s).$$

- We will also use the following mapping to truncate vector value-function:

$$h_m:\mathbb{R}^N\to\mathbb{R}^N, \quad h_m(\lambda) = \begin{cases} \lambda & \text{if } |\lambda|\le m,\\ m\,\dfrac{\lambda}{|\lambda|} & \text{if } |\lambda|>m, \end{cases} \quad \text{where } m>0.$$

Taking into account the growth and the coercivity assumptions (1.3) and (1.4), we need to work in the variable exponent Sobolev space $\dot{E}^{\pi(\,\cdot\,)}(\Omega)$ defined below (notice that the exponent $\pi(\,\cdot\,)$ itself is related to u by $\pi(\,\cdot\,) := p(\,\cdot\,,u(\,\cdot\,))$, so the solutions and different data will possess different integrability properties). For the sake of completeness, we also recall the definition of variable exponent Lebesgue and Sobolev spaces $L^{\pi(\,\cdot\,)}(\Omega)$ and $W^{1,\pi(\,\cdot\,)}(\Omega)$. In the sequel, we will use the same notation $L^{\pi(\,\cdot\,)}(\Omega)$ for the space $(L^{\pi(\,\cdot\,)}(\Omega))^N$ of vector-valued functions.

Definition 2.1. Let $\pi:\Omega\to[1,\infty)$ be a measurable function.

- $L^{\pi(\,\cdot\,)}(\Omega)$ is the space of all measurable functions $f:\Omega\to\mathbb{R}$ such that the modular

$$\rho_{\pi(\,\cdot\,)}(f) := \int_\Omega |f|^{\pi(x)}\,dx < \infty.$$

If p_+ is finite, this space is equipped with the Luxembourg norm

$$\|f\|_{L^{\pi(\,\cdot\,)}(\Omega)} := \inf\Big\{\lambda>0;\ \rho_{\pi(\,\cdot\,)}\Big(\frac{f}{\lambda}\Big)\le 1\Big\}.$$

- $W^{1,\pi(\,\cdot\,)}(\Omega)$ is the space of all functions $f \in L^{\pi(\,\cdot\,)}(\Omega)$ such that the gradient of f (taken in the sense of distributions) belongs to $L^{\pi(\,\cdot\,)}(\Omega)$. If p_+ is finite, the space $W^{1,\pi(\,\cdot\,)}(\Omega)$ is equipped with the norm
$$\|u\|_{W^{1,\pi(\,\cdot\,)}(\Omega)} := \|u\|_{L^{\pi(\,\cdot\,)}(\Omega)} + \|\nabla u\|_{L^{\pi(\,\cdot\,)}(\Omega)}.$$
$W_0^{1,\pi(\,\cdot\,)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,\pi(\,\cdot\,)}(\Omega)$.

Further, $\dot{E}^{\pi(\,\cdot\,)}(\Omega)$ is the set of all $f \in W_0^{1,1}(\Omega)$ such that $\nabla f \in L^{\pi(\,\cdot\,)}(\Omega)$. This space is equipped with the norm
$$\|u\|_{\dot{E}^{\pi(\,\cdot\,)}(\Omega)} := \|\nabla u\|_{L^{\pi(\,\cdot\,)}(\Omega)}.$$
When $1 < p_- \le \pi(\,\cdot\,) \le p_+ < \infty$, all the above spaces are separable and reflexive Banach spaces.

Generally, $W_0^{1,\pi(\,\cdot\,)}(\Omega) \subsetneq \dot{E}^{\pi(\,\cdot\,)}(\Omega)$. In the present paper, we assume that $\pi(x) = p(x, u(x))$ verify the log-Hölder continuity assumption (2.1) below. Furthermore, we denote
$$\pi_\varepsilon(x) := p(x, u_\varepsilon(x)).$$

Proposition 2.1 (see [1, Proposition 2.3]). *For all measurable functions $\pi : \Omega \to [p_-, p_+]$, the following properties hold:*

(i) *$L^{\pi(\,\cdot\,)}(\Omega)$ and $W^{1,\pi(\,\cdot\,)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *$L^{\pi'(\,\cdot\,)}(\Omega)$ can be identified with the dual space of $L^{\pi(\,\cdot\,)}(\Omega)$, and the following Hölder type inequality holds:*
$$\forall\, f \in L^{\pi(\,\cdot\,)}(\Omega),\ \ g \in L^{\pi'(\,\cdot\,)}(\Omega), \qquad \left| \int_\Omega f\, g\, dx \right| \le 2\|f\|_{L^{\pi(\,\cdot\,)}(\Omega)} \|g\|_{L^{\pi'(\,\cdot\,)}(\Omega)},$$
with $\frac{1}{\pi'(x)} + \frac{1}{\pi(x)} = 1$ for all $x \in \Omega$.

(iii) *One has $\rho_{\pi(\,\cdot\,)}(f) = 1$ if and only if*
$$\|f\|_{L^{\pi(\,\cdot\,)}(\Omega)} = 1;$$
further, if $\rho_{\pi(\,\cdot\,)}(f) \le 1$, then
$$\|f\|^{p_+}_{L^{\pi(\,\cdot\,)}(\Omega)} \le \rho_{\pi(\,\cdot\,)}(f) \le \|f\|^{p_-}_{L^{\pi(\,\cdot\,)}(\Omega)};$$
if $\rho_{\pi(\,\cdot\,)}(f) \ge 1$, then
$$\|f\|^{p_-}_{L^{\pi(\,\cdot\,)}(\Omega)} \le \rho_{\pi(\,\cdot\,)}(f) \le \|f\|^{p_+}_{L^{\pi(\,\cdot\,)}(\Omega)}.$$

In particular, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^{\pi(\,\cdot\,)}(\Omega)$, then $\|f_n\|_{L^{\pi(\,\cdot\,)}(\Omega)}$ tends to zero (resp., to infinity) if and only if $\rho_{\pi(\,\cdot\,)}(f_n)$ tends to zero (resp., to infinity) as $n \to \infty$.

The following lemma prove that the space $W_0^{1,\pi(\,\cdot\,)}(\Omega)$ is stable by truncation (see [1, Lemma 2.9]).

Lemma 2.1. *If $u \in W_0^{1,\pi(\,\cdot\,)}(\Omega)$ then $T_k(u) \in W_0^{1,\pi(\,\cdot\,)}(\Omega)$ for all $k > 0$.*

Notice that $\dot{E}^{\pi(\,\cdot\,)}(\Omega)$ is also stable by truncation, since $W_0^{1,1}(\Omega)$ is stable by truncation and $|\nabla T_k(u)| \le |\nabla u| \in L^{\pi(\,\cdot\,)}(\Omega)$, whenever $u \in \dot{E}^{\pi(\,\cdot\,)}(\Omega)$.

From the results of Fan and Zhikov (see [1, Corollary 2.6]), we deduce the following

Lemma 2.2. *Assume that $\pi : \Omega \to [p_-, p_+]$ has a representative which can be extended to a continuous function up to the boundary $\partial\Omega$ and satisfying the log-Hölder continuity assumption:*
$$\exists\, L > 0,\ \ \forall\, x, y \in \overline{\Omega},\ \ x \ne y, \quad -\big(\log|x-y|\big)|\pi(x) - \pi(y)| \le L. \tag{2.1}$$
Then $\mathcal{D}(\Omega)$ is dense in $\dot{E}^{\pi(\,\cdot\,)}(\Omega)$. In particular, the spaces $\dot{E}^{\pi(\,\cdot\,)}(\Omega)$ and $W_0^{1,\pi(\,\cdot\,)}(\Omega)$ are Lipschitz homeomorphic and hence they may be identified.

Young measures and nonlinear weak-* convergence

Throughout the paper, we denote by δ_c the Dirac measure on $\mathbb{R}^d$ ($d \in \mathbb{N}$) concentrated at the point $c \in \mathbb{R}^d$.

In the following theorem, we compile the results of Ball [4], Pedregal [14] and Hungerbühler [10] which are needed for our purposes (we limit the statement to the case of a bounded domain Ω). It should be noted that the results (ii), (iii) below, expressed in terms of the convergence in measure, are very convenient for the applications we have in mind.

Theorem 2.1.

(i) *Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, and $(v_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued functions, $d \in \mathbb{N}$, such that $(v_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_x$ of probability measures on $\mathbb{R}^d$ ($d \in \mathbb{N}$), weakly measurable in x with respect to the Lebesgue measure on Ω, such that for all Carathéodory functions $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, one has*

$$\lim_{k\to\infty} \int_\Omega F(x, v_{n_k})\,dx = \int_\Omega \int_{\mathbb{R}^d} F(x,\lambda)\,d\nu_x(\lambda)\,dx, \tag{2.2}$$

whenever the sequence $(F(\,\cdot\,, v_n(\,\cdot\,)))_{n\in\mathbb{N}}$ is equi-integrable on Ω. In particular,

$$v(x) := \int_{\mathbb{R}^d} \lambda\,d\nu_x(\lambda) \tag{2.3}$$

is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ in $L^1(\Omega)$. The family $(\nu_x)_{x\in\Omega}$ is called the Young measure generated by the subsequence $(v_{n_k})_{k\in\mathbb{N}}$ as $k \to \infty$.

(ii) *If Ω is of finite measure and $(\nu_x)_{x\in\Omega}$ is the Young measure generated by a sequence $(v_n)_{n\in\mathbb{N}}$, then $\nu_x = \delta_{v(x)}$ for a.e. $x \in \Omega \iff v_n$ converges in measure on Ω to v as $n \to \infty$.*

(iii) *If Ω is of finite measure, $(u_n)_{n\in\mathbb{N}}$ generates a Dirac Young measure $(\delta_{u(x)})_x$ on $\mathbb{R}^{d_1}$, and $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_x$ on $\mathbb{R}^{d_2}$, then the sequence $(u_n, v_n)_{n\in\mathbb{N}}$ generates the Young measure $(\delta_{u(x)} \otimes \nu_x)_x$ on $\mathbb{R}^{d_1+d_2}$.*

Whenever a sequence $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_x$, following the terminology of [9], we say that $(v_n)_{n\in\mathbb{N}}$ is nonlinear weak-* convergent, and $(\nu_x)_x$ is the nonlinear weak-* limit of the sequence $(v_n)_{n\in\mathbb{N}}$. In the case when $(v_n)_{n\in\mathbb{N}}$ possesses a nonlinear weak-* convergent subsequence, we say that it is nonlinear weak-* compact. Theorem 2.10(i) in [1] means that any equi-integrable sequence of measurable functions is nonlinear weak-*compact on Ω.

Lemma 2.3 (see [1, Theorem 3.11] and [2, Step 2 of proof of Theorem 2.6])**.** *Assume that $(u_n)_{n\in\mathbb{N}}$ converges a.e. on Ω to some function u, then*

$$|p(x,u_n(x)) - p(x,u(x))| \text{ converges in measure to } 0 \text{ on } \Omega, \text{ and for all bounded subsets } K \text{ of } \mathbb{R}^N,$$
$$\sup_{\xi\in K} \big|a(x,u_n(x),\xi) - a(x,u(x),\xi)\big| \text{ converges in measure to } 0 \text{ on } \Omega.$$

In the sequel, we will give a useful convergence result.

Lemma 2.4. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graph such that $\beta_n \to \beta$ in the sense of graphs (i.e., for all $(x,y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider $(z_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$, two sequences of $L^1(\Omega)$ such that $w_n \in \beta_n(z_n)\mathcal{L}^N$ a.e. in Ω. If $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$, then $z \in \mathrm{dom}(\beta)\mathcal{L}^N$ a.e. in Ω.*

The main tool of the proof of the above lemma is the "biting lemma of Chacon" (see [6]).

Lemma 2.5. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and $(f_n)_{n\geq 1}$ be a bounded sequence in $L^1(\Omega)$. Then there exist $f \in L^1(\Omega)$, a subsequence $(f_{n_k})_{n_k\geq 1}$ and a sequence of the measurable set $(E_j)_{j\in\mathbb{N}^*}$, $E_j \subset \Omega$, $\forall\, j \in \mathbb{N}^*$ with $E_{j+1} \subset E_j$ and $\lim_{j\to\infty} |E_j| = 0$ such that for any $j \in \mathbb{N}^*$, $f_{n_k} \rightharpoonup f$ in $L^1(\Omega \setminus E_j)$ as $n_k \to \infty$.*

Proof of Lemma 2.4. Since the sequence $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$, using the "biting lemma of Chacon", there exist $w \in L^1(\Omega)$, a subsequence $(w_{n_k})_{n_k\geq 1}$, a sequence of mesurable sets $(E_j)_{j\in\mathbb{N}^*}$, $E_j \subset \Omega$, $\forall\, j \in \mathbb{N}^*$ with $E_{j+1} \subset E_j$ and $\lim |E_j| = 0$ and for all $j \in \mathbb{N}^*$, $w_{n_k} \rightharpoonup w$ in $L^1(\Omega \setminus E_j)$, as $n_k \to \infty$. Since $z_{n_k} \rightharpoonup z$ in $L^1(\Omega)$ and so in $L^1(\Omega \setminus E_j)$, $\forall j \in \mathbb{N}$ and $\beta_{n_k} \to \beta$ in the sense of graph, we have $w \in \beta(z)$ a.e. in $\Omega \setminus E_j$. Thus, $z \in \operatorname{dom}(\beta)$ a.e. in $\Omega \setminus E_j$. Finally, we obtain $z \in \operatorname{dom}(\beta)$ a.e. in Ω. $\square$

For all measurable functions $u : \Omega \to \mathbb{R}$ we write

$$\big\{|u| \leq k \ \ (< k,\ \ > k,\ \ \geqslant k,\ \ = k)\big\} \ \text{ or } \ \big[\,|u| \leq k \ \ (< k,\ \ > k,\ \ \geqslant k,\ \ = k)\,\big]$$

for the set

$$\big\{x \in \Omega;\ \ |u(x)| \leq k \ \ (< k,\ \ > k,\ \ \geqslant k,\ \ = k)\big\},$$

and $\operatorname{meas}(\Omega)$ or $|\Omega|$ denote the measure of the set Ω.

Let us set

$$\operatorname{int}(\operatorname{dom}\beta) = (m, M) \ \text{ with } \ -\infty \leq m \leq 0 \leq M \leq \infty.$$

3 Existence and partial uniqueness of the renormalized solution

We give our notion of solution of the problem $P(\beta, f)$ due to Igbida et *al.* in [11].

Definition 3.1. For $f \in L^1(\Omega)$, a renormalized solution of problem $P(\beta, f)$ is a couple (u, w) with u a measurable function such that $T_k(u) \in \dot{E}^{\pi(\,\cdot\,)}(\Omega)$ for all $k > 0$ and $u \in \operatorname{dom}(\beta)\mathcal{L}^N$ a.e. in Ω, $w \in L^1(\Omega)$ and $w \in \beta(u)\mathcal{L}^N$ a.e. in Ω; and there exists a measure $\mu \in \mathcal{M}_b^{\pi(\,\cdot\,)}(\Omega)$ such that $\mu \perp \mathcal{L}^N$, μ^+ is concentrated on $[u = M] \cap [u \neq \infty]$, μ^- is concentrated on $[u = m] \cap [u \neq -\infty]$ such that

$$\int_\Omega w h(u)\varphi\, dx + \int_\Omega a(x, u, \nabla u)\nabla\big(h(u)\varphi\big)\, dx + \int_\Omega h(u)\varphi\, d\mu = \int_\Omega f h(u)\varphi\, dx \tag{3.1}$$

for all $\varphi \in \dot{E}^{\pi(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$, and

$$\lim_{M\to\infty} \int_{[M<|u|<M+1]} a(x, u, \nabla u) \cdot \nabla u\, dx = 0.$$

All the terms of (3.1) are well defined. Since $h(u)\varphi \in \dot{E}^{\pi(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$, the first integral of the left-hand and right-hand sides of (3.1) are well defined. The second integral of the left-hand side is well defined thanks to (1.3). The third integral of the left-hand side is also well defined, since the measure μ is diffuse.

Remark 3.1. If $M = \infty$ and $-\infty < m$ (resp. $m = -\infty$ and $M < \infty$), then $\mu_+ \equiv 0$ (resp. $\mu_- \equiv 0$). Thus, (3.1) holds true with $\mu \equiv \mu_+$ (resp. $\mu \equiv \mu_-$).

If $M = \infty$ and $m = -\infty$, then the domain of β is equal to $\mathbb{R}$ and relation (3.1) becomes

$$\int_\Omega a(x, u, \nabla u)\nabla(h(u)\varphi)\, dx + \int_\Omega w h(u)\varphi\, dx = \int_\Omega f h(u)\varphi\, dx$$

for all $\varphi \in \dot{E}^{\pi(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. In the case where the domain of β is bounded, the renormalization with h is not necessary in Definition 3.1. We can take $h \equiv 1$.

Now, we are going to prove the following existence result.

Theorem 3.1. *Assume that* (1.1)–(1.5) *hold and* $f \in L^1(\Omega)$. *Then there exists at least one renormalized solution to the problem* $P(\beta, f)$.

Proof. The proof of Theorem 3.1 is divided into two steps.

Step 1. The approximate problem
For every $\varepsilon > 0$, we consider the Yosida regularization β_ε of β given by

$$\beta_\varepsilon = \frac{1}{\varepsilon}\big(I - (I+\varepsilon\beta)^{-1}\big).$$

Due to [5], there exists a non negative, convex and lower semicontinuous function j defined on $\mathbb{R}$ such that $\beta = \partial j$.

To regularize β, we consider

$$j_\varepsilon(s) = \min_{r\in\mathbb{R}}\Big\{\frac{1}{2\varepsilon}|s-r|^2 + j(r)\Big\},\ \ \forall\, s\in\mathbb{R},\ \ \forall\,\varepsilon>0.$$

Using Proposition 2.11 from [5], we have

$$\begin{cases} \mathrm{dom}(\beta)\subset\mathrm{dom}(j)\subset\overline{\mathrm{dom}(j)}=\overline{\mathrm{dom}(\beta)},\\ j_\varepsilon(s)=\dfrac{\varepsilon}{2}|\beta_\varepsilon(s)|^2+j(J_\varepsilon(s)),\ \ \text{where}\ \ J_\varepsilon:=(I+\varepsilon\beta)^{-1},\\ j_\varepsilon\ \ \text{is convex, Frechet-differentiable and}\ \ \beta_\varepsilon=\partial j_\varepsilon,\\ j_\varepsilon\uparrow j\ \ \text{as}\ \ \varepsilon\downarrow 0. \end{cases}$$

Moreover, for any $\varepsilon>0$, β_ε is nondecreasing, Lipschitz continuous function and $\beta_\varepsilon(0)=0$. Now, we consider the following problem:

$$P(\beta_\epsilon, f)\qquad \begin{cases} \beta_\varepsilon(u_\varepsilon)-\operatorname{div} a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)=f & \text{in } \Omega,\\ u_\varepsilon=0 & \text{on } \partial\Omega, \end{cases}$$

where $a_\varepsilon(x,z,\xi)=a(x,z,\xi)+\varepsilon|\xi|^{p_+-2}\xi$ and $f\in L^1(\Omega)$, a_ε satisfies assumptions (1.1)–(1.4) with $p(x,z)$ replaced by the constant exponent p_+ (see [13, Lemma 3.1]). Since $p_+\ge p_->N$, $L^1(\Omega)\subset W^{-1,(p_+)'}(\Omega)$, we have $f\in W^{-1,(p_+)'}(\Omega)$. Therefore, there exists a weak solution $u_\varepsilon\in W_0^{1,p_+}(\Omega)$ of the problem $P(\beta_\varepsilon,f)$ in the sense

$$\beta_\varepsilon(u_\varepsilon)-\operatorname{div} a_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)=f\ \ \text{in } \mathcal{D}'(\Omega), \tag{3.2}$$

thanks to [1, Theorem 3.11] and [13, Remark 3.2].

Step 2. A priori estimates
This part is divided into several assertions and lemmas.

Assertion 3.1. *The sequence $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*

Proof. For all $\varphi\in\mathcal{D}(\Omega)$, using (3.2), we obtain

$$\int_\Omega\Big[\beta_\varepsilon(u_\varepsilon)\varphi+a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla\varphi+\varepsilon|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\nabla\varphi\Big]\,dx=\int_\Omega f\varphi\,dx. \tag{3.3}$$

Taking $\varphi=T_k(u_\varepsilon)$ in (3.3), we get

$$\int_\Omega\Big[\beta_\varepsilon(u_\varepsilon)T_k(u_\varepsilon)+a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla T_k(u_\varepsilon)+\varepsilon|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\nabla T_k(u_\varepsilon)\Big]\,dx=\int_\Omega fT_k(u_\varepsilon)\,dx.$$

Since all the terms of the left-hand side of the above equality are nonnegative, we deduce that

$$\int_\Omega\beta_\varepsilon(u_\varepsilon)T_k(u_\varepsilon)\,dx\le\int_\Omega fT_k(u_\varepsilon)\,dx,$$

which implies that

$$\int_\Omega \beta_\varepsilon(u_\varepsilon)T_k(u_\varepsilon)\,dx \le k\|f\|_{L^1(\Omega)}.$$

Dividing the above inequality by k and letting k tend to 0, we obtain

$$\int_\Omega |\beta_\varepsilon(u_\varepsilon)|\,dx \le \|f\|_{L^1(\Omega)}. \qquad \square$$

Assertion 3.2. *One has*

$$\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)}\,dx \le K_1 \tag{3.4}$$

and

$$\|u_\varepsilon\|_{W_0^{1,p_-}(\Omega)} \le K_2. \tag{3.5}$$

Proof. Using (1.4) with variable exponent $p(x,u_\varepsilon(x))$ on $a(x,u_\varepsilon,\nabla u_\varepsilon)$ and (3.3), u_ε satisfies

$$\int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon\,dx + \frac{1}{C_2}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)}\,dx + \varepsilon\int_\Omega |\nabla u_\varepsilon|^{p_+}\,dx = \int_\Omega f u_\varepsilon\,dx. \tag{3.6}$$

Applying Young's inequality to the right-hand side of (3.6) and the fact that

$$f \in L^1(\Omega) \subset W^{-1,(p_+)'}(\Omega) \subset W^{-1,(p_-)'}(\Omega),$$

we get

$$\begin{aligned}
\int_\Omega f u_\varepsilon\,dx &\le \int_\Omega |f||u_\varepsilon|\,dx \le \|f\|_{W^{-1,(p_-)'}(\Omega)}\|u_\varepsilon\|_{W_0^{1,p_-}(\Omega)} = \|f\|_{W^{-1,(p_-)'}(\Omega)}\|\nabla u_\varepsilon\|_{L^{p_-}(\Omega)}\\
&= \Big(\frac{2\,C_2}{p_-}\Big)^{\frac{1}{p_-}}\|f\|_{W^{-1,(p_-)'}(\Omega)}\Big(\frac{p_-}{2\,C_2}\Big)^{\frac{1}{p_-}}\|\nabla u_\varepsilon\|_{L^{p_-}(\Omega)}\\
&\le \frac{1}{(p_-)'}\Big(\frac{2\,C_2}{p_-}\Big)^{\frac{(p_-)'}{p_-}}\|f\|^{(p_-)'}_{W^{-1,(p_-)'}(\Omega)} + \frac{1}{2\,C_2}\|\nabla u_\varepsilon\|^{p_-}_{L^{p_-}(\Omega)}.
\end{aligned} \tag{3.7}$$

Moreover, as $p_- < \pi_\varepsilon(\,\cdot\,)$, we have

$$\|\nabla u_\varepsilon\|^{p_-}_{L^{p_-}(\Omega)} = \int_\Omega |\nabla u_\varepsilon|^{p_-}\,dx \le \operatorname{meas}(\Omega) + \int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)}\,dx. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), it follows that

$$\int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon\,dx + \frac{1}{2\,C_2}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)}\,dx + \varepsilon\int_\Omega |\nabla u_\varepsilon|^{p_+}\,dx \le Const(p_-,\Omega,f), \tag{3.9}$$

where

$$Const(p_-,\Omega,f) = \frac{1}{(p_-)'}\Big(\frac{2\,C_2}{p_-}\Big)^{\frac{(p_-)'}{p_-}}\|f\|^{(p_-)'}_{W^{-1,(p_-)'}(\Omega)} + \frac{\operatorname{meas}(\Omega)}{2\,C_2}\,.$$

From (3.9), we deduce that

$$\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)}\,dx \le 2\,C_2\,Const(p_-,\Omega,f) := K_1. \tag{3.10}$$

Thus, thanks to (3.8) and (3.10), we infer

$$\|u_\varepsilon\|_{W_0^{1,p_-}(\Omega)} \le K_2. \qquad \square$$

Remark 3.2. Using (3.5) and the compact embedding $W_0^{1,p_-}(\Omega) \hookrightarrow L^{p_-}(\Omega)$, for some subsequence still labelled with ε and some function u, one gets

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \text{ in } W_0^{1,p_-}(\Omega) \text{ as } \varepsilon \to 0,\\ \nabla u_\varepsilon &\rightharpoonup \nabla u \text{ in } L^{p_-}(\Omega) \text{ as } \varepsilon \to 0, \qquad (3.11)\\ u_\varepsilon &\to u \text{ in } L^{p_-}(\Omega) \text{ as } \varepsilon \to 0,\\ u_\varepsilon &\to u \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \qquad (3.12)\end{aligned}$$

Assertion 3.3. *The sequence $(\nabla u_\varepsilon)_{\varepsilon>0}$ converges to a Young measure $\nu_x(\lambda)$ on $\mathbb{R}^N$ in the sense of nonlinear weak-* convergence and*

$$\nabla u = \int\limits_{\mathbb{R}^N} \lambda\, d\nu_x(\lambda). \tag{3.13}$$

Proof. From (3.11) and (3.12), up to a subsequence still labelled with ε, u_ε converges a.e. in Ω to a limit u and ∇u_ε weakly converges to ∇u in $L^{p_-}(\Omega)$. Furthermore, $(\nabla u_\varepsilon)_{\varepsilon>0}$ is bounded, so it is equi-integrable on Ω. Thus, using the representation of a weakly convergent sequences in $L^1(\Omega)$ in terms of Young measures (see Theorem 2.1 and formula (2.3)), we can write

$$\nabla u = \int\limits_{\mathbb{R}^N} \lambda\, d\nu_x(\lambda). \qquad \square$$

Assertion 3.4. $|\lambda|^{\pi(\cdot)}$ *is integrable with respect to the measure $\nu_x(\lambda)\,dx$ on $\mathbb{R}^N \times \Omega$ and $\nabla u \in L^{\pi(\cdot)}(\Omega)$.*

Proof. We know that π_ε converges in measure to π. Using Theorem 2.1 (ii) and (iii), it follows that $(\pi_\varepsilon, \nabla u_\varepsilon)_{\varepsilon>0}$ converges in $\mathbb{R}\times\mathbb{R}^N$ to the Young measure $\mu_x = \delta_{\pi(x)} \otimes \nu_x$. Thus, we can apply the weak convergence properties (2.2) to the Carathéodory function

$$F_m : (x, \lambda_0, \lambda) \in \Omega \times (\mathbb{R}\times\mathbb{R}^N) \longmapsto |h_m(\lambda)|^{\lambda_0}$$

with $m \in \mathbb{N}$, where h_m is defined in the preliminary, to get

$$\begin{aligned} \int\limits_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{\pi(\cdot)}\, d\nu_x(\lambda)\,dx &= \int\limits_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)} |h_m(\lambda)|^{\lambda_0}\, d\mu_x(\lambda_0,\lambda)\,dx\\ &= \int\limits_{\Omega}\int\limits_{\mathbb{R}\times\mathbb{R}^N} F_m(x,\lambda_0,\lambda)\, d\mu_x(\lambda_0,\lambda)\,dx = \lim_{\varepsilon\to 0}\int\limits_{\Omega} F_m(x, \pi_\varepsilon(x), \nabla u_\varepsilon(x))\,dx\\ &= \lim_{\varepsilon\to 0}\int\limits_{\Omega} |h_m(\nabla u_\varepsilon)|^{\pi_\varepsilon(\cdot)}\,dx \le \lim_{\varepsilon\to 0}\int\limits_{\Omega} |\nabla u_\varepsilon|^{\pi_\varepsilon(\cdot)}\,dx \le K_1.\end{aligned}$$

Since $h_m(\lambda) \to \lambda$ as $m \to \infty$, using the Lebesgue dominated convergence theorem, as $m \mapsto h_m(\lambda)$ is increasing, we deduce that

$$\int\limits_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(\cdot)}\, d\nu_x(\lambda)\,dx \le K_1.$$

Hence, $|\lambda|^{\pi(\cdot)}$ is integrable with respect to the measure $\nu_x(\lambda)\,dx$ in $\mathbb{R}^N \times \Omega$.

Now, we prove that $\nabla u \in L^{\pi(\cdot)}(\Omega)$. Using (3.13), the Jensen inequality and the last inequality, we get

$$\int\limits_{\Omega} |\nabla u|^{\pi(\cdot)}\,dx = \int\limits_{\Omega}\Big|\int\limits_{\mathbb{R}^N} \lambda\, d\nu_x(\lambda)\Big|^{\pi(\cdot)} dx \le \int\limits_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(\cdot)}\, d\nu_x(\lambda)\,dx \le K_1.$$

Thus, $\nabla u \in L^{\pi(\cdot)}(\Omega)$. $\square$

Assertion 3.5. *The sequence* $(\Phi_\varepsilon)_{\varepsilon>0}$*, defined by* $\Phi_\varepsilon := a(x,u_\varepsilon,\nabla u_\varepsilon)$*, is equi-integrable on* Ω*.*

Proof. Using (1.3) with the exponent $\pi_\varepsilon(\,\cdot\,)$, we obtain

$$|a(x,u_\varepsilon,\nabla u_\varepsilon)|^{\pi'_\varepsilon(\,\cdot\,)} \le C_1\big(|\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)} + \mathcal{M}(x)\big).$$

The above inequality gives us

$$\begin{aligned}|a(x,u_\varepsilon,\nabla u_\varepsilon)| &\le C\Big(\big(1+|\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)}\big)+\mathcal{M}(x)\Big)^{\frac{1}{\pi'_\varepsilon(\,\cdot\,)}}\\ &\le C\Big((1+\mathcal{M}(x))^{\frac{1}{\pi'_\varepsilon(\,\cdot\,)}} + |\nabla u_\varepsilon|^{\frac{\pi_\varepsilon(\,\cdot\,)}{\pi'_\varepsilon(\,\cdot\,)}}\Big)\\ &\le C\Big(1+\mathcal{M}(x)+|\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)-1}\Big).\end{aligned}$$

For all sets $E\subset\Omega$,

$$\int_E |a(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx \le C\int_E (1+\mathcal{M}(x))\,dx + C'\big\|\,|\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)-1}\big\|_{L^{\pi'_\varepsilon(\,\cdot\,)}(\Omega)}\|\chi_E\|_{L^{\pi_\varepsilon(\,\cdot\,)}(\Omega)}, \tag{3.14}$$

where $C' = const(p_-)$. The first term of the right-hand side of the last inequality is small for meas(E) small enough, since $1+\mathcal{M}\in L^1(\Omega)$.

According to Proposition 2.1, we obtain

$$\|\chi_E\|_{L^{\pi_\varepsilon(\,\cdot\,)}(\Omega)} \le \max\Big\{\rho_{\pi_\varepsilon(\,\cdot\,)}(\chi_E)^{\frac{1}{p_+}};\rho_{\pi_\varepsilon(\,\cdot\,)}(\chi_E)^{\frac{1}{p_-}}\Big\} = \max\Big\{(\text{meas}(E))^{\frac{1}{p_+}},(\text{meas}(E))^{\frac{1}{p_-}}\Big\}.$$

Analogously,

$$\begin{aligned}\big\|\,|\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)-1}\big\|_{L^{\pi'_\varepsilon(\,\cdot\,)}(\Omega)} &\le \max\Big\{\Big(\rho_{\pi'_\varepsilon(\,\cdot\,)}\big(|\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)-1}\big)^{\frac{1}{(p')_+}}\Big),\Big(\rho_{\pi'_\varepsilon(\,\cdot\,)}\big(|\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)-1}\big)^{\frac{1}{(p')_-}}\Big)\Big\}\\ &= \max\Big\{\Big(\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)}\Big)^{\frac{1}{(p')_+}},\Big(\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(\,\cdot\,)}\Big)^{\frac{1}{(p')_-}}\Big\}.\end{aligned}$$

Using (3.4) and (3.14), $\int_E |a(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx$ is small for meas(E) small enough.

Hence, $(\Phi_\varepsilon)_{\varepsilon>0}$ is equi-integrable. □

Assertion 3.6 (see [13], Assertion 3.4)**.** *The weak limit* Φ *of* $(\Phi_\varepsilon)_{\varepsilon>0}$ *(or of a subsequence) belongs to* $L^{\pi'(\,\cdot\,)}(\Omega)$ *and*

$$\Phi(x) = \int_{\mathbb{R}^N} a(x,u(x),\lambda)\,d\nu_x(\lambda).$$

Assertion 3.7.

$$\int_\Omega \Phi\cdot\nabla u\,dx \ge \int_{\Omega\times\mathbb{R}^N} a(x,u(x),\lambda)\cdot\lambda\,d\nu_x(\lambda)\,dx. \tag{3.15}$$

Proof. For all $\varphi\in\mathcal{D}(\Omega)$ in (3.2), we have

$$\int_\Omega \beta_\varepsilon(u_\varepsilon)\varphi\,dx + \int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi\,dx + \varepsilon\int_\Omega |\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\nabla\varphi\,dx = \int_\Omega f\varphi\,dx. \tag{3.16}$$

First of all, we recall that $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$. Then, up to a subsequence still labelled with ε, there exists $z\in\mathcal{M}_b(\Omega)$ such that

$$\beta_\varepsilon(u_\varepsilon) \rightharpoonup^* z \ \text{ in } \mathcal{M}_b(\Omega) \text{ as } \varepsilon\to 0. \tag{3.17}$$

Thus, letting ε tend to 0 in (3.16), we obtain

$$\int_{\Omega} \varphi\, dz + \int_{\Omega} \Phi \cdot \nabla \varphi\, dx = \int_{\Omega} f \varphi\, dx, \tag{3.18}$$

by virtue of (3.17), Assertion 3.6 and (3.9).

Using the density argument, we can replace φ with u_ε in (3.16) to get

$$\int_{\Omega} \beta_\varepsilon(u_\varepsilon) u_\varepsilon\, dx + \int_{\Omega} a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon\, dx + \varepsilon \int_{\Omega} |\nabla u_\varepsilon|^{p_+}\, dx = \int_{\Omega} f u_\varepsilon\, dx. \tag{3.19}$$

Moreover, $u \in W_0^{1,p_-}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$ and $p(\,\cdot\,,\,\cdot\,)$ is locally uniformly log-Hölder continuous, then the exponent $\pi(\,\cdot\,)$ verifies (2.1). Therefore, $\mathcal{D}(\Omega)$ is dense in $\dot{E}^{\pi(\,\cdot\,)}(\Omega)$, so, we change φ by u in (3.18) to obtain

$$\int_{\Omega} u\, dz + \int_{\Omega} \Phi \cdot \nabla u\, dx = \int_{\Omega} f u\, dx. \tag{3.20}$$

By Fatou's lemma, we get

$$\liminf_{\varepsilon \to 0} \int_{\Omega} \beta_\varepsilon(u_\varepsilon) u_\varepsilon\, dx \geq \int_{\Omega} u\, dz. \tag{3.21}$$

Furthermore, the sequence $(f u_\varepsilon)_{\varepsilon > 0}$ converges a.e. in Ω to fu and

$$|f u_\varepsilon| \leq |f| \|u_\varepsilon\|_{L^\infty(\Omega)}.$$

Since $(u_\varepsilon)_{\varepsilon>0}$ is also uniformly bounded in $L^\infty(\Omega)$, applying Lebesgue dominated convergence Theorem, we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega} f u_\varepsilon\, dx = \int_{\Omega} f u\, dx. \tag{3.22}$$

Combining (3.21) and (3.22), it follows that

$$\liminf_{\varepsilon \to 0} \int_{\Omega} f u_\varepsilon\, dx - \int_{\Omega} u\, dz \geq \liminf_{\varepsilon \to 0} \int_{\Omega} (f u_\varepsilon - \beta_\varepsilon(u_\varepsilon) u_\varepsilon)\, dx.$$

Using (3.19), (3.20), (3.22), the last inequality and the definition of Φ_ε, we infer

$$\int_{\Omega} \Phi \cdot \nabla u\, dx \geq \liminf_{\varepsilon \to 0} \int_{\Omega} \left(\Phi_\varepsilon \cdot \nabla u_\varepsilon + \varepsilon |\nabla u_\varepsilon|^{p_+} \right) dx \geq \liminf_{\varepsilon \to 0} \int_{\Omega} \Phi_\varepsilon \cdot \nabla u_\varepsilon\, dx.$$

Hence,

$$\int_{\Omega} \Phi \cdot \nabla u\, dx \geq \liminf_{\varepsilon \to 0} \int_{\Omega} \Phi_\varepsilon \cdot \nabla u_\varepsilon\, dx. \tag{3.23}$$

By Lemma 2.1 in [1], $m \mapsto a(x, u_\varepsilon, h_m(\nabla u_\varepsilon)) \cdot h_m(\nabla u_\varepsilon)$ is increasing and converges to $a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon$ for m large enough. Then

$$a(x, u_\varepsilon, h_m(\nabla u_\varepsilon)) \cdot h_m(\nabla u_\varepsilon) \leq a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon.$$

Therefore, using (3.23) and Theorem 2.1, we have

$$\begin{aligned}
\int_{\Omega} \Phi \cdot \nabla u\, dx &\geq \liminf_{\varepsilon \to 0} \int_{\Omega} \Phi_\varepsilon \cdot \nabla u_\varepsilon\, dx \\
&\geq \lim_{\varepsilon \to 0} \int_{\Omega} a(x, u_\varepsilon, h_m(\nabla u_\varepsilon)) \cdot h_m(\nabla u_\varepsilon)\, dx \\
&= \int_{\Omega \times \mathbb{R}^N} a(x, u, h_m(\lambda)) \cdot h_m(\lambda)\, d\nu_x(\lambda)\, dx.
\end{aligned} \tag{3.24}$$

Using in (3.24) the Lebesgue dominated convergence Theorem as m tends to ∞, we get (3.15). $\square$

Assertion 3.8 (see [13], Assertion 3.6 and Assertion 3.7)**.**

(i) *The following "div-curl" inequality holds:*

$$\int\limits_{\Omega\times\mathbb{R}^N} \big(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x))\big)(\lambda-\nabla u(x))\,d\nu_x(\lambda)\,dx \le 0.$$

(ii) $\Phi(x)=a(x,u(x),\nabla u(x))$ *a.e.* $x\in\Omega$ *and* ∇u_ε *converges to* ∇u *in measure on* Ω *as* $\varepsilon\to 0$.

Lemma 3.1.

(i) $(\nabla u_\varepsilon)_{\varepsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$.

(ii) $(u_\varepsilon)_{\varepsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$.

(iii) $u\in \operatorname{dom}(\beta)$ $\mathcal{L}^N$*-a.e. in* Ω *and* $T_k(u)\in \dot{E}^{\pi(\cdot)}(\Omega)$.

Proof.
(i) Since $p_->1$, we have

$$\int\limits_\Omega |\nabla u_\varepsilon|\,dx \le \int\limits_\Omega \big(1+|\nabla u_\varepsilon|^{p_-}\big)\,dx \le \operatorname{meas}(\Omega)+C,$$

with C being a positive constant depending on K_2 and p_-. Thus, as Ω is bounded, (i) follows.
(ii) We firstly recall that $W_0^{1,p_-}(\Omega)\hookrightarrow L^{p_-}(\Omega)$ (compact embedding). So, there exists a positive constant C such that

$$\|u_\varepsilon\|_{L^{p_-}(\Omega)} \le C\|u_\varepsilon\|_{W_0^{1,p_-}(\Omega)}.$$

So,

$$\int\limits_\Omega |u_\varepsilon|\,dx \le \int\limits_\Omega \big(1+|u_\varepsilon|^{p_-}\big)\,dx \le Const(\Omega,K_2,p_-).$$

(iii) Since $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$ and $u_\varepsilon\to u$ in $L^1(\Omega)$ as $\varepsilon\to 0$, it follows that $u\in\operatorname{dom}(\beta)$ $\mathcal{L}^N$-a.e. in Ω, due to Lemma 2.4. Moreover, $u\in W_0^{1,p_-}(\Omega)\subset W_0^{1,1}(\Omega)$, due to Assertion 3.2. Thus Assertion 3.4 yields $u\in \dot{E}^{\pi(\cdot)}(\Omega)$. Since $\dot{E}^{\pi(\cdot)}(\Omega)$ is stable by truncation and $u\in \dot{E}^{\pi(\cdot)}(\Omega)$, it follows that $T_k(u)\in \dot{E}^{\pi(\cdot)}(\Omega)$. □

Lemma 3.2. *For all* $\varphi\in\mathcal{D}(\Omega)$ *and* $h\in C_c^1(\mathbb{R})$,

$$\int\limits_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(h(u_\varepsilon)\varphi)\,dx \longrightarrow \int\limits_\Omega a(x,u,\nabla u)\cdot\nabla(h(u)\varphi)\,dx \ \text{ as } \ \varepsilon\to 0. \tag{3.25}$$

Proof. First of all,

$$a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(h(u_\varepsilon)\varphi) \longrightarrow a(x,u,\nabla u)\cdot\nabla(h(u)\varphi) \ \text{ a.e. in } \Omega \text{ as } \varepsilon\to 0,$$

by (3.12) and Assertion 3.8(ii). It remains to prove that $(a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(h(u_\varepsilon)\varphi))_{\varepsilon>0}$ is equi-integrable, and one can use Vitali's convergence theorem to obtain the convergence in $L^1(\Omega)$.

Let $E\subset\Omega$, it follows from Young's inequality that

$$\begin{aligned}\int\limits_E \big|a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(h(u_\varepsilon)\varphi)\big|\,dx &\le \int\limits_E \frac{|a(x,u_\varepsilon,\nabla u_\varepsilon)|^{\pi'_\varepsilon(\cdot)}}{\pi'_\varepsilon(\cdot)}\,dx + \int\limits_E \frac{|\nabla(h(u_\varepsilon)\varphi)|^{\pi_\varepsilon(\cdot)}}{\pi_\varepsilon(\cdot)}\,dx\\ &\le \int\limits_E |a(x,u_\varepsilon,\nabla u_\varepsilon)|^{\pi'_\varepsilon(\cdot)}\,dx + \int\limits_E |\nabla(h(u_\varepsilon)\varphi)|^{\pi_\varepsilon(\cdot)}\,dx.\end{aligned} \tag{3.26}$$

Using (1.3) for the first term of the right-hand side of (3.26), we have

$$\int_E |a(x,u_\varepsilon,\nabla u_\varepsilon)|^{\pi'_\varepsilon(\cdot)} \leq \int_E C_1\big(\mathcal{M}(x)+|\nabla u_\varepsilon|^{\pi_\varepsilon(\cdot)}\big)\,dx.$$

So, $\int_E |a(x,u_\varepsilon,\nabla u_\varepsilon)|^{\pi'_\varepsilon(\cdot)}\,dx$ is small for meas(E) small enough, since $\mathcal{M}+|\nabla u_\varepsilon|^{\pi_\varepsilon(\cdot)} \in L^1(\Omega)$.

For the second term of the right-hand side of (3.26), we first recall that

$$\nabla(h(u_\varepsilon)\varphi) = h(u_\varepsilon)\nabla\varphi + h'(u_\varepsilon)\varphi\nabla u_\varepsilon.$$

Since $h \in C_c^1(\mathbb{R})$ and $\varphi \in L^\infty(\Omega)$, we have $|h'(u_\varepsilon)\varphi| \leq C_3$ and $|h(u_\varepsilon)| \leq C_4$. It follows that

$$|\nabla(h(u_\varepsilon)\varphi)| \leq |h(u_\varepsilon)\nabla\varphi| + |h'(u_\varepsilon)\varphi\nabla u_\varepsilon| \leq C_4|\nabla\varphi| + C_3|\nabla u_\varepsilon|.$$

We recall that

$$\frac{1}{2^p}(a+b)^p \leqslant \frac{1}{2}(a^p+b^p)$$

for all $a,b>0$ and $p>1$. Thus, for all set $E\subset\Omega$,

$$\begin{aligned}\int_E |\nabla(h(u_\varepsilon)\varphi)|^{\pi_\varepsilon(\cdot)}\,dx &\leq \int_E 2^{\pi_\varepsilon(\cdot)-1}\Big[\big(C_4|\nabla\varphi|\big)^{\pi_\varepsilon(\cdot)} + \big(C_3|\nabla u_\varepsilon|\big)^{\pi_\varepsilon(\cdot)}\Big]\,dx \\ &\leq \int_E 2^{p_+-1}\Big[1+\big(C_4|\nabla\varphi|\big)^{p_+} + \big(C_3|\nabla u_\varepsilon|\big)^{\pi_\varepsilon(\cdot)}\Big]\,dx. \end{aligned} \tag{3.27}$$

From Assertion 3.2 and the density argument between $\mathcal{D}(\Omega)$ and $W_0^{1,p_+}(\Omega)$, it follows that $(C_3|\nabla u_\varepsilon|)^{\pi_\varepsilon(\cdot)} \in L^1(\Omega)$ and $(C_4|\nabla\varphi|)^{p_+} \in L^1(\Omega)$. So, the left-hand side of (3.27) is small for meas(E) small enough. Therefore, $\int_E |a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(h(u_\varepsilon)\varphi)|\,dx$ is small for meas(E) small enough. Hence, $(a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(h(u_\varepsilon)\varphi))_{\varepsilon>0}$ is equi-integrable. Finally, (3.25) follows from Vitali's convergence theorem. □

Lemma 3.3. *(u,w) is a solution of the problem $P(\beta,f)$.*

Proof. First of all, we need to pass to the limit in $\beta_\varepsilon(u_\varepsilon)$.

Let us consider $\varphi\in\mathcal{D}(\Omega)$, $h\in C_c^1(\mathbb{R})$ and $h(u_\varepsilon)\varphi$ as a test function in (3.3). We have

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(h(u_\varepsilon)\varphi)\,dx + \varepsilon\int_\Omega |\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\cdot\nabla(h(u_\varepsilon)\varphi)\,dx + \int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi\,dx = \int_\Omega f h(u_\varepsilon)\varphi\,dx. \tag{3.28}$$

Since $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$, up to a subsequence still labelled with ε, there exists $z\in\mathcal{M}_b(\Omega)$ such that

$$\beta_\varepsilon(u_\varepsilon) \rightharpoonup^* z \ \text{ in } \mathcal{M}_b(\Omega) \text{ as } \varepsilon\to 0$$

and

$$h(u_\varepsilon)\varphi \to h(u)\varphi \ \text{ in } C_0(\Omega) \text{ as } \varepsilon\to 0$$

for all $\varphi\in\mathcal{D}(\Omega)$.

We also recall that $(\mathcal{M}_b(\Omega))' = C_0(\Omega)$. Then

$$\lim_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi\,dx = \int_\Omega h(u)\varphi\,dz. \tag{3.29}$$

Using the Lebesgue dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0}\int_\Omega fh(u_\varepsilon)\varphi\,dx=\int_\Omega fh(u)\varphi\,dx. \tag{3.30}$$

For the first term of the left-hand side of (3.28), we have

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(h(u_\varepsilon)\varphi)\,dx=\int_\Omega a(x,u,\nabla u)\cdot\nabla(h(u)\varphi)\,dx, \tag{3.31}$$

due to Lemma 3.2. For the second term of the left-hand side of (3.28), we get

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega |\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\cdot\nabla(h(u_\varepsilon)\varphi)\,dx=0. \tag{3.32}$$

So, using (3.29), (3.30), (3.31) and (3.32), from (3.28) we infer that

$$\int_\Omega h(u)\varphi\,dz=\int_\Omega fh(u)\varphi\,dx-\int_\Omega a(x,u,\nabla u)\cdot\nabla(h(u)\varphi)\,dx \tag{3.33}$$

for all $\varphi\in\mathcal{D}(\Omega)$. □

Remark 3.3. Since $u\in W_0^{1,p_-}(\Omega)\subset C^{0,\alpha}(\overline{\Omega})$ for $p_->N$ and $p(\,\cdot\,,\,\cdot\,)$ is locally uniformly log-Hölder continuous, $p(\,\cdot\,,u(\,\cdot\,)):=\pi(\,\cdot\,)$ verifies (1.5). Therefore, $\mathcal{D}(\Omega)$ is dense in $\dot{E}^{\pi(\,\cdot\,)}(\Omega)$. Thus, relation (3.33) holds true with $\varphi\in\dot{E}^{\pi(\,\cdot\,)}(\Omega)\cap L^\infty(\Omega)$.

In other words,

$$\int_\Omega h(u)\varphi\,dz=\int_\Omega fh(u)\varphi\,dx-\int_\Omega a(x,u,\nabla u)\cdot\nabla(h(u)\varphi)\,dx \tag{3.34}$$

for all $\varphi\in\dot{E}^{\pi(\,\cdot\,)}(\Omega)\cap L^\infty(\Omega)$, which implies that $z\in\mathcal{M}_b^{\pi(\,\cdot\,)}(\Omega)$, since, by Lemma 2.2, the spaces $\dot{E}^{\pi(\,\cdot\,)}(\Omega)$ and $W_0^{1,\pi(\,\cdot\,)}(\Omega)$ can be identified.

Now, we give a Radon–Nikodym decomposition result of the measure z.

Lemma 3.4. *The Radon–Nikodym decomposition of the measure z, given by* (3.34) *with respect to $\mathcal{L}^N$,*

$$z=w\mathcal{L}^N+\mu\ \text{ with }\ \mu\perp\mathcal{L}^N,$$

satisfies the following properties:

$$\begin{cases} w\in\beta(u)\mathcal{L}^N\text{-a.e. in }\Omega,\ \ w\in L^1(\Omega),\ \ \mu\in\mathcal{M}_b^{\pi(\,\cdot\,)}(\Omega),\\ \mu^+\ \text{ is concentrated on }[u=M]\cap[u\neq\infty]\text{ and}\\ \mu^-\ \text{ is concentrated on }[u=m]\cap[u\neq-\infty].\end{cases}$$

Proof. For the proof of Lemma 3.4, we use the arguments of [11, Lemma 3.2].

Let $(z_\varepsilon)_{\varepsilon>0}$ be a subsequence of $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ such that $z_\varepsilon\rightharpoonup^* z$ in $\mathcal{M}_b(\Omega)$.

Since, for any $\varepsilon>0$, $z_\varepsilon\in\partial j_\varepsilon(u_\varepsilon)$, we have

$$j(t)\geq j_\varepsilon(t)\geq j_\varepsilon(u_\varepsilon)+(t-u_\varepsilon)z_\varepsilon\mathcal{L}^N\text{-a.e. in }\Omega,\ \ \forall\,t\in\mathbb{R},$$

for any $h\in C_c^1(\mathbb{R})$, $h\geq 0$ and $\psi\geq 0$, it follows that

$$\psi h(u_\varepsilon)j(t)\geq\psi h(u_\varepsilon)j_\varepsilon(u_\varepsilon)+(t-u_\varepsilon)\psi h(u_\varepsilon)z_\varepsilon\mathcal{L}^N\text{-a.e. in }\Omega,\ \ \forall\,t\in\mathbb{R}.$$

Moreover, for any $0<\varepsilon<\widetilde{\varepsilon}$,

$$\psi h(u_\varepsilon)j(t)\geq\psi h(u_\varepsilon)j_{\widetilde{\varepsilon}}(u_\varepsilon)+(t-u_\varepsilon)\psi h(u_\varepsilon)z_\varepsilon\mathcal{L}^N\text{-a.e. in }\Omega,\ \ \forall\,t\in\mathbb{R},$$

and integrating over Ω, we get

$$\int_\Omega \psi h(u_\varepsilon) j(t)\,dx \geq \int_\Omega \psi h(u_\varepsilon) j_{\widetilde{\varepsilon}}(u_\varepsilon)\,dx + \int_\Omega (t-u_\varepsilon)\psi h(u_\varepsilon) z_\varepsilon\,dx.$$

As $\varepsilon \to 0$, using Fatou's Lemma, we obtain

$$\int_\Omega \psi h(u) j(t)\,dx \geq \int_\Omega \psi h(u) j_{\widetilde{\varepsilon}}(u)\,dx + \liminf_{\varepsilon\to 0}\int_\Omega (t-u_\varepsilon)\psi h(u_\varepsilon) z_\varepsilon\,dx.$$

Now, for any $\psi \in C_c^1(\Omega)$ and $t \in \mathbb{R}$, let $\widetilde{h}(r) = (t-r)h(r)$, we arrive at

$$\lim_{\varepsilon\to 0}\int_\Omega (t-u_\varepsilon)\psi h(u_\varepsilon) z_\varepsilon\,dx = \lim_{\varepsilon\to 0}\int_\Omega \widetilde{h}(u_\varepsilon)\psi z_\varepsilon\,dx = \int_\Omega (t-u)h(u)\psi\,dz.$$

So,

$$\int_\Omega \psi h(u) j(t)\,dx \geq \int_\Omega \psi h(u) j_{\widetilde{\varepsilon}}(u)\,dx + \int_\Omega (t-u)h(u)\psi\,dz.$$

As $\widetilde{\varepsilon} \to 0$, using Fatou's Lemma, we get

$$\int_\Omega \psi h(u) j(t)\,dx \geq \int_\Omega \psi h(u) j(u)\,dx + \int_\Omega (t-u)h(u)\psi\,dz.$$

From the last inequality, we infer

$$h(u)j(t) \geq h(u)j(u) + (t-u)h(u)z \ \text{ in } \mathcal{M}_b(\Omega),\ \ \forall\, t \in \mathbb{R}. \tag{3.35}$$

Using the Radon–Nikodym decomposition of z, we find that $z = w\mathcal{L}^N + \mu$ with $\mu \perp \mathcal{L}^N$, $w \in L^1(\Omega)$, and then, comparing the regular and singular parts of (3.35), for any $h \in C_c^1(\mathbb{R})$, we obtain

$$h(u)j(t) \geq h(u)j(u) + (t-u)h(u)w\mathcal{L}^N(\Omega)\text{-a.e. in } \Omega,\ \ \forall\, t \in \mathbb{R} \tag{3.36}$$

and

$$(t-u)h(u)\mu \leq 0 \ \text{ in } \mathcal{M}_b(\Omega),\ \ \forall\, t \in \overline{\mathrm{dom}(j)}. \tag{3.37}$$

From (3.36), we get

$$j(t) \geq j(u) + (t-u)w\mathcal{L}^N(\Omega) \ \text{ a.e. in } \Omega,\ \ \forall\, t \in \mathbb{R}.$$

So, $w \in \partial j(u)\mathcal{L}^N(\Omega)$. Relation (3.37) implies that for any $t \in \overline{\mathrm{dom}(j)}$,

$$\mu \geq 0 \ \text{ in } [u \in (t,+\infty) \cap \mathrm{supp}(h)] \tag{3.38}$$

and

$$\mu \leq 0 \ \text{ in } [u \in (-\infty,t) \cap \mathrm{supp}(h)]. \tag{3.39}$$

In particular, this implies that

$$\mu\big([m < u < M]\big) = 0.$$

Furthermore, if $m \neq -\infty$ (resp. $M \neq \infty$), then (3.38) (resp. (3.39)) implies that

$$\mu^+ \ \text{ is concentrated on } [u = M] \cap [u \neq \infty] \text{ (resp. } \mu^- \text{ is concentrated on } [u = m] \cap [u \neq -\infty]).$$

By the construction of measure z, it is obvious that

$$\mu([u = \pm\infty]) = 0. \qquad \square$$

Furthermore, using the Radon–Nikodym decomposition of measure z, the first term of (3.34) becomes

$$\int_\Omega h(u)\varphi\,dz = \int_\Omega h(u)w\varphi\,dx + \int_\Omega h(u)\varphi\,d\mu. \tag{3.40}$$

Combining (3.34) and (3.40), we infer

$$\int_\Omega h(u)w\varphi\,dx + \int_\Omega a(x,u,\nabla u)\cdot\nabla(h(u)\varphi)\,dx + \int_\Omega h(u)\varphi\,d\mu = \int_\Omega fh(u)\varphi\,dx \tag{3.41}$$

for all $\varphi \in \dot{E}^{\pi(\,\cdot\,)}(\Omega)\cap L^\infty(\Omega)$.

To end the proof of Theorem 3.1, it remains to prove that

$$\lim_{M\to\infty} \int_{[M<|u|<M+1]} a(x,u,\nabla u)\cdot\nabla u\,dx = 0. \tag{3.42}$$

Since $|\nabla u| \in L^{\pi(\,\cdot\,)}(\Omega)$, using (1.3) with a variable exponent $\pi(\,\cdot\,)$, $a(x,u,\nabla u) \in L^{\pi'(\,\cdot\,)}(\Omega)$, and it follows from Hölder type inequality with a variable exponent that $a(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$. Moreover, using Lemma (3.1)(ii) and Fatou's Lemma, $\int_\Omega |u|\,dx < \infty$. Therefore,

$$\int_\Omega |u|\,dx \ge \int_{[|u|\ge M]} |u|\,dx \ge M \operatorname{meas}\big([|u|\ge M]\big).$$

Then

$$\operatorname{meas}\big([|u|\ge M]\big) \le \frac{1}{M}\int_\Omega |u|\,dx \le \frac{C}{M} \tag{3.43}$$

for any $M>0$ and C being a positive constant not depending of M. Now, we have

$$\int_{[M<|u|<M+1]} a(x,u,\nabla u)\cdot\nabla u\,dx \le \int_{[|u|>M]} a(x,u,\nabla u)\cdot\nabla u\,dx \le \int_{[|u|\ge M]} a(x,u,\nabla u)\cdot\nabla u\,dx.$$

Since $\operatorname{meas}([|u|\ge M]) \to 0$ as $M\to\infty$, taking into account (3.43) and the fact that $a(x,u,\nabla u)\cdot\nabla u \in L^1(\Omega)$, we get

$$\lim_{M\to\infty} \int_{[|u|\ge M]} a(x,u,\nabla u)\cdot\nabla u\,dx = 0.$$

Hence,

$$\lim_{M\to\infty} \int_{[M<|u|<M+1]} a(x,u,\nabla u)\cdot\nabla u\,dx = 0.$$

Finally, using Lemma 3.1, Lemma 3.4, (3.41) and (3.42), we deduce that (u,w) is a solution of problem $P(\beta,f)$.

This concludes the proof of the existence result. □

In order to prove the partial uniqueness result, we make the following hypotheses on the function a, namely, the local Lipschitz continuity with respect to z.

For all bounded subsets K of $\mathbb{R}\times\mathbb{R}^N$, there exists a constant $C(K)$ such that

$$\text{a.e. } x\in\Omega, \quad \text{for all } (z,\eta),(\widetilde{z},\eta)\in K, \quad \big|a(x,z,\eta) - a(x,\widetilde{z},\eta)\big| \le C(K)|z-\widetilde{z}|. \tag{3.44}$$

Remark 3.4. Let (u,w) be a solution of the problem $P(\beta,f)$, then $u\in C(\overline{\Omega})$, since $u \in W_0^{1,p_-}(\Omega)$ and $p_- > N$. Moreover, if u is a Lipschitz continuous function, then $u \in W^{1,\infty}(\Omega)$.

Indeed, Ω is an open bounded domain with a smooth boundary $\partial\Omega$, so, the space of Lipschitz functions $C^{0,1}(\overline{\Omega})$ and $W^{1,\infty}(\Omega)$ are homeomorphic and they can be identified. The uniqueness in the sense of Theorem 3.1 seems difficult to demonstrate. Therefore, our partial uniqueness result reduces to the case where the domain of β is bounded.

Theorem 3.2. *Suppose* (1.1)–(1.5) *and* (3.44) *are satisfied, and* $\mathcal{M}$ *in* (1.3) *can be taken as a constant. Assume that* $-\infty < m \leq 0 \leq M < \infty$*. Moreover, assume that* $f \in L^1(\Omega)$ *such that the problem* $P(\beta, f)$ *has a solution* (u, w) *in the sense that* u *is a measurable and Lipschitz continuous function with* $T_k(u) \in \dot{E}^{\pi(\cdot)}(\Omega)$ *for all* $k > 0$ *and* $u \in \operatorname{dom}(\beta)\mathcal{L}^N$ *a.e. in* Ω*,* $w \in L^1(\Omega)$ *and* $w \in \beta(u)\mathcal{L}^N$ *a.e. in* Ω*; and there exists a measure* $\mu \in \mathcal{M}_b^{\pi(\cdot)}(\Omega)$ *such that* $\mu \perp \mathcal{L}^N$*,* μ^+ *is concentrated on* $[u = M]$*,* μ^- *is concentrated on* $[u = m]$ *and*

$$\int_\Omega w\varphi\,dx + \int_\Omega a(x, u, \nabla u)\nabla\varphi\,dx + \int_\Omega \varphi\,d\mu = \int_\Omega f\varphi\,dx, \tag{3.45}$$

for all $\varphi \in \dot{E}^{\pi(\cdot)}(\Omega) \cap L^\infty(\Omega)$*.*

Then any other solution $(\widetilde{u}, \widetilde{w})$ *of the problem* $P(\beta, f)$ *in the sense of equality* (3.45) *partially coincides with* (u, w)*, i.e.,*

$$w = \widetilde{w} \text{ a.e. in } \Omega \quad \text{and} \quad \mu = \widetilde{\mu}.$$

Remark 3.5. Establishing a partial uniqueness result without Lipschitz continuous assumption on u and condition (3.44) seems to be a rather difficult task, since there is no a priori guarantee that distinct solutions u_1 and u_2 are in a same test space. The uniqueness result is valid for $W^{1,\infty}$-solutions.

Proof of Theorem 3.2. First of all, we complete the proof of the existence by proving the relation (3.45). Here, we consider the function h_k, where k is a positive constant such that

$$\begin{cases} h_k \in C_c^1(\mathbb{R}), \quad h_k(s) \geq 0, \quad \forall\, s \in \mathbb{R}, \\ h_k(s) = 1 \text{ if } |s| < k \text{ and } h_k(s) = 0 \text{ if } |s| \geq k. \end{cases}$$

Since the domain of β is bounded and equality (3.41) holds for any $h \in C_c^1(\mathbb{R})$, we take $h(s) = h_k(s) = 1$ for all $s \in [m, M] \subsetneq [-k, k] = \operatorname{supp}(h_k)$, which implies

$$\int_\Omega w\varphi\,dx + \int_\Omega a(x, u, \nabla u)\cdot\nabla\varphi\,dx + \int_\Omega \varphi\,d\mu = \int_\Omega f\varphi\,dx,$$

for all $\varphi \in \dot{E}^{\pi(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Now, we prove the partial uniqueness result (for more details, see [2, Proof of Theorem 2.8] and [13, Proof of Theorem 3.7]).

Let (u_1, w_1) be a solution of the problem $P(\beta, f)$, where u_1 is a Lipschitz continuous function, and (u_2, w_2) is another solution to problem $P(\beta, f)$ in the sense of equality (3.45).

Let $\phi := \frac{1}{k}T_k(u_1 - u_2)$, then ϕ is an admissible test function in the formulations for both (u_1, w_1) and (u_2, w_2). Thus, with this test function, we have

$$\begin{aligned} \int_\Omega w_1 \frac{1}{k} T_k(u_1 - u_2)\,dx + \frac{1}{k}\int_\Omega a(x, u_1, \nabla u_1)\cdot\nabla(u_1 - u_2)\chi_{[0<|u_1-u_2|<k]}\,dx \\ + \int_\Omega \frac{1}{k} T_k(u_1 - u_2)\,d\mu_1 = \int_\Omega f\,\frac{1}{k} T_k(u_1 - u_2)\,dx \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} \int_\Omega w_2 \frac{1}{k} T_k(u_1 - u_2)\,dx + \frac{1}{k}\int_\Omega a(x, u_2, \nabla u_2)\cdot\nabla(u_1 - u_2)\chi_{[0<|u_1-u_2|<k]}\,dx \\ + \int_\Omega \frac{1}{k} T_k(u_1 - u_2)\,d\mu_2 = \int_\Omega f\,\frac{1}{k} T_k(u_1 - u_2)\,dx. \end{aligned} \tag{3.47}$$

We substract (3.46) and (3.47) to get

$$\frac{1}{k}\int_\Omega \big(a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_2)\big)\cdot\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}\,dx \\ +\int_\Omega (w_1-w_2)\,\frac{1}{k}\,T_k(u_1-u_2)\,dx+\int_\Omega \frac{1}{k}\,T_k(u_1-u_2)(d\mu_1-d\mu_2)=0. \quad (3.48)$$

By I we denote the first term of the left-hand side of (3.48). It is known that

$$\big(a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_2)\big)\nabla(u_1-u_2) \\ =\big(a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_1)\big)\nabla(u_1-u_2)+\underbrace{\big(a(x,u_2,\nabla u_1)-a(x,u_2,\nabla u_2)\big)\nabla(u_1-u_2)}_{\geq 0}.$$

Hence

$$I=I_k+\int_\Omega \big(a(x,u_2,\nabla u_1)-a(x,u_2,\nabla u_2)\big)\,\frac{1}{k}\,\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}\,dx,$$

where

$$I_k=\int_\Omega \big(a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_1)\big)\,\frac{1}{k}\,\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}\,dx.$$

Let us show that $I_k\to 0$ as $k\to 0$. Since u_1 is bounded, u_2 is also bounded on the set $[0<|u_1-u_2|<k]$. Thus,

$$\begin{aligned} |I_k| &\leq \frac{1}{k}\int_{[0<|u_1-u_2|<k]} \big|a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_1)\big|\,|\nabla u_1-\nabla u_2|\,dx \\ &\leq \frac{1}{k}\int_{[0<|u_1-u_2|<k]} C\big(\|u_1\|_{L^\infty(\Omega)},\|\nabla u_1\|_{L^\infty(\Omega)}\big)|u_1-u_2|\,|\nabla u_1-\nabla u_2|\,dx \quad \text{(by using (3.44))} \\ &\leq C\big(\|u_1\|_{L^\infty(\Omega)},\|\nabla u_1\|_{L^\infty(\Omega)}\big)\int_{[0<|u_1-u_2|<k]} |\nabla u_1-\nabla u_2|\,dx\longrightarrow 0 \ \text{ as } \ k\to 0. \end{aligned} \quad (3.49)$$

Notice that $\lim\limits_{k\to 0}\operatorname{meas}([0<|u_1-u_2|<k])=0$ and $|\nabla u_1-\nabla u_2|\in L^1(\Omega)$.

For the second term of the left-hand side of (3.48), we have

$$\lim_{k\to 0}\int_\Omega (w_1-w_2)\,\frac{1}{k}\,T_k(u_1-u_2)\,dx=\int_\Omega (w_1-w_2)\operatorname{sign}_0(u_1-u_2)\,dx=\int_\Omega |w_1-w_2|\,dx, \quad (3.50)$$

and for the third term of the left-hand side of (3.48),

$$\lim_{k\to 0}\int_\Omega \frac{1}{k}\,T_k(u_1-u_2)(d\mu_1-d\mu_2) \\ =\int_\Omega \operatorname{sign}_0(u_1-u_2)(d\mu_1-d\mu_2)=\int_\Omega |d\mu_1-d\mu_2|=\int_\Omega |d(\mu_1-\mu_2)|. \quad (3.51)$$

Finally, letting k tend to 0 in (3.48) and taking into account inequalities (3.49), (3.50) and (3.51), we obtain

$$\lim_{k\to 0}\int_\Omega \big(a(x,u_2,\nabla u_1)-a(x,u_2,\nabla u_2)\big)\,\frac{1}{k}\,\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}\,dx \\ +\int_\Omega |w_1-w_2|\,dx+\int_\Omega |d(\mu_1-\mu_2)|=0. \quad (3.52)$$

Since all the terms of equality (3.52) are nonnegative, we deduces that

$$\int_\Omega |w_1-w_2|\,dx=0 \ \text{ and } \ \int_\Omega |d(\mu_1-\mu_2)|=0.$$

Hence

$$w_1=w_2 \ \text{ a.e. in } \Omega \quad \text{and} \quad \mu_1=\mu_2. \qquad \square$$

References

[1] B. Andreianov, M. Bendahmane and S. Ouaro, Structural stability for variable exponent elliptic problems, I: the $p(x)$-Laplacian kind problems. *Nonlinear Anal.* **73** (2010), no. 1, 2–24.

[2] B. Andreianov, M. Bendahmane and S. Ouaro, Structural stability for variable exponent elliptic problems. II. The $p(u)$-Laplacian and coupled problems. *Nonlinear Anal.* **72** (2010), no. 12, 4649–4660.

[3] S. N. Antontsev and J. F. Rodrigues, On stationary thermo-rheological viscous flows. *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **52** (2006), no. 1, 19–36.

[4] J. M. Ball, A version of the fundamental theorem for Young measures. *PDEs and continuum models of phase transitions (Nice, 1988)*, 207–215, Lecture Notes in Phys., 344, *Springer, Berlin*, 1989.

[5] H. Brezis, *Opérateurs Maximaux Monotones et Semigroupes de Contractions Dans les Espaces de Hilbert.* North Holland, Amsterdam, 1973.

[6] J. K. Brooks and R. V. Chacon, Continuity and compactness of measures. *Adv. in Math.* **37** (1980), no. 1, 16–26.

[7] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* **66** (2006), no. 4, 1383–1406.

[8] R. J. DiPerna and P.-L. Lions, On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)* **130** (1989), no. 2, 321–366.

[9] R. Eymard, T. Gallouët and R. Herbin, Finite volume methods. *Handbook of numerical analysis, Vol. VII*, 713–1020, Handb. Numer. Anal., VII, *North-Holland, Amsterdam*, 2000.

[10] N. Hungerbühler, A refinement of Ball's theorem on Young measures. *New York J. Math.* **3** (1997), 48–53.

[11] N. Igbida, S. Ouaro and S. Soma, Elliptic problem involving diffuse measure data. *J. Differential Equations* **253** (2012), no. 12, 3159–3183.

[12] I. Nyanquini, S. Ouaro and S. Soma, Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data. *An. Univ. Craiova Ser. Mat. Inform.* **40** (2013), no. 2, 174–198.

[13] S. Ouaro and N. Sawadogo, Suitable radon measure for nonlinear Dirichlet boundary $p(u)$-Laplacian problem. *Matematiche (Catania)* **77** (2022), no. 1, 3–31.

[14] P. Pedregal, *Parametrized Measures and Variational Principles.* Progress in Nonlinear Differential Equations and their Applications, 30. Birkhäuser Verlag, Basel, 1997.

[15] M. Růžička, *Electrorheological Fluids: Modelling and Mathematical Theory.* Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, 2002.

[16] J. Tiirola, Image denoising using directional adaptive variable exponents model. *J. Math. Imaging Vision* **57** (2017), no. 1, 56–74.

(Received 23.08.2024; accepted 06.02.2025)

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