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ANALYTICAL STUDY OF AN ELECTROMECHANICAL EQUATION MODEL OF A DRILLING SYSTEM

Abstract. Through this work, we present a general study of certain mechanical systems, particularly electromechanical problems related to drilling systems. The approach employed combines Schaefer's fixed point theorem with the Bressan–Colombo selection theorem for lower semicontinuous maps with decomposable values. We show that a first-order differential inclusion system admits solutions.

As a second result, a Filippov–Gronwall-type estimate is established for the solutions to this problem. We conclude the work with several examples that illustrate the obtained results.

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1 Introduction

Differential inclusion systems extend the classical framework of differential equations and describe a wide range of practical systems. Their study has received significant attention (see [4, 13, 20, 23, 24, 27, 28]). Concerning mechanical and electromechanical problems, the study of drilling systems requires accurate mathematical models that capture the interaction between the mechanical and electrical components of the process. In particular, the electromechanical model of a drilling system, which combines the torsional dynamics of the drill string with the dynamic behavior of the induction motor, has become an essential tool for both analysis and control design. Induction motors are widely used in drilling rigs because of their robustness, cost-effectiveness, and high reliability under demanding operational conditions. By considering the motor's electrical and mechanical equations alongside the drill string dynamics, researchers are able to reproduce complex phenomena such as stick–slip oscillations, bit–rock interaction, and resonance, which are common sources of inefficiency and equipment damage in drilling operations. The development of such models, as proposed in [11, 25], provides a rigorous framework for studying system stability, designing vibration suppression techniques, and improving the overall efficiency and safety of drilling processes. The equations of the induction motor proposed in [15, 19], supplemented with the drilling resistance torque $\mathcal{M}_f$, are given by the following system:

$$\begin{cases} \dfrac{di}{d\iota} + \mathcal{R}i = \mathcal{S}\mathcal{B}(\sin\theta)\dot{\theta}, \\ \mathcal{L}\dfrac{dj}{d\iota} + \mathcal{R}j = \mathcal{S}\mathcal{B}(\cos\theta)\dot{\theta}, \\ \mathcal{I}\ddot{\theta} \in -\beta\mathcal{S}\mathcal{B}(i\sin\theta + j\cos\theta) + \mathcal{M}_f\Big(\dfrac{\mathcal{R}}{\mathcal{L}} + \dot{\theta}\Big), \end{cases} \tag{$\mathcal{P}_0$}$$

where θ is a rotation angle of the drill about the magnetic field created by the stator, which rotates with a constant angular speed $\frac{\mathcal{R}}{\mathcal{L}}$, $i(\iota)$, $j(\iota)$ are the currents in rotor windings, where $\mathcal{R}$ denotes the resistance of the windings, $\mathcal{L}$ is their inductance, $\mathcal{B}$ is the induction of magnetic field, and $\mathcal{S}$ is the area of one winding, the inertia torque of the drill is represented by $\mathcal{I}$, while β is a proportionality factor. The angular velocity of the drill with respect to a fixed coordinate system is given by $\varpi = \dot{\theta} + \frac{\mathcal{R}}{\mathcal{L}}$. The resistance torque $\mathcal{M}_f$ is assumed to be of the Coulomb type [9, 22]. However, unlike the classical Coulomb friction law, which is characterized by a symmetric discontinuous profile, the friction torque $\mathcal{M}_f$ exhibits a non-symmetric discontinuous characteristic, as illustrated below:

$$\mathcal{M}_f(\omega) = \begin{cases} -\mathcal{T}_0 & \text{for } \omega > 0, \\ [-\mathcal{T}_0, M\mathcal{T}_0] & \text{for } \omega = 0, \\ M\mathcal{T}_0 & \text{for } \omega < 0. \end{cases}$$

For $\mathcal{T}_0 \geq 0$, the constant M is considered to be sufficiently large, which implies that the drilling process takes place only when $\varpi > 0$. This property prevents ϖ from changing sign during the transient regime of real drilling systems. The system may thus become stuck at $\omega = 0$ for a long enough period of time. These phenomena frequently arise in drilling operations and are studied within the framework of the system $\mathcal{P}_0$.

Using a simple change of variables given (see [14]) by

$$\begin{cases} s(\iota) = -\dot{\theta}, \\ v(\iota) = \dfrac{\mathcal{L}}{\mathcal{S}\mathcal{B}}(i\cos\theta - j\sin\theta), \\ u(\iota) = \dfrac{\mathcal{L}}{\mathcal{S}\mathcal{B}}(i\sin\theta + j\cos\theta), \end{cases}$$

the problem $\mathcal{P}_0$ is transformed to a first order differential inclusion system of the form

$$\begin{cases} \dot{s}(\iota) \in av - \widetilde{\mathcal{M}}_f(s), \\ \dot{v}(\iota) = -cv - s - us, \\ \dot{u}(\iota) = -cu + vs, \end{cases} \tag{$\mathcal{P}_1$}$$

where

$$a = \frac{\beta(\mathcal{SB})^2}{\mathcal{IL}}, \quad c = \frac{\mathcal{R}}{\mathcal{L}} \text{ and } \widetilde{\mathcal{M}}_f = \begin{cases} -\gamma M & \text{for } s > c, \\ [-\gamma M, \gamma] & \text{for } s = c, \\ \gamma & \text{for } s < c. \end{cases}$$

Also, according to Chua, the system with discontinues characteristic [16–18] is given by

$$\begin{cases} \dot{u}_1(\iota) \in -\alpha(m_1+1)u_1(\iota) + \alpha u_2(\iota) - \alpha(m_0 - m_1)Sign(u_1(\iota)), \\ \dot{u}_2(\iota) = u_1(\iota) - u_2(\iota) + u_3(\iota), \\ \dot{u}_3(\iota) = -\beta u_2(\iota) - \gamma u_3(\iota), \end{cases} \tag{$\mathcal{P}_2$}$$

where α, β, γ, m_0, m_1 are parameters of the system $\mathcal{P}_2$, and $Sign(x_1)$ is defined by

$$Sign(u_1) = \begin{cases} +1 & \text{for } u_1 > 0, \\ [-1,1] & \text{for } u_1 = 0, \\ -1 & \text{for } u_1 < 0. \end{cases}$$

In this paper we consider the problem of differential inclusion of first order system of the form:

$$\begin{cases} \dot{u}_1(\iota) \in f_1(t, u_1, u_2, u_3) - \lambda^* F(t, u_1), \\ \dot{u}_2(\iota) = f_2(t, u_1, u_2, u_3), \\ \dot{u}_3(\iota) = f_3(t, u_1, u_2, u_3), \end{cases} \quad \text{a.e. } t \in [0, \mathcal{T}] \tag{1.1}$$

with the initial values

$$u_1(0) = u_2(0) = u_3(0) = 0, \tag{1.2}$$

where $\lambda^* \in \mathbb{R}_+$, $f_i(t, u_1, u_2, u_3) \in C([0,\mathcal{T}] \times \mathbb{R}^3, \mathbb{R})$ with $i \in \{1,2,3\}$, $F : [0,\mathcal{T}] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

In this study, we focus on the mathematical aspects of this type of equations. The functions $Sign$ and $\widetilde{\mathcal{M}}f$ are considered in their general forms. The aim of this paper is to present the general case of problems $\mathcal{P}1$ and $\mathcal{P}_2$. We establish sufficient conditions for the existence of solutions to problem (1.1), (1.2), relying on Schaefer's fixed point theorem together with the Bressan–Colombo selection theorem for lower semicontinuous maps with decomposable values. In addition, we establish a Filippov–Gronwall-type inequality for the corresponding solutions. To conclude, we provide examples based on drilling system models that demonstrate the applicability of our theoretical findings.

The structure of this article is as follows. In Section 2, we present the notation and preliminary results, including lemmas and theorems essential for establishing our main results. Section 3 contains the main results. Finally, we conclude by giving conditions for the existence of solutions for two important systems: the dynamics of the Watt governor with dry friction and the Chua system with a discontinuous characteristic [14].

2 Preliminaries

We first recall some notations. Namely, we denote by $C([0,\mathcal{T}], \mathbb{R}^3)$ the Banach space formed by all continuous functions $u(\cdot) : [0,\mathcal{T}] \to \mathbb{R}^3$ equipped with the supremum norm $\|u\|_\infty = \sup\{\|u(\iota)\| \text{ for all } \iota \in [0,\mathcal{T}]\}$, where $\|\cdot\|$ is the norm in the space $\mathbb{R}^3$.

We consider the Banach space $L^1([0,\mathcal{T}], \mathbb{R})$, consisting of all Lebesgue integrable functions $\varphi : [0,T] \to \mathbb{R}$, equipped with the norm $\|\varphi\|_{L^1} = \int\limits_0^{\mathcal{T}} |\varphi(\iota)|\, d\iota$, and let $AC([0,\mathcal{T}], \mathbb{R}^3)$ denote the space of absolutely continuous functions on $[0,\mathcal{T}]$ with values in $\mathbb{R}^3$.

Consider $A \subset [0,\mathcal{T}] \times \mathbb{R}^3$. The set A is $\mathcal{L} \otimes \mathcal{B}$ measurable precisely when it belongs to σ-algebra generated by sets of the type $\mathcal{I} \times \mathcal{D}$, where $\mathcal{I}$ is a Lebesgue measurable subset of $[0,\mathcal{T}]$ and $\mathcal{D}$ is the Borel set in $\mathbb{R}^3$.

Definition 2.1. A subset $\mathcal{A}$ of $L^1([0,\mathcal{T}],\mathbb{R})$ is said to be decomposable if, for all $\mu,\nu \in \mathcal{A}$ and $\mathcal{I} \subset [0,\mathcal{T}] = I$ measurable, the function $\mu\chi_{\mathcal{I}} + \nu\chi_{I-\mathcal{I}} \in \mathcal{A}$, where χ denotes the characteristic function.

Let $(\mathcal{X}, d)$ be a metric space induced from the normed space $(X, \|\cdot\|)$. We denote $\rceil E(\mathcal{X}) = \{A \in \mathcal{P}(\mathcal{X}) : \ A \neq \phi\}$, $Cl(\mathcal{X}) = \{A \in \rceil E(\mathcal{X}) : \ A \text{ is closed}\}$, $Bnd(\mathcal{X}) = \{A \in \rceil E(\mathcal{X}) : \ A \text{ is bounded}\}$, $Cmp(\mathcal{X}) = \{A \in \rceil E(\mathcal{X}) : \ A \text{ is compact}\}$, $Cv(\mathcal{X}) = \{A \in \rceil E(\mathcal{X}) : \ A \text{ is convex}\}$.

For two closed subsets $\mathcal{A},\mathcal{B} \subset \mathcal{X}$, the Pompeiu–Hausdorff distance is given by

$$\mathcal{H}_d(\mathcal{A},\mathcal{B}) = \max\left\{d^*(\mathcal{A},\mathcal{B}), d^*(\mathcal{A},\mathcal{B})\right\},$$

where $d^*(\mathcal{A},\mathcal{B}) = \sup\{d(a,\mathcal{B}), a \in \mathcal{A}\}$.

Consider a separable Banach space. Suppose $\mathcal{C} \subset \mathcal{E}$ is a nonempty closed set, and $F : \mathcal{C} \to \mathcal{P}_{cl}(\mathcal{E})$ is a set-valued mapping. The mapping F is said to be lower semicontinuous (l.s.c.) if, for every open set $\mathcal{O} \subset \mathcal{E}$, the set $\{u \in \mathcal{C} : F(u) \cap \mathcal{O} \neq \varnothing\}$ is open. A point $u \in \mathcal{C}$ is called a fixed point of F if $u \in F(u)$.

Background material on multivalued mappings is available in the books of Aubin and Cellina [1], Aubin and Frankowska [2], Deimling [6], Górniewicz [10], and Hu & Papageorgiou [12].

Definition 2.2. Let X be a separable metric space and let $N : X \to \mathcal{P}_{cl}(L^1([0,\mathcal{T}],\mathbb{R}))$ be a multivalued operator. We say that N has the property $(\mathcal{BC})$ if the following conditions are satisfied:

1) N is lower semi-continuous (l.s.c),

2) The values of N are decomposable.

Let $F : [0,\mathcal{T}] \times \mathbb{R} \to Cmp(\mathbb{R})$ be a multivalued map. We define the associated operator $\mathcal{F} : C([0,\mathcal{T}],\mathbb{R}) \to \mathcal{P}_0(L^1([0,\mathcal{T}],\mathbb{R}))$ by

$$\mathcal{F}(u_1) = \left\{w \in L^1([0,\mathcal{T}],\mathbb{R}) : \ \ w(\iota) \in F(\iota, u_1(\iota)) \ \text{ for a.e. } \ \iota \in [0,\mathcal{T}]\right\}.$$

The operator $\mathcal{F}$ induced by F is called the Niemytzki operator. We say that F is of (l.s.c.) type if $\mathcal{F}$ is lower semicontinuous and takes values that are nonempty, closed, and decomposable.

In what follows, we present several selection theorems for multivalued functions, which will play a central role in our arguments.

Lemma 2.1 ([3])**.** *Let $\mathcal{X}$ be a separable metric space and let $\mathcal{N} : \mathcal{X} \to Cl(L^1([0,\mathcal{T}],\mathbb{R}))$ be a multivalued operator having property $(\mathcal{BC})$. Then $\mathcal{N}$ admits a continuous selection, i.e., there exists a continuous single-valued mapping $g : \mathcal{X} \to L^1([0,\mathcal{T}],\mathbb{R})$ such that $g(u) \in \mathcal{N}(u)$ for every $u \in \mathcal{X}$.*

Definition 2.3. A multivalued map $\mathcal{G} : J \to Cmp(\mathcal{X})$ is called measurable if for each $u \in \mathcal{X}$, the mapping $\iota \to d(u, \mathcal{G}(\iota))$ is measurable on J.

Lemma 2.2 (see [10, Theorem 19.7])**.** *Let $\mathcal{E}$ be a separable metric space and G be a multivalued map with nonempty closed values. Then $\mathcal{G}$ has a measurable selection.*

Lemma 2.3 ([21])**.** *Let $\mathcal{G} : [0,b] \to Cl(\mathbb{R})$ be a measurable multivalued map, and let $u : [0,b] \to \mathbb{R}$ be a measurable function. Assume that there exist $p \in L^1(J,\mathbb{R})$ such that $\mathcal{G}(\iota) \subseteq p(\iota)\overline{B}(0,1)$ for a.e. $\iota \in [0,b]$, where $\overline{B}(0,1)$ denotes the closed ball in $\mathbb{R}$. Then there exists a measurable selection g of $\mathcal{G}$ such that*

$$|u(\iota) - g(\iota)| \leq d(u(\iota), \mathcal{G}(\iota)) \ \ \textit{for a.e. } \ \iota \in [0,b].$$

Consider the following problem:

$$\begin{cases} u_1' \in f_1(\iota, u_1, u_2, u_3) - \lambda^* g(u_1), & \\ u_2' = f_2(\iota, u_1, u_2, u_3), & \text{a.e. } \ t \in [0,\mathcal{T}] \\ u_3' = f_3(\iota, u_1, u_2, u_3), & \\ u_1(0) = u_2(0) = u_3(0) = 0, & \end{cases} \tag{2.1}$$

where $\lambda^* \in \mathbb{R}_+$, $f_i \in C([0,\mathcal{T}] \times \mathbb{R}^3, \mathbb{R})$ with $i \in \{1,2,3\}$, and $g : C([0,\mathcal{T}],\mathbb{R}) \to L^1([0,\mathcal{T}],\mathbb{R})$ is a continuous map.

Definition 2.4. We say that $u = \langle u_1, u_2, u_3\rangle \in C([0,\mathcal{T}],\mathbb{R}^3)$ solves (2.1) if u is given by

$$u(\iota) = \int_0^{\iota} f(s,u)\,ds - \lambda^* \int_0^{\iota} G(u^*)\,ds,$$

where $u^* = \langle u_1, 0, 0\rangle$, $G = \langle g, 0, 0\rangle \in C(\mathbb{R},\mathbb{R}^3)$, and $f = \langle f_1, f_2, f_3\rangle \in C([0,\mathcal{T}]\times\mathbb{R}^3,\mathbb{R}^3)$.

Definition 2.5. A function $u = \langle u_1, u_2, u_3\rangle \in C([0,\mathcal{T}],\mathbb{R}^3)$ is said to be a solution of problem (1.1), (1.2), if there exists a function $g : C([0,\mathcal{T}],\mathbb{R}) \to L^1([0,\mathcal{T}],\mathbb{R})$ such that

$$g(u_1(\iota)) \in F(\iota, u_1(\iota)) \;\text{ for all }\; u_1 \in C([0,\mathcal{T}],\mathbb{R}),$$

$$u(\iota) = \int_0^{\iota} f(s,u)\,ds - \lambda^* \int_0^{\iota} G(u^*)\,ds \;\text{ a.e. }\; \iota \in [0,\mathcal{T}],$$

with $u^* = \langle u_1, 0, 0\rangle$, $G = \langle g, 0, 0\rangle \in C(\mathbb{R},\mathbb{R}^3)$, and $f \in C([0,\mathcal{T}]\times\mathbb{R}^3,\mathbb{R}^3)$.

3 The main results

In this section, we employ Schaefer's fixed point theorem in combination with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values to prove the existence of solutions to problem (1.1), (1.2). Furthermore, we establish some Filippov-type results related to this problem.

Existence of Solutions

Before studying the existence of solutions to problem (1.1), (1.2), we introduce the following assumptions:

Hypothesis 1:

$F : [0,\mathcal{T}]\times\mathbb{R} \to Cmp(\mathbb{R})$ is a multivalued map verifying:

$$(H_1)\ \begin{cases} \text{(a)} \quad (\iota,u) \to F(\iota,u) \;\text{ is } \mathcal{L}\otimes\mathcal{B}\text{-measurable.} \\ \text{(b)} \quad u \to F(\iota,u) \;\text{ is (l.s.c) for a.e. } \iota \in [0,1]. \end{cases}$$

(H_2) The multifunction F is integrably bounded, that is, there exists a nonnegative function $m \in L^1([0,\mathcal{T}],\mathbb{R}^+)$ such that, for almost every $\iota \in [0,\mathcal{T}]$ and for all u in $\mathbb{R}$, one has

$$\|F(\iota,u)\| = \sup\left\{\|v\|_\infty : \;\; v \in F(\iota,u)\right\} \leqslant m(\iota).$$

In our result, we obtain the existence of a solution to (1.1), (1.2) by considering that the function $f(\iota,x)$ satisfies the following hypotheses:

(H_3) There exists a continuous function $M(\iota) > 0$ such that

$$\|f(\iota,x) - f(\iota,y)\| \le M(\iota)\|x-y\|.$$

(H_4) There exists $i_0 \in \{1,2,3\}$ such that

$$f_{i_0}(t,0,0,0) \neq 0 \;\text{ for all }\; \iota \in E,$$

or the multivalued map F satisfies $0 \notin F(t,0)$ for all $\iota \in [0,\mathcal{T}]$.

Denote

$$\sup_{t\in[0,T]} M(\iota) = M, \qquad \sup_{\iota\in[0,\mathcal{T}]} \|f(\iota,0)\| = \widetilde{M}.$$

Lemma 3.1 ([8]). *Let $F : [0, \mathcal{T}] \times \mathbb{R} \to Cmp(\mathbb{R})$ be a multivalued map. If the assumptions (H_1) and (H_2) hold, then F is of the l.s.c type.*

Theorem 3.1. *If hypotheses (H_1)–(H_4) are satisfied, then problem (1.1), (1.2) has at least a non-trivial solution whenever condition*

$$1 - \mathcal{T}M > 0 \tag{3.1}$$

holds.

Proof. It follows from Lemma 3.1 and Hypotheses (H_1) and (H_2) that the multivalued map F is of lower semi-continuous type. Then, Lemma 2.1 guarantees the existence of a continuous function $g : C([0, \mathcal{T}], \mathbb{R}) \to L^1([0, \mathcal{T}], \mathbb{R})$ such that $g(u_1) \in \mathcal{F}(u_1)$ for all $u_1 \in C([0, \mathcal{T}], \mathbb{R})$.

We consider the operator $\Lambda : C([0, \mathcal{T}], \mathbb{R}^3) \to C([0, \mathcal{T}], \mathbb{R}^3)$ defined by

$$\Lambda(u)(\iota) = \int_0^\iota f(s, u)\, ds - \lambda^* \int_0^\iota G(u^*)\, ds.$$

Note that if $u \in C([0, \mathcal{T}], \mathbb{R}^3)$ is a fixed point of Λ, then u solves problem (1.1), (1.2).

Next, we establish the compactness of Λ.

Step 1: The continuity of operator Λ.

Let $\{u_n = \langle u_1^n, u_2^n, u_3^n\rangle\}$ be a sequence such that $u_n \to u$ in $C([0, \mathcal{T}], \mathbb{R}^3)$. Define $u_n^* = \langle u_1^n, 0, 0\rangle$ and $u^* = \langle u_1, 0, 0\rangle$. Then $u_n^* \to u^*$. So,

$$\big\|\Lambda(u_n)(\iota) - \Lambda(u)(\iota)\big\| \le \int_0^\iota \|f(s, u_n) - f(s, u)\|\, ds + \lambda^* \int_0^\iota \|G(u_n^*) - G(u^*)\|\, ds.$$

By the continuity of f and G, we deduce that $\|\Lambda(u_n) - \Lambda(u)\|_\infty \to 0$ as $n \to \infty$.

Step 2: T maps bounded subsets of $C([0, \mathcal{T}], \mathbb{R}^3)$ into bounded subsets.

Equivalently, there exists c such that for every $h \in \Lambda(u), u \in B_r = \{u \in C([0, \mathcal{T}], \mathbb{R}^3) : \|u\|_\infty \le r\}$, one has $\|h\|_\infty \le c$. Using (H_2) and (H_3), for every $\iota \in [0, \mathcal{T}]$, we have

$$\begin{aligned}
\|h(\iota)\| = \bigg\| \int_0^\iota f(s, u)\, ds - \lambda^* \int_0^\iota G(u^*)\, ds \bigg\| &\le \int_0^{\mathcal{T}} \|f(s, u)\| ds + \lambda^* \int_0^{\mathcal{T}} \|G(u^*)\|\, ds \\
&\le \int_0^{\mathcal{T}} \big(\|f(s, u) - f(s, 0)\| + \|f(s, 0)\|\big)\, ds + \lambda^* \int_0^{\mathcal{T}} m(s)\, ds \\
&\le \mathcal{T}(M\|u\|_\infty + \widetilde{M}) + \lambda^* \int_0^{\mathcal{T}} m(s)\, ds \le \mathcal{T}(Mr + \widetilde{M}) + \lambda^* \int_0^{\mathcal{T}} m(s)\, ds = c.
\end{aligned}$$

Then

$$\|h\|_\infty \le c.$$

Step 3: Λ maps bounded subsets B_r of $C([0, \mathcal{T}], \mathbb{R}^3)$ into equicontinuous subsets.

Take $\iota_1, \iota_2 \in [0, \mathcal{T}]$ with $\iota_1 < \iota_2$, then, for every $h \in \Lambda(u)$ with $u \in B_r$, we obtain

$$\|h(\iota_2) - h(\iota_1)\| \le \int_{\iota_1}^{\iota_2} \|f(s, u)\|\, ds + \lambda^* \int_{\iota_1}^{\iota_2} \|g(u^*)\|\, ds.$$

As $\iota_2 \to \iota_1$, the right-hand side of the previous inequality tends to zero. Combining Steps 1 to 3 and the Arzelà–Ascoli theorem, we can conclude that Λ is completely continuous.

To complete the application of Schaefer's theorem, we must demonstrate

Step 4: The set $\Xi = \{u \in C([0,\mathcal{T}],\mathbb{R}^3) : \lambda u = \Lambda(u) \text{ for some } \lambda > 1\}$ is bounded.
Let $u \in \Xi$. Then $\lambda u = \Lambda(u)$ for some $\lambda > 1$, and

$$u(\iota) = \lambda^{-1}\Big(\int_0^\iota f(s,u)\,ds - \lambda^* \int_0^\iota G(u^*)\,ds\Big),$$

which implies by (H_2) and (3.4) that for each $\iota \in [0,\mathcal{T}]$, we have

$$\begin{aligned}
\|u(\iota)\| &\le |\lambda^{-1}|\Big(\int_0^\iota \|f(s,u)\|\,ds + \lambda^* \int_0^\iota \|G(u^*)\|\,ds\Big)\\
&\le \int_0^{\mathcal{T}} \big(\|f(s,u) - f(s,0)\| + \|f(s,0)\|\big)\,ds + \lambda^* \int_0^{\mathcal{T}} m(s)\,ds,\\
\|u(\iota)\| &\le \frac{\mathcal{T}\widetilde{M} + \lambda^* \int_0^{\mathcal{T}} m(s)\,ds}{1-\mathcal{T}M} = K.
\end{aligned}$$

Thus $\|u\|_\infty \le K$. This shows that Ξ is bounded.

By Schaefer's theorem (see [26, p. 29]), we conclude that Λ admits a fixed point u. Hence, problem (1.1), (1.2) has a solution. □

We now establish a Filippov–Gronwall type inequality for solutions of problem (1.1), (1.2).

Filippov's Theorem

For the proof of our continuous version of Filippov's theorem related to problem (1.1), (1.2), the following hypotheses will be required.

Hypothesis 2:

(H_5) The function $F : [0,\mathcal{T}] \times \mathbb{R} \to Cmp(\mathbb{R})$,

- (a) for all $u_1 \in \mathbb{R}$, the map $\iota \to F(\iota, u_1)$ is measurable,
- (b) the map $\alpha : \iota \to d(g(\iota), F(\iota, u_1))$ is integrable,

(H_6) There exists a function $p \in L^1([0,\mathcal{T}],\mathbb{R}^+)$ such that

$$H_d(F(\iota,u), F(\iota,\overline{u})) \le p(\iota)|u - \overline{u}| \text{ for almost all } \iota \in [0,\mathcal{T}].$$

Let $u = \langle u_1, u_2, u_3\rangle \in C([0,\mathcal{T}],\mathbb{R}^3)$ be a solution of the following problem:

$$\begin{cases} u_1' = f_1(\iota, u_1, u_2, u_3) - \lambda^* g(\iota), & \\ u_2' = f_2(\iota, u_1, u_2, u_3), & \text{a.e. } \iota \in [0,\mathcal{T}], \\ u_3' = f_3(\iota, u_1, u_2, u_3), & \\ u_1(0) = u_2(0) = u_3(0) = 0. \end{cases} \tag{3.2}$$

Remark 3.1. From Assumptions (H_5)(a) and (H_6), it follows that the multivalued mapping $\iota \to F(\iota, u_1)$ is measurable. Consequently, applying Lemmas 1.4 and 1.5 from [7], we deduce that $\alpha(\iota) = d(g(\iota), F(\iota, u_1))$ is measurable.

Theorem 3.2. *Suppose that hypotheses* (H_5)*–*(H_6) *are satisfied. If* $\lambda^*\|p\|_{L^1} < 1$*, then problem* (1.1), (1.2) *has at least one solution* $v = \langle v^1, v^2, v^3\rangle$ *for all* $t \in [0, \mathcal{T}]$*, satisfying the estimate*

$$\|v(\iota) - u(\iota)\| \leq \Phi(\iota),$$

where

$$\Phi(\iota) \leq \lambda^* \Big(\frac{\|\alpha\|_\infty \|p\|_{L^1}}{1 - \lambda^*\|p\|_{L^1}}\Big) + \|\alpha\|_\infty.$$

Proof. We define a sequence of functions $(v_n)_{n\in\mathbb{N}}$ and prove that it converges to a solution of problem (1.1), (1.2) on $[0, \mathcal{T}]$.

Let $G_0 = \langle g_0, 0, 0\rangle = \langle g, 0, 0\rangle$, $f = \langle f_1, f_2, f_3\rangle$ and $v_0(\iota) = \langle v_0^1, v_0^2, v_0^3\rangle = u(\iota)$ for all $t \in [0, \mathcal{T}]$, i.e.,

$$v_0(\iota) = \int_0^\iota f(s)\, ds - \lambda^* \int_0^\iota G_0(s)\, ds, \quad \iota \in [0, \mathcal{T}].$$

Define the multivalued map

$$U_1 : [0, \mathcal{T}] \to \mathcal{P}(\mathbb{R}), \quad \iota \to F(\iota, v_0^1(\iota)) \cap (g(\iota) + \alpha(\iota)B(0, 1)).$$

Since g and α are measurable, Theorem III.4.1 in [5] tells us that the ball $\alpha(\iota)B(0, 1)$ is measurable. Moreover, $F(\iota, v_0^1(\iota))$ is measurable. We claim that U_1 is nonempty. It is clear that

$$d(0, F(\iota, 0)) \leq d(0, g(t)) + d\big(g(\iota), F(\iota, v_0^1(t))\big) + H_d(F(\iota, v_0^1(\iota)), F(\iota, 0)) \leq |g(\iota)| + \alpha(\iota) + p(\iota)|v_0^1(\iota)|.$$

Therefore, for every $w \in F(\iota, v_0^1(\iota))$ we have

$$\|w\| \leq d(0, F(\iota, 0)) + H_d(F(\iota, v_0^1(\iota)), F(\iota, 0)) \leq |g(\iota)| + \alpha(\iota) + 2p(\iota)|v_0^1(\iota)| = M(\iota)$$

for all $\iota \in [0, \mathcal{T}]$. We deduce that for all $\iota \in [0, \mathcal{T}]$,

$$F(\iota, v_0^1(\iota)) \subset M(\iota)B(0, 1).$$

From Lemma 2.3, there exists a function k that is a measurable selection of $F(\iota, v_0^1(\iota))$ such that

$$|k(\iota) - g(\iota)| \leq d\big(g(\iota), F(\iota, v_0^1(\iota))\big) = \alpha(\iota).$$

Therefore, $k(\iota) \in U_1(\iota)$, confirming the claim. Consequently, the intersection multivalued operator $U_1(\iota)$ is measurable (see [2, 5, 10]). Applying Lemma 2.2 (the Kuratowski–Ryll–Nardzewski selection theorem), we obtain a measurable selection $\iota \to g_1(\iota)$ associated with U_1.

Let $G_1 = \langle g_1, 0, 0\rangle$. Consider

$$v_1(\iota) = \int_0^\iota f(s)\, ds - \lambda^* \int_0^\iota G_1(s)\, ds, \quad \iota \in [0, \mathcal{T}].$$

For each $\iota \in [0, \mathcal{T}]$, we have

$$\|v_1(\iota) - v_0(\iota)\| \leq \lambda^* \int_0^\iota |g_0(s) - g_1(s)|\, ds.$$

Hence

$$\|v_1(\iota) - v_0(\iota)\| \leq \lambda^* \|\alpha\|_\infty \ \text{ for all } \ \iota \in [0, \mathcal{T}].$$

According to Lemma 1.4 in [7], the set $F(\iota, v_1^1(\iota))$ is measurable. In addition, by Theorem III.4.1 in [5], the ball $g_1(\iota) + p(\iota)|v_1^1(\iota) - v_0^1(\iota)|B(0, 1)$ is measurable.

Define

$$U_2 : [0, \mathcal{T}] \to \mathcal{P}(\mathbb{R}), \quad \iota \to F(\iota, v_1^1(\iota)) \cap (g_1(\iota) + p(\iota)|v_1^1(\iota) - v_0^1(\iota)|B(0, 1)).$$

This set is nonempty. Indeed, since g_1 is measurable, Lemma 2.3 provides a measurable selection k of $F(\iota, v_1^1(\iota))$ satisfying

$$|k(\iota) - g_1(\iota)| \leq d(g_1(\iota), F(\iota, v_1^1(\iota))).$$

Then, using (H_6), we obtain

$$|k(\iota) - g_1(\iota)| \leq H_d(F(\iota, v_0^1(\iota)), F(\iota, v_1^1(\iota))) \leq p(\iota)|v_1^1(\iota) - v_0^1(\iota)|.$$

Thus $k(\iota) \in U_2(\iota)$, which proves the claim. Since the intersection operator $U_2(t)$ is measurable (see [2,5,10]), Lemma 2.2, guarantees the existence of a measurable selection $\iota \to g_2(\iota)$ for U_2. Hence

$$|g_2(\iota) - g_1(\iota)| \leq p(t)|v_1^1(\iota) - v_0^1(\iota)|.$$

Let $G_2 = \langle g_2, 0, 0\rangle$. Define

$$v_2(\iota) = \int_0^\iota f(s)\,ds - \lambda^* \int_0^\iota G_2(s)\,ds, \quad \iota \in [0, \mathcal{T}].$$

For each $\iota \in [0, \mathcal{T}]$, we have

$$\|v_2(\iota) - v_1(\iota)\| \leq \lambda^* \int_0^\iota |g_2(s) - g_1(s)|\,ds \leq \lambda^* \int_0^\iota p(s)|v_1^1(s) - v_0^1(s)|ds \leq (\lambda^*)^2\|\alpha\|_\infty \|p\|_{L^1}.$$

Let

$$U_3 : [0, \mathcal{T}] \to \mathcal{P}(\mathbb{R}), \quad \iota \to F(\iota, v_2^1(\iota)) \cap (g_2(\iota) + p(\iota)|v_2^1(\iota) - v_1^1(\iota)|B(0,1)).$$

By the same reasoning applied to U_2, we conclude that U_3 is a measurable multivalued mapping with nonempty values. Accordingly, there exists a measurable selection $g_3(\iota)$ for U_3, which permits the definition

$$v_3(\iota) = \int_0^\iota f(s)\,ds - \lambda^* \int_0^\iota G_3(s)\,ds, \quad \iota \in [0, \mathcal{T}],$$

with $G_3 = \langle g_3, 0, 0\rangle$, and for each $\iota \in [0, \mathcal{T}]$, we have

$$\|v_3(\iota) - v_2(\iota)\| \leq \lambda^* \int_0^t |g_3(s) - g_2(s)|\,ds \leq \lambda^* \int_0^\iota p(s)|v_2^1(\iota) - v_1^1(\iota)|\,ds \leq (\lambda^*\|p\|_{L^1})^2\|\alpha\|_\infty.$$

Iterating the above argument for $n = 0, 1, 2, 3, \ldots$, we arrive at the bound

$$|v_n(t) - v_{n-1}(t)| \leq (\lambda^*\|p\|_{L^1})^{n-1}\|\alpha\|_\infty. \tag{3.3}$$

Assume that (3.3) holds for some n, we now prove it for $n+1$.
Let

$$U_{n+1} : [0, \mathcal{T}] \to \mathcal{P}(\mathbb{R}), \quad \iota \to F(\iota, v_n^1(\iota)) \cap (g_n(\iota) + p(\iota)|v_n^1(\iota) - v_{n-1}^1(\iota)|B(0,1)).$$

As U_{n+1} is a nonempty measurable multivalued map, there exists a measurable selection $g_{n+1}(\iota) \in U_{n+1}(\iota)$. Hence, for $n \in \mathbb{N}$, we can define

$$v_{n+1}(\iota) = \int_0^\iota f(s)\,ds - \lambda^* \int_0^\iota G_{n+1}(s)\,ds, \quad \iota \in [0, \mathcal{T}],$$

with $G_{n+1} = \langle g_{n+1}, 0, 0\rangle$, and for each $\iota \in [0, \mathcal{T}]$, we have

$$\|v_{n+1}(\iota) - v_n(\iota)\| \leq \lambda^* \int_0^\iota |g_{n+1}(s) - g_n(s)|\,ds \leq \lambda^* \int_0^\iota p(s)|v_n^1(s) - v_{n-1}^1(s)|\,ds.$$

Hence

$$\|v_{n+1}(\iota) - v_n(\iota)\| \le (\lambda^* \|p\|_{L^1})^n \|\alpha\|_\infty.$$

Consequently, (3.3) holds for all $n \in \mathbb{N}$. Since $\lambda^* \|p\|_{L^1} < 1$, the sequence $\{v_n\}$ is Cauchy in $C([0,\mathcal{T}],\mathbb{R}^3)$, converging uniformly to some $v \in C([0,\mathcal{T}],\mathbb{R}^3)$. In addition, by the definition of U_n, for $n \in \mathbb{N}$, we have

$$|g_{n+1}(\iota) - g_n(\iota)| \le p(\iota)|v_n^1(\iota) - v_{n-1}^1(\iota)| \text{ for } n \in \mathbb{N}, \text{ a.e. } \iota \in [0,\mathcal{T}].$$

Hence, the sequence $\{g_n(\iota) : n \in \mathbb{N}\}$ is Cauchy in $\mathbb{R}$; therefore, it converges almost everywhere to a measurable function $\{\widetilde{g}(\iota)\}$ in $\mathbb{R}$. Moreover, since $g_0 = g$, for each $\iota \in [0,\mathcal{T}]$, we have

$$\begin{aligned}
|g_n(\iota)| &\le \sum_{i=1}^{n} |g_i(\iota) - g_{i-1}(\iota)| + |g_0(\iota)| \\
&\le \sum_{i=2}^{n} p(\iota)|v_{i-1}^1(\iota) - v_{i-2}^1(\iota)| + |g(\iota)| \\
&\le \sum_{i=1}^{\infty} p(\iota)|v_i^1(\iota) - v_{i-1}^1(\iota)| + \alpha(\iota) + |g(\iota)|, \\
|g_n| &\le \frac{p(\iota)\|\alpha\|_\infty}{1 - \lambda^* \|p\|_{L^1}} + \alpha(\iota) + |g(\iota)|.
\end{aligned}$$

We deduce that g_n converges to $\widetilde{g}$ in $L_1([0,\mathcal{T}],\mathbb{R})$. Consequently,

$$v(\iota) = \int_0^\iota f(s)\,ds - \lambda^* \int_0^\iota \widetilde{G}(s)\,ds, \quad \iota \in [0,\mathcal{T}],$$

with $\widetilde{G} = \langle \widetilde{g}, 0, 0\rangle$ is a solution to problem (1.1), (1.2) with the conditions $v_1(0) = v_1(0) = v_1(0) = 0$.

Finally, we prove that the solution $v(\iota)$ for all $\iota \in [0,\mathcal{T}]$ verifies the estimate

$$\|v(\iota) - u(\iota)\| \le \Phi(\iota).$$

We have

$$\begin{aligned}
\|v(\iota) - u(\iota)\| &\le \lambda^* \int_0^\iota |g(s) - \widetilde{g}(s)|\,ds \\
&\le \lambda^* \Big(\int_0^\iota |g(s) - g_n(s)|\,ds + \int_0^\iota |g_n(s) - \widetilde{g}(s)|\,ds \Big).
\end{aligned}$$

As $n \to \infty$, we conclude that, for all $t \in [0,\mathcal{T}]$, we have

$$|v(\iota) - u(\iota)| \le \lambda^* \int_0^\iota \frac{\|\alpha\|_\infty}{1 - \lambda^* \|p\|_{L^1}} p(s) + \alpha(s)\,ds \le \lambda^* \Big(\frac{\|\alpha\|_\infty \|p\|_{L^1}}{1 - \lambda^* \|p\|_{L^1}} \Big) + \|\alpha\|_\infty. \qquad \square$$

4 Examples

Example 4.1. We consider the system of differential equations modeling the Watt governor with dry friction [14] under the action of an external time-dependent excitation $g(t)$ such that $|g(t)| > 1$ on the period $[0,\mathcal{T}]$:

$$\begin{cases} y_1' \in -Ay_1 + y_2 - Sign(y_1) + g(t), \\ y_2' = -By_1 + y_3, \\ y_3' = -y_1, \end{cases} \qquad \text{a.e. } t \in [0,\mathcal{T}] \tag{$\mathcal{P}_3$}$$

with the initial values

$$y_1(0) = y_2(0) = y_3(0) = 0,$$

where $\lambda^* = 1$. The mapping $F(y_1) = Sign(y_1) - g(t)$, satisfies $0 \notin F(t,0)$ for all $\iota \in [0,\mathcal{T}]$, and

$$\begin{cases} f_1(t,y_1,y_2,y_3) = -Ay_1 + y_2, \\ f_2(t,y_1,y_2,y_3) = -By_1 + y_3, \\ f_3(t,y_1,y_2,y_3) = -y_1. \end{cases}$$

Therefore,

$$\|f(x) - f(y)\| \leq \max\big(1, |A|+1, |B|+1\big)\|x-y\|.$$

For $w \in F$, we have

$$|w| \leqslant m(t) = |g(t)|.$$

By a simple calculation, for $r = 1$, we get $M = \max(1, |A|+1, |B|+1)$, $\widetilde{M} = 0$, and

$$\begin{cases} c = \mathcal{T}\big(\max(1, |A|+1, |B|+1)r\big) + \int\limits_0^{\mathcal{T}} |g(s)|\, ds, \\ K = \dfrac{\int\limits_0^{\mathcal{T}} |g(s)|\, ds}{1 - \mathcal{T}\max(1, |A|+1, |B|+1)} \end{cases}$$

if $\max(1, |A|+1, |B|+1) < 1/\mathcal{T}$. We conclude that problem $(\mathcal{P}_3)$ has at least a non-trivial solution on the interval $[0,\mathcal{T}]$.

Example 4.2. Consider the Chua system with discontinues characteristic [14], with an initial boundary value problem for all $t \in [0,\mathcal{T}]$ under a constant affine perturbation η of the set-valued function F:

$$\begin{cases} x_1' \in -\alpha(m_1+1)x_1 + \alpha x_2 - \alpha(m_0 - m_1)Sign(x_1) - \eta, \\ x_2' = x_1 - x_2 + x_3, \\ x_3' = -\beta x_2 - \gamma x_3, \\ x_1(0) = x_2(0) = x_3(0) = 0. \end{cases} \tag{$\mathcal{P}_4$}$$

where the parameters $\alpha, \beta, \gamma, m_0, m_1 \in \mathbb{R}$, while the constant forcing term $\eta > |\alpha(m_0 - m_1)|$, with $\alpha(m_0 - m_1) \neq 0$.

We have

$$\lambda^* = |\alpha(m_0 - m_1)| \text{ and } F(y_1) = Sign(x_1) + \frac{\eta}{|\alpha(m_0 - m_1)|},$$

satisfies $0 \notin F(t,0)$ for all $\iota \in [0,\mathcal{T}]$. We have

$$\begin{cases} f_1(t,x_1,x_2,x_3) = -\alpha(m_1+1)x_1 + \alpha x_2, \\ f_2(t,x_1,x_2,x_3) = x_1 - x_2 + x_3, \\ f_3(t,x_1,x_2,x_3) = -\beta x_2 - \gamma x_3. \end{cases}$$

Therefore,

$$\|f(x) - f(y)\| \leq \max\big(|\alpha(m_1+1)| + |\alpha|, 3, |\beta| + |\gamma|\big)\|x-y\|.$$

If $w \in F$, we have

$$|w| \leqslant m(t) = 1 + \left|\frac{\eta}{\alpha(m_0 - m_1)}\right|.$$

For $r = 1$, we get $M = \max(|\alpha(m_1+1)| + |\alpha|, 3, |\beta| + |\gamma|)$, $\widetilde{M} = 0$, and

$$\begin{cases} c = \mathcal{T}\big(\max(|\alpha(m_1+1)| + |\alpha|, 3, |\beta| + |\gamma|) + |\eta| + |\alpha(m_0 - m_1)|\big), \\ K = \dfrac{\mathcal{T}(|\alpha(m_0 - m_1)| + |\eta|)}{1 - \mathcal{T}\max(|\alpha(m_1+1)| + |\alpha|, 3, |\beta| + |\gamma|)}. \end{cases}$$

Then problem $(\mathcal{P}_4)$ has at least one non-trivial solution on $[0,\mathcal{T}]$ if $\max(|\alpha(m_1+1)| + |\alpha|, 3, |\beta| + |\gamma|) < 1/\mathcal{T}$.

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