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ON EXACT LIMITS FOR THE NEGATIVITY OF THE GREEN FUNCTION OF A TWO-POINT BOUNDARY VALUE PROBLEM

Abstract. The question of the positivity of the Green function $G(x, s, \lambda)$ of the two-point boundary value problem

$$(-1)^k u^{(n)} - \lambda \int\limits_0^l u(s)\, d_s r(x, s) = f(x), \quad x \in [0, l], \quad B_k u = 0, \tag{$*$}$$

where

$$B_m u := \big(u(0), u'(0), \ldots, u^{(n-m-1)}(0), u(l), -u'(l), u''(l), \ldots, (-1)^{m-1} u^{(m-1)}(l)\big),$$

with non-decreasing $r(x, \cdot)$ for almost all $x \in [0, l]$ is reduced to estimating the eigenvalues of auxiliary boundary value problems. The Green function $G(x, s, \lambda)$ is positive if and only if

$$-\min\{\lambda_{k-1}, \lambda_{k+1}\} \le \lambda < \lambda_k$$

(there are also small clarifying details that are not needed for the ordinary differential equation $(-1)^{k+1} u^{(n)} - \lambda p(x) u = f(x)$). Here, λ_m is the smallest positive eigenvalue of the boundary value problem

$$(-1)^m u^{(n)} - \lambda \int\limits_0^l u(s)\, d_s r(x, s) = 0, \quad B_m u = 0$$

($m \in \{0, \ldots, n\}$).

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1 Introduction

The two-term equation $u^{(n)}+p(x)u = f(x)$ has been the subject of many works. Of particular note are the works of J. Mikusinsky [17] and A. Levin [15], devoted to the properties of solving a homogeneous equation, similar to the alternation of zeros for a second-order equation. These properties serve as a basis for studying the spectrum of a differential operator and for substantiating the oscillation properties [5] of the Green function. They are used in the works on the oscillatory properties of higher-order differential equations.

The question of the positivity of the Green function is included in the range of problems mentioned above. We note its importance not only in problems of comparison, existence and estimates of solutions to nonlinear equations, but also in such problems as the existence of a positive eigenfunction of a boundary value problem and analogues of the Jacobi criterion in the calculus of variations.

The solution of the question of the sign-definiteness of the Green function of a two-point boundary value problem essentially depends on the sign of $p(x)$. The boundary value problem is reduced to an equation of the second kind $z - \lambda K z = f$ in the space of Lebesgue integrable functions. If $\lambda > 0$ and the operator K is positive, then the necessary and sufficient condition for the positivity of the Green function is $\lambda < \lambda_k$, where $\lambda_k = 1/r(K)$ and $r(K)$ is the spectral radius of the operator K. This leads to the theorems on integral and differential inequalities, giving necessary and sufficient conditions for positivity. For the opposite sign of $p(x)$, the use of this equation leads only to rough sufficient conditions.

1.1 Briefly about the result

Let $1 \le k \le n-1$ ($n \ge 2$). Consider the two-point $(n-k, k)$-boundary value problem (BVP in what follows)

$$(-1)^{k+1}u^{(n)}(x) - \lambda p(x)u(x) = f(x), \quad x \in [0, l], \tag{1.1}$$

$$B_k u = 0, \tag{1.2}$$

where

$$B_m u := \Big(u(0), u'(0), \ldots, u^{(n-m-1)}(0), u(l), -u'(l), u''(l), \ldots, (-1)^{m-1}u^{(m-1)}(l)\Big)$$

(symbol := means *equal by definition*; $m \in \{0, 1, \ldots, n\}$). By the Green function of BVP $\{(1.1), (1.2)\}$ we mean a function $G(x, s, \lambda)$ such that the solution to this problem is written in the form

$$u(x) = \int_0^l G(x, s, \lambda) f(s)\, ds.$$

Our goal is, considering $p(x) \ge 0$, to establish the exact boundaries of the interval for the parameter λ, for which the Green function $G(x, s, \lambda)$ is *negative*. Without the multiplier $(-1)^{k+1}$, we would talk about the positivity of the function $(-1)^k G(x, s, \lambda)$. We will obtain the corresponding result for a more general functional differential equation

$$\mathcal{L}_\lambda u(x) := (-1)^{k+1}u^{(n)}(x) - \lambda \int_0^l u(s)\, d_s r(x, s) = f(x), \quad x \in [0, l], \tag{1.3}$$

assuming $r(x, \cdot)$ to be *non-decreasing*, which corresponds to the case $p(x) \ge 0$ for equation (1.1). The coefficient $(-1)^k$ in equation $(*)$ is replaced here by $(-1)^{k+1}$. This is done for the convenience of the proof and does not affect the essence of the matter.

Let λ_m ($m \in \{0, \ldots, n\}$) be the least positive eigenvalue of BVP

$$\mathcal{L}_{\lambda,m} u := (-1)^m u^{(n)} - \lambda \int_0^l u(s)\, d_s r(x, s) = 0, \quad B_m u = 0. \tag{1.4}$$

If there are no such numbers, then by definition $\lambda_m = +\infty$. In this paper, we show that the necessary and sufficient conditions for the existence and *negativity* of the Green function of BVP $\{(1.3),(1.2)\}$ are the inequalities (for more details, see Theorems 4.1, 4.2, 4.3 below)

$$-\lambda_k < \lambda \leq \min\{\lambda_{k-1}, \lambda_{k+1}\}.$$

It is *important to emphasize* that the problem of estimating eigenvalues λ_m is solved very efficiently by using differential or integral inequality theorems (see Theorem 3.1 below).

In [4], the same result was obtained for an equation of the form $\mathcal{L}u = \lambda u$ under the condition that the interval $[0,l]$ is the disconjugacy interval of the ordinary differential equation $\mathcal{L}u = 0$. The difference consists in replacing the operator $u^{(n)}$ by an arbitrary linear differential operator $\mathcal{L}$ with disconjugacy property and replacing $u(x)$ by $\int_0^l u(s)\, d_s r(x,s)$. Our result can probably be extended to the case of an arbitrary ordinary operator on disconjugacy interval $[0,l]$, since it expands into a product of first-order operators [16].

For a second-order equation, the presented problem was considered in [8] (delay equation), [1,11,20]. Subsequently, the problem was studied for particular cases in [12] (third-order equation), [13] ($(n-1,1)$-BVP), [14] (even order). The methods for obtaining effective conditions for the positivity of the Green functions for a wide class of linear boundary values are developed in [2]. We note the work [19], in which the method based on the relationship of boundary value problems with different boundary conditions was widely used.

The following material is divided into three parts. The first part (Section 2) provides supporting information. The second part (Section 3) is auxiliary. It studies the properties of BVP (1.4) and the estimation of eigenvalues, as well as one auxiliary three-point problem. The third part (Section 4) deals with the actual main BVP $\{(1.3),(1.2)\}$.

1.2 Notation, assumptions, definitions

- For general concepts related to the boundary value problems for equation (1.3), we refer to the monograph [3].

  Denote $I = [0,l]$. As usual, we use the notation $L(I)$ for the Lebesgue space of integrable on I functions. $AC^{n-1}(I)$ is the space of functions that have an absolutely continuous on I derivative of order $n-1$. Norms in these spaces are usual.

- To emphasize that we are not talking about the value of the function at point x, but about the function itself, as an object, we write $f(\,\cdot\,)$ instead of $f(x)$, or just f. The inequalities for functions $f \leq g$ are understood pointwise, and for measurable functions almost everywhere. For finite-dimensional vectors, we understand the inequalities $\alpha \leq \beta$ component-by-component. Similarly, we will understand the inequalities $u \neq 0$, $f \neq 0$, etc., which mean that these are nonzero elements of the corresponding function spaces. For clarity, we will sometimes write $u(x) \not\equiv 0$. The inequality $(f,\alpha) \geq (0,0)$ (or just $(f,\alpha) \geq 0$) means $f \geq 0$, $\alpha \geq 0$.

  The inequality $f \geq\neq 0$ ($f \leq\neq 0$) means non-negativity (non-positivity) of a function on the segment $[0,l]$ (almost everywhere for a measurable function), excluding the case of identical zero (equivalence to zero).

- For a better understanding of the material, it is desirable to pay attention to the fact that the value of k should be considered fixed, and the value of m is arbitrary. However, we will need $m = k-1$ and $m = k+1$.

- We use the abbreviation BVP for the term *boundary value problem*.

- Operator $Q : AC^{n-1}(I) \to L(I)$ is defined by

$$Qu(x) := \int_0^l u(s)\, d_s r(x,s), \quad x \in I.$$

Here, for almost all $x \in I$, the function $r(x,\cdot)$ is non-decreasing; for all $s \in I$, the function $r(\cdot,s)$ is measurable and $r(x,0) = 0$, $r(\cdot,l) \in L(I)$, $r(\cdot,l) \neq 0$. The operator Q is positive in the sense that it maps the cone of non-negative functions in $AC^{n-1}(I)$ to the cone of non-negative functions in $L(I)$.

Additionally, in the case of $k = n-1$, we assume that

$$r(x,0+) = r(x,0), \quad x \in I, \tag{1.5}$$

and in the case of $k = 1$, we assume that

$$r(x,l) = r(x,l-), \quad x \in I. \tag{1.6}$$

This assumption is made because the Green function does not depend on the limits $r(x,0+)$ and $r(x,l-)$. However, the solutions of the homogeneous equation depend on this. For the equation (1.1), this is not relevant.

- Let $\mathcal{E} \subset AC^{n-1}(I)$ be the set of functions satisfying the conditions

$$u(0) = \cdots = u^{(n-k-2)}(0) = 0, \quad u(l) = \cdots = u^{(k-2)}(l) = 0. \tag{1.7}$$

If $k = n-1$ or $k = 1$, the corresponding group of equalities disappears. In the main Section 4, all solutions are considered exclusively on the set $\mathcal{E}$.

- Everywhere below, $G(x,s,\lambda)$ represents the Green function of BVP $\{(1.3),(1.2)\}$.

- The function

$$\Gamma(x,s,\lambda) := \frac{G(x,s,\lambda)}{s^k(l-s)^{n-k}} \tag{1.8}$$

is the Green function of BVP

$$x^k(l-x)^{n-k}\big((-1)^{k+1}u^{(n)}(x) - \lambda Qu(x)\big) = f(x), \ \ B_k u = 0.$$

The solution to BVP

$$(-1)^m u^{(n)} - \lambda Qu = f, \ \ B_m u = \alpha \tag{1.9}$$

under condition of unique solvability has the form $u = G_{\lambda,m} f + U_{\lambda,m}\alpha$, where $G_{\lambda,m}$ is the Green operator for this problem and $U_{\lambda,m}\alpha$ is the solution of the semi-homogeneous BVP $\mathcal{L}_{\lambda,m} u = 0$, $B_m u = \alpha$.

Definition 1.1. We call BVP (1.9) *positively solvable* if $(G_{\lambda,m}, U_{\lambda,m}) \geq 0$, that is, $f \geq 0$, $\alpha \geq 0$ implies $u \geq 0$.

The Green function of a two-point problem in the case of sign conservation usually has a property that can be called $(n-m,m)$-positivity (negativity), according to the following definition.

Definition 1.2. A function $G(x,s)$ is called $(n-m,m)$-positive ($(n-m,m)$-negative) if $f \geq\neq 0$ ($f \leq\neq 0$) implies

$$\int_0^l G(x,s)f(s)\,ds \geq \varepsilon x^{n-m}(l-x)^m$$

for some $\varepsilon > 0$ and all $x \in [0,l]$.

It will be convenient for us to use a similar concept for functions.

Definition 1.3. A function $u \in AC^{n-1}(I)$ is called $(n-m,m)$-positive ($(n-m,m)$-negative) if for some $\varepsilon > 0$ and $x \in I$, $u(x) \geq \varepsilon x^{n-m}(l-x)^m$ (resp. $u(x) \leq -\varepsilon x^{n-m}(l-x)^m$).

Remark 1.1. Note that if $B_m u = 0$, then $(n-m,m)$-positivity is equivalent to the inequalities $u(x) > 0$ on $(0,l)$,

$$u^{(n-m)}(0) > 0, \ \ (-1)^m u^{(m)}(l) > 0.$$

The definition of $(n-m,m)$-positivity has the following generalization.

Definition 1.4. Let $u_0 \in AC^{n-1}(I)$, $u_0 \geq\neq 0$. An operator $G : L(I) \to AC^{n-1}(I)$ is said to be u_0-positive if for any $f \geq\neq 0$ there exists $\varepsilon > 0$ such that $Gf \geq \varepsilon u_0$.

Note that $(n-m,m)$-positivity is u_0-positivity if we set $u_0(x) = x^{m-n}(l-x)^m$.

2 Supporting information

2.1 Estimation of the spectral radius of a positive compact operator

The concepts used in this section are well known (see, for example, [6]). Let $\mathcal{K}$ be a total cone[1] in the Banach space E and $A : E \to E$ be a linear compact operator, positive with respect to $\mathcal{K}$, that is, $A\mathcal{K} \subset \mathcal{K}$. Let $r(A)$ be the spectral radius of the operator A.

Theorem 2.1 (M. Krein, M. Rutman [7])**.** *If the spectrum of A contains nonzero points, then $r = r(A)$ is an eigenvalue of the operator A and its adjoint. The operator A has a positive eigenvector $v_0 \in K$, $Av_0 = rv_0$, and the adjoint A^* has a positive eigenvector $\psi \in K^*$, $A^*\psi = r\psi$.*

Definition 2.1 ([6])**.** An operator $A : E \to E$ is said to be u_0-bounded from above if for any $x \in E$ there exists $\beta > 0$ such that $Ax \leq \beta u_0$.

Lemma 2.1 ([9, Corollary from Lemma 2])**.** *Let A be u_0-bounded from above, where $u_0 \in K$, and let there exist $v \in K$ satisfying the inequality $v - Av \geq \gamma u_0$ for some $\gamma > 0$. Then $r(A) < 1$.*

2.2 Basic BVP

If $\lambda = 0$, we get the simplest BVP, the properties of which can also be applied to the main problem for small absolute values of λ:

$$(-1)^m u^{(n)} = z, \quad B_m u = \alpha. \tag{2.1}$$

The solution to problem (2.1) has the form

$$u = H_m z + V_m \alpha,$$

where H_m is an integral operator and $V_m\alpha$ is a polynomial of degree at most $n-1$.

Lemma 2.2. *Let $(z,\alpha) \geq\neq 0$, $u = H_m z + V_m\alpha$. Then*

(1) *if $1 \leq m \leq n-1$, then $u(\cdot)$ is $(n-m,m)$-positive,*

(2) *if $m = 0$ and $\alpha \geq\neq 0$, then $u(x) \geq \varepsilon x^{n-1}$, $x \in I$,*

(3) *if $m = n$ and $\alpha \geq\neq 0$, then $u(x) \geq \varepsilon (l-x)^{n-1}$, $x \in I$.*

Proof. For $m = 0$ and $m = n$. the assertion is obvious. Let $1 \leq m \leq n-1$.

If $\alpha = 0$, that is, $u = H_m z$, the statement can be obtained from the interpolation formula $u(x) = (z(\xi)/n!)\, x^{n-m}(l-x)^m$ in case of continuous z. However, the assertion also follows from representation (2.2) for the Green function $H_m(x,s)$ given below.

If $z = 0$, that is, $u = V_m\alpha$, see, for example, [10, Lemma 15]. □

The operator H_m is integral one, its kernel is the Green function $H_m(x,s)$, for which the following expression was obtained in [18]:

$$H_m(x,s) = \frac{x^{n-m-1}(l-x)^{m-1}(l-s)^{n-m-1}s^{m-1}}{l^{n-2}(n-1)!}\, G_{11}(x,s)\varphi(x,s), \tag{2.2}$$

[1] A cone $\mathcal{K}$ is said to be *total* if the closure of its linear span coincides with the entire space E [6].

where $G_{11}(x,s)$ is the Green function of BVP $-u''=f$, $u(0)=u(l)=0$, and

$$\varphi(x,s)=\begin{cases}\displaystyle\sum_{i=0}^{m-1} C_{n-2-i}^{n-m-1}\Big(\frac{l(s-x)}{s(l-x)}\Big)^i, & \text{if } 0\le x\le s\le l,\\ \displaystyle\sum_{i=0}^{n-m-1} C_{n-2-i}^{m-1}\Big(\frac{l(x-s)}{x(l-s)}\Big)^i, & \text{if } 0\le s\le x\le l.\end{cases}$$

Note that

$$\varphi(x,0)=\sum_{i=0}^{n-m-1} C_{n-2-i}^{m-1}=C_{n-1}^{m},$$

$$G_{11}(x,s)=\frac{1}{l}\begin{cases}x(l-s), & \text{if } x\le s,\\ s(l-x), & \text{if } s\le x.\end{cases}$$

The following assertion is derived directly from formula (2.2).

Lemma 2.3. *There is a limit*

$$\lim_{s\to 0}\frac{H_m(x,s)}{s^m}=\frac{x^{n-m-1}(l-x)^m}{l^m(n-1)!}\,C_{n-1}^{m}. \tag{2.3}$$

The convergence here is uniform on $[0,l]$ *if* $m<n-1$*. In the case* $m=n-1$*, uniform convergence takes place on* $[\nu,l]$ *for any* $\nu\in(0,l]$*.*

2.3 Limit values of the Green function

Lemma 2.4. *Let* $k<n-1$*. There is a limit (uniform convergence)*

$$g(x)=\lim_{s\to 0}\frac{G(x,s,\lambda)}{s^k}\,. \tag{2.4}$$

The function $g(x)$ *is a solution to the problem*

$$\mathcal{L}_\lambda u=0,\ \ u(0)=\cdots=u^{(n-k-2)}(0)=0,\ \ u^{(n-k-1)}(0)\neq 0,\ \ u(l)=\cdots=u^{(k-1)}(l)=0. \tag{2.5}$$

Proof. The function $u_s(x)=G(x,s,\lambda)/s^k$ is the solution to the equation

$$u_s(x)=-\lambda\int_0^l H_k(x,t)Qu_s(t)\,dt-\frac{H_k(x,s)}{s^k}\,. \tag{2.6}$$

According to (2.3), the last term converges to the function $-Mx^{n-k-1}(l-x)^k$ as $s\to 0$ in the uniform norm (in $C(I)$), where

$$M=\frac{l^{n-k-1}}{l^{n-1}(n-1)!}\,\varphi(x,0)=\frac{1}{l^k(n-1)!}\,C_{n-1}^{k}.$$

The integral operator in (2.6) is compact in $C(I)$, equation (2.6) is uniquely solvable, so $u_s(x)\to u(x)$, where $u=-\lambda H_kQu-Mx^{n-k-1}(l-x)^k$. Hence $u(x)$ is the solution to the BVP (2.5). □

Lemma 2.5. *If* $k=n-1$ *and the condition* (1.5) *is satisfied, then the limit* (2.4) *exists and the limit function satisfies the conditions*

$$\mathcal{L}_\lambda u=0,\ \ u(0+)\neq 0,\ \ u(l)=\cdots=u^{(n-2)}(l)=0. \tag{2.7}$$

Proof. Let us show that the solution u_s of equation (2.6) for $k = n-1$ converges at $s \to 0$ to the solution u of the equation

$$u = -\lambda H_{n-1} Q u + (-M)(l-x)^{n-1}$$

uniformly on $[\nu, l]$ for any $\nu \in (0, l)$. Then u will be a solution to BVP (2.7).

For $x \in [\nu, l]$, let us denote

$$Q_{[\nu,l]} v(x) := \int_{\nu}^{l} v(\tau)\, d_\tau r(x,\tau), \quad Q_{[0,\nu]} v(x) := \int_{0}^{\nu} v(\tau)\, d_\tau r(x,\tau).$$

Solution $u_{s,\nu}$ of the equation

$$u_{s,\nu}(x) = -\lambda \int_0^l H_{n-1}(x,t) Q_{[\nu,l]} u_{s,\nu}(t)\, dt - \frac{H_{n-1}(x,s)}{s^{n-1}}, \;\; x \in [\nu, l],$$

converges as $s \to 0$ uniformly on $[\nu, l]$ to the solution $u_{0,\nu}$ of the equation

$$u_{0,\nu}(x) = -\lambda \int_0^l H_{n-1}(x,t) Q_{[\nu,l]} u_{0,\nu}(t)\, dt - M(l-x)^{n-1}, \;\; x \in [\nu, l].$$

Let us estimate the differences $u - u_{0,\nu}$ and $u_s - u_{s,\nu}$. They satisfy the equations

$$u - u_{0,\nu} = -\lambda H_{n-1} Q_{[\nu,l]}(u - u_{0,\nu}) - \lambda H_{n-1} Q_{[0,\nu]} u$$

and

$$u_s - u_{s,\nu} = -\lambda H_{n-1} Q_{[\nu,l]}(u_s - u_{s,\nu}) - \lambda H_{n-1} Q_{[0,\nu]} u_s,$$

respectively. Since

$$\int_0^\nu |u_s(\tau)|\, d_\tau r(t,\tau) \le \max |u_s|\, r(x,\nu),$$

from condition (1.5) it follows that $u - u_{0,\nu}$ and $u_s - u_{s,\nu}$ uniformly tend to zero as $\nu \to 0$.

The standard ε-procedure proves the uniform convergence of $u_s \to u$ on $[\nu, l]$. □

The statement symmetric to Lemma 2.4 is true because the $(n-k, k)$-problem is transformed into a $(k, n-k)$-problem by changing the variable $x \to l - x$.

Lemma 2.6. *Let $k > 1$. There is a limit (uniform convergence)*

$$g(x) = \lim_{s \to l} \frac{G(x,s,\lambda)}{(l-s)^{n-k}}.$$

The function $g(x)$ is a solution to BVP

$$\mathcal{L}_\lambda u = 0, \;\; u(l) = \cdots = u^{(k-2)}(l) = 0, \;\; u^{(k-1)}(l) \ne 0, \;\; u(0) = \cdots = u^{(n-k-1)}(0) = 0.$$

Lemmas 2.4 and 2.6 show the existence of the following two limits:

$$g_0(x) := \lim_{s \to 0} \frac{G(x,s,\lambda)}{s^k}, \tag{2.8}$$

$$g_1(x) := \lim_{s \to l} \frac{G(x,s,\lambda)}{(l-s)^{n-k}}. \tag{2.9}$$

The limits (2.8), (2.9) can be replaced by

$$\Gamma(x, 0+, \lambda) = \frac{g_0(x)}{l^{n-k}} \;\text{ and }\; \Gamma(x, l-, \lambda) = \frac{g_1(x)}{l^k}, \tag{2.10}$$

respectively, where the function Γ is defined by equality (1.8).

3 Two-point and three-point BVP's

The two-point BVP (1.9) is needed to estimate the eigenvalue λ_m. This two-point BVP, as well as the three-point BVP considered below, are auxiliary for solving the main problem, to which the next Section 4 is devoted.

The two-point problem $\mathcal{L}_{\lambda,m}u = f$, $B_m u = \alpha$ $(1 \le m \le n-1)$ was studied in [10] for a more general singular BVP of the form

$$(-1)^m x^m (l-x)^{n-m} u^{(n)} - \int_0^l u(s)\, d_s r(x,s) = f(x), \quad B_m u = \alpha.$$

Our BVP (1.9) is reduced to this form by multiplying both sides of the equation by $x^m(l-x)^{n-m}$. In [10], Theorem 3.1 we need is obtained (see below). However, due to the difference in the definition of the solution in the singular and given BVPs, and also, in connection with the need to study the three-point problem, we briefly present a scheme for studying both problems.

3.1 Two-point BVP

We consider not only the proper two-point BVP for $1 \le m \le n-1$. We also need the cases $m = 0$ and $m = n$. Due to the symmetry (change of the variable $x \to l-x$), the properties of the $(n-m,m)$-problem coincide with the properties of the $(m, n-m)$-problem. Therefore, it suffices, for example, to consider only the case $m = 0$, omitting $m = n$.

Using the substitution $u = H_m z + V_m \alpha$, we reduce the BVP $\mathcal{L}_{\lambda,m}u = f$, $B_m u = \alpha$ to the integral equation

$$z - \lambda Q H_m z = \lambda Q V_m \alpha + f \tag{3.1}$$

with a positive compact operator $K_m := QH_m$. Let $r(K_m)$ be the spectral radius of the operator K_m. Operator K_m is integral one with the kernel

$$K_m(x,s) := \int_0^l H_m(t,s)\, d_t r(x,t). \tag{3.2}$$

Indeed,

$$QH_m(x) = \int_0^l d_t r(x,t) \int_0^l H_m(t,s) z(s)\, ds = \int_0^l K_m(x,s) z(s)\, ds.$$

Lemma 3.1 (positive solvability)**.** *Let $\lambda > 0$, $\lambda r(K_m) < 1$. Then BVP (1.9) is positively solvable, and if $(f,\alpha) \ge\ne 0$ and $u(\cdot)$ is the solution to (1.9), then for some $\varepsilon > 0$:*

(1) *if $1 \le m \le n-1$, then $u(\cdot)$ is $(n-m,m)$-positive,*

(2) *if $m = 0$ and $\alpha \ge\ne 0$, then $u(x) \ge \varepsilon x^{n-1}$, $x \in I$,*

(3) *if $m = n$ and $\alpha \ge\ne 0$, then $u(x) \ge \varepsilon (l-x)^{n-1}$, $x \in I$.*

Proof. The solution of (1.9) is $u = H_m z + V_m \alpha$, where z is the solution of (3.1). By Lemma 2.2, $QV_m \alpha \ge 0$. Therefore, $z \ge 0$, and $z \ge f$. Now we refer to Lemma 2.2. □

Lemma 3.2 (eigenfunction is positive)**.** *The eigenvalue λ_m is inverse to the spectral radius $r(K_m)$:*

$$\lambda_m r(K_m) = 1.$$

If $\lambda = \lambda_m$, then BVP (1.4) has a solution $u \ge\ne 0$. If $1 \le m \le n-1$, then $u(\cdot)$ is $(n-m,m)$-positive. If $m = 0$, then $u^{(i)} \ge 0$, $i = 0, \dots, n-1$. If $m = n$, then $(-1)^i u^{(i)} \ge 0$, $i = 0, \dots, n-1$.

Proof. From (1.4) and (3.1) we have $u = H_m z$ and $z - \lambda K_m z = 0$. By virtue of Theorem 2.1, the equation $z - \lambda_m K_m z = 0$ has a non-trivial non-negative solution, and $\lambda_m r(K_m) = 1$. Therefore, BVP (1.4) also has a non-trivial non-negative solution $u = H_m z$.

The $(n-m, m)$-positivity follows from Lemma 2.2. □

Theorem 3.1 (cf. [10, Theorem 2]). *Let $1 \le m \le n-1$. The following statements are equivalent:*

(1) *BVP $\mathcal{L}_{\lambda,m} u = f$, $B_m u = \alpha$ is uniquely solvable, and if $(f,\alpha) \geq\neq 0$, then $u(\cdot)$ is $(n-m,m)$-positive.*

(2) $\lambda < \lambda_m$.

(3) *There is a solution $u \ge 0$ of the inequalities $\mathcal{L}_\lambda u = \psi \ge 0$, $B_m u = \alpha \ge 0$, and $(\psi, \alpha) \ne 0$.*

(4) *There exists $u \ge 0$ such that $u - \lambda H_m Q u = g \ge 0$, $Qg \ne 0$.*

Theorem 3.1 can be considered as a consequence of [10, Theorem 2]. The difference is that the *working space* is wider in the singular problem. However, all the functions in Theorem 3.1 are in the subset $AC^{n-1}(I)$, and it remains valid. For the cases $m = 0$ and $m = n$, we also need an analogue of Theorem 3.1. In view of symmetry, it suffices to consider the case $m = 0$.

Theorem 3.2. *Let $m = 0$. The following statements are equivalent:*

(1) *BVP $\mathcal{L}_{\lambda,m} u = f$, $B_m u = \alpha$ is uniquely solvable; if $(f,\alpha) \ge 0$, then $u^{(i)}(x) \ge 0$, $i = 0, \dots, n-1$; if $\alpha \ne 0$, then $u(x) \ge \varepsilon x^{n-1}$ for some $\varepsilon > 0$.*

(2) $\lambda < \lambda_m$.

(3) *There is a solution $u \ge 0$ of the inequalities $\mathcal{L}_\lambda u = \psi \ge 0$, $B_m u = \alpha \ge 0$, and $\alpha \ne 0$.*

Proof. Only the implication $3 \to 2$ needs proof. Note that the spectral radii of the operators $K_0 = QH_0$ and $H_0 Q$ coincide. We have

$$(-1)^{k+1} u^{(n)} = \lambda Q u + \psi = z, \quad u = H_0 z + V_0 \alpha, \quad u - \lambda H_0 Q u = H_0 \psi + V_0 \alpha.$$

By Lemma 2.2, for some $\varepsilon > 0$ the right side of $H_0 \psi + V_0 \alpha \ge V_0 \alpha \ge \varepsilon x^{n-1}$. Since the operator $H_0 Q$ is u_0-bounded from above, where $u_0(x) = x^{n-1}$, then $\lambda r(QH_0) < 1$ by virtue of Lemma 2.1. □

For the cases $m = 0$ and $m = n$, we need the following assertion. Due to symmetry, it suffices to consider only one of the cases $m = 0$ or $m = n$.

Theorem 3.3. *Let $m = n$. Suppose there exists a positive on $[0, l)$ solution to BVP $\mathcal{L}_\lambda u = f \ge 0$, $B_n u = 0$. Then $\lambda \le \lambda_n$.*

Proof. From (3.1) we have $u = H_n z$ and

$$z - \lambda K_n z = f.$$

The eigenfunction φ of the adjoint operator K_n^* corresponding to the spectral radius $r = r(K_n)$ satisfies

$$r\varphi(s) = \int_0^l K_n(x,s)\varphi(x)\,dx.$$

It is non-negative and does not decrease since $K_n(x,s)$ does not decrease with respect to s due to (3.2). Let $\varphi(s) > 0$ on $(l-\varepsilon, l]$. Since $u(x) > 0$ on $[0, l)$, we have $z(x) = (-1)^n u^{(n)}(x) \not\equiv 0$ on $[l-\varepsilon, l]$. This implies $\int_0^l \varphi(x) z(x)\,dx > 0$. Introducing the notation

$$\langle f, g\rangle := \int_0^l f(x) g(x)\,dx,$$

we write it briefly: $\langle \varphi, z \rangle > 0$.

Since

$$\langle \varphi, K_n z \rangle = \langle K_n^* \varphi, z \rangle = r \langle \varphi, z \rangle,$$

we have

$$\langle \varphi, z \rangle - \lambda r \langle \varphi, z \rangle = \langle \varphi, f \rangle \geq 0.$$

Hence $1 - \lambda r \geq 0$ and $\lambda \leq \lambda_n$. □

3.2 Three-point BVP

Let $\xi \in (0, l)$, $n \geq 3$. Consider the boundary value problem

$$\mathcal{L}_\lambda u = f, \quad B_{k,\xi} u = 0, \tag{3.3}$$

where the vector functional $B_{k,\xi}$ is defined by the equality

$$B_{k,\xi} u := \Big(u(0), u'(0), \ldots, u^{(n-k-2)}(0), u(\xi), u'(\xi), u(l), -u'(l), u''(l), \ldots, (-1)^{k-2} u^{(k-2)}(l) \Big).$$

If $k = n - 1$, there is no condition group at the left end. Similarly, for $k = 1$, there is no group of conditions for $x = l$.

Let $H_{k,\xi}$ be the Green operator of the BVP $(-1)^{k+1} u^{(n)} = z$, $B_{k,\xi} u = 0$, i.e., the solution to this problem is $u = H_{k,\xi} z$.

Lemma 3.3. *Let $u = H_{k,\xi} z$ and $z \geq\neq 0$. Moreover, if $k = 1$, then suppose additionally that $z(x) \not\equiv 0$ on $[0, \xi]$, and if $k = n - 1$, then suppose additionally that $z(x) \not\equiv 0$ on $[\xi, l]$. Then $u(x) \geq \varepsilon x^{n-k-1} (x - \xi)^2 (l - x)^{k-1}$, $x \in I$, for some $\varepsilon > 0$.*

Proof. If z is continuous, it suffices to mention the interpolation formula

$$u(x) = \frac{z(c)}{n!}\, x^{n-k-1} (x - \xi)^2 (l - x)^{k-1}, \quad c \in (0, l).$$

The inequality $u \geq 0$ is also true in the general case. The assumption that u has a zero of higher multiplicity at one of the points 0, ξ, l leads to a contradiction. Indeed, in this case, u will be a solution to another interpolation problem for which $u \leq 0$.

Similarly, the presence of other zeros of $u(\,\cdot\,)$ is excluded. □

Note that in the extreme cases $k = 1$ or $k = n - 1$, when the boundary conditions disappear at one of the ends, the solution may vanish identically between the remaining zeros even for $z \geq\neq 0$.

Substituting $u = H_{k,\xi} z$ into (3.3) gives the equation

$$z - \lambda Q H_{k,\xi} z = f. \tag{3.4}$$

Let $r_\xi = r(Q H_{k,\xi})$ be the spectral radius of the operator $Q H_{k,\xi}$.

Lemma 3.4. *Let $\lambda > 0$ and $\lambda r(Q H_{k,\xi}) < 1$. Then BVP (3.3) has a unique solution $u(x)$.*

If $f \geq\neq 0$, and in the case of $k = 1$, $f(x) \not\equiv 0$ holds on $[0, \xi]$, and in the case of $k = n - 1$, the inequality $f(x) \not\equiv 0$ holds on $[\xi, l]$, then for some $\varepsilon > 0$, $u(x) \geq \varepsilon x^{n-k-1} (x - \xi)^2 (l - x)^{k-1}$, $x \in I$.

Proof. The solution of (3.3) is $u = H_{k,\xi} z$, where $z = (I - \lambda Q H_{k,\xi})^{-1} f$ (see equation (3.4)). Therefore, $z \geq f$. Now, we refer to Lemma 3.3. □

Let $\lambda_{k,\xi}$ be the smallest positive value λ for which BVP

$$(-1)^{k+1} u^{(n)} - \lambda Q u = 0, \quad B_{k,\xi} u = 0 \tag{3.5}$$

has a non-trivial solution. If there are no such numbers, then, by definition, $\lambda_{k,\xi} = +\infty$.

Lemma 3.5. $\lambda_{k,\xi} = 1/r(Q H_{k,\xi})$ *(if $r(Q H_{k,\xi}) = 0$, then $\lambda_{k,\xi} = +\infty$). For $\lambda = \lambda_{k,\xi}$, problem (3.5) has a non-trivial non-negative solution.*

The proof is the same as that of Lemma 3.2.

4 The main boundary value problem

4.1 Main results

We return to the problem of the negativity of the Green function of BVP $\{(1.3),(1.2)\}$. Let us write it in a short form with nonzero boundary conditions

$$\mathcal{L}_\lambda u = f, \quad B_k u = \alpha. \tag{4.1}$$

4.1.1 Positive solvability

For BVP (4.1) positive solvability is impossible. Instead, we will use the restricted notion of $\mathcal{E}$-positive solvability. Recall that $\mathcal{E} \subset AC^{n-1}(I)$ is the set of functions satisfying conditions (1.7).

Definition 4.1. We call BVP (4.1) $\mathcal{E}$-positively solvable if it is uniquely solvable and $u \geq 0$ follows from $f \leq 0$, $\alpha \geq 0$, $u \in \mathcal{E}$.

The relations $\alpha \geq 0$, $u \in \mathcal{E}$ mean that all components of the vector $B_k u$ are equal to zero except, possibly, $u^{(n-k-1)}(0) \geq 0$ and $(-1)^{k-1}u^{(k-1)}(l) \geq 0$. Actually, problem (4.1) is considered only on the set $\mathcal{E}$. Note that $\mathcal{E}$-positive solvability implies the negativity of the Green function $G(x,s,\lambda)$.

Definition 4.2. The equation $\mathcal{L}_\lambda u = 0$ is $\mathcal{E}$-disconjugate on the interval $[0,l]$ if any of its nontrivial solutions belonging to the set $\mathcal{E}$ has at most $n-1$ zeros on $[0,l]$, counting multiple zeros as many times as their multiplicity.

Note that since the solution $u \in \mathcal{E}$ already has $n-2$ zeros, counting multiplicities, it can only have one simple zero in $(0,l)$. In this case, the sum of multiplicities of zeros at the points 0 and l is equal to $n-2$.

Theorem 4.1 (Main theorem)**.** *Let $\lambda \geq 0$. The following statements are equivalent:*

(1) *Problem* (4.1) *is $\mathcal{E}$-positively solvable, and if $(-f,\alpha) \geq\neq 0$, $u \in \mathcal{E}$, then the solution $u(\cdot)$ is $(n-k,k)$-positive.*

(2) *The equation $\mathcal{L}_\lambda u = 0$ is $\mathcal{E}$-disconjugate on* $[0,l]$.

(3) $\lambda < \min\{\lambda_{k-1},\lambda_{k+1}\}$.

(4) *The Green function $G(x,s,\lambda)$ is $(n-k,k)$-negative and*

$$(-1)^k\Gamma^{(k)}(l,0+,\lambda) < 0, \quad \Gamma^{(n-k)}(0,l-,\lambda) < 0 \tag{4.2}$$

(recall that $s^k(l-s)^{n-k}\Gamma(x,s,\lambda) = G(x,s,\lambda)$). Derivatives in (4.2) *are taken with respect to the first argument.*

The proof of the equivalence of the first three statements is given in Theorems 4.4–4.8. Equivalence of statements (3) and (4) is contained in Theorems 4.9 and 4.10.

Remark 4.1. In the case of $\lambda < 0$, BVP (4.1), by multiplying the equation by -1, turns into BVP (1.9) when $m = k$:

$$(-1)^k u^{(n)} + \lambda Q u = -f, \quad B_k u = \alpha.$$

Therefore, a necessary and sufficient condition for $\mathcal{E}$-positive solvability is the pair of inequalities

$$-\lambda_k < \lambda < \min\{\lambda_{k-1},\lambda_{k+1}\}.$$

Remark 4.2. For the statement (4) to be equivalent to the first three in the cases $k = n-1$ and $k = 1$, assumptions (1.5) and (1.6) are essential. For the ordinary differential equation $(-1)^{k+1}u^{(n)} - \lambda p(x)u = 0$, this remark is not relevant, since $(n,0)$- and $(0,n)$-BVPs are uniquely solvable ($\lambda_0 = \lambda_n = \infty$).

Remark 4.3. Estimates of the eigenvalues λ_m are efficiently performed by using theorems on differential and integral inequalities (Theorems 3.1 and 3.2) (see estimates in [10] and in Section 5.1).

4.1.2 Extreme case

Theorem 4.1 gives the conditions for the strong positivity of the Green function in the sense of inequalities (4.2). Here, we consider a more subtle situation of simple negativity of the Green function.

Theorem 4.2. *Let $\lambda \geq 0$. Suppose $2 \leq k \leq n-2$. The following statements are equivalent:*

(1) *Problem* (4.1) *is $\mathcal{E}$-positively solvable, and if $(-f, \alpha) \geq\neq 0$, $u \in \mathcal{E}$, then $u(x) > 0$, $x \in (0, l)$.*

(2) $\lambda \leq \min\{\lambda_{k-1}, \lambda_{k+1}\}$.

(3) $G(x, s, \lambda)$ *is $(n-k, k)$-negative.*

(4) $G(x, s, \lambda) \leq 0$.

Proof. We use the scheme $2 \to 3 \to 4 \to 2$; $1 \to 2 \to 1$. The implication $2 \to 3$ follows from Theorem 4.11, $3 \to 4$ is obvious, $4 \to 2$ follows from Lemma 4.5. The implication $1 \to 2$ follows from Lemma 4.5, $2 \to 1$ from Lemma 4.2 and the statement (3). □

Remark 4.4. For the equation $(-1)^{k+1}u^{(n)} - \lambda p(x)u = f$ in the cases $k = 1$ and $k = n-1$, Theorem 4.2 remains true since $\lambda_0 = \lambda_n = \infty$.

Remark 4.5. In the cases $k = 1$ and $k = n-1$, the sufficiency of the inequality $\lambda \leq \min\{\lambda_{k-1}, \lambda_{k+1}\}$ remains valid.

There is a conjecture about the validity of necessity (of course, under conditions (1.5), (1.6)). In any case, the following assertion is true.

Theorem 4.3. *If $k = n-1$, $G(x, s, \lambda) \leq 0$, $\Gamma^{(n-1)}(l, 0+, \lambda) = 0$ and on the interval $(0, l)$ $\Gamma(x, 0+, \lambda) < 0$, then $\lambda = \lambda_n$.*

Proof. By virtue of Lemma 2.5, the function $g_0(x) = l\Gamma(x, 0+, \lambda)$ (see (2.10)) is an eigenfunction of BVP $\mathcal{L}_\lambda u = 0$, $B_n u = 0$ and satisfies the conditions of Theorem 3.3. □

Without the positivity condition for the function g_0 in Theorem 4.3, the question of the inequality $\lambda \leq \lambda_n$ remains unresolved.

4.2 Proof of the main theorem

Everywhere below, it is assumed that all solutions are in the set $\mathcal{E}$, i.e., satisfy conditions (1.7). In the case of unique solvability of BVP $\mathcal{L}_\lambda u = 0$, $B_k u = 0$, the intersection of the set of solutions of the homogeneous equation $\mathcal{L}_\lambda u = 0$ with the set $\mathcal{E}$ is two-dimensional. Let us define the basis u_0, u_1 in this intersection by the boundary conditions

$$\begin{pmatrix} u_0^{(n-k-1)}(0) & u_1^{(n-k-1)}(0) \\ (-1)^{k-1}u_0^{(k-1)}(l) & (-1)^{k-1}u_1^{(k-1)}(l) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.3}$$

4.2.1 Necessity

Here, we show that statement (3) of Theorem 4.1 follows from the first and second statements. Due to symmetry, it suffices to consider only one of the inequalities $\lambda < \lambda_{k-1}$ or $\lambda < \lambda_{k+1}$.

Theorem 4.4. *Let statement (1) of Theorem* 4.1 be satisfied. Then $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$.

Proof. Let us prove, for example, $\lambda < \lambda_{k+1}$. It follows from statement (1) of Theorem 4.1 that $u_0(x) > 0$ on $(0, l)$, and $(-1)^k u_0^{(k)}(l) > 0$.

By Theorem 3.1 (3.2), for $m = k+1$ we have $\lambda < \lambda_{k+1}$. □

Theorem 4.5. *Let statement (2) of Theorem 4.1 be satisfied. Then $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$.*

Proof. The $\mathcal{E}$-disconjugacy implies the unique solvability of BVP (1.9) for $m = k-1$ and $m = k+1$, since a nontrivial solution of the homogeneous problem $\mathcal{L}_\lambda u = 0$, $B_m u = 0$ would have n zeros on $[0, l]$.

Let u be a solution to the problem $\mathcal{L}_\lambda u = 0$, $u \in \mathcal{E}$, $u^{(n-k-1)}(0) = 0$, $u^{(n-k)}(0) = 1$. Since u has $n-1$ zeros at the ends of the interval, $u(x) > 0$ in $(0, l)$. By Theorem 3.1 (3.2), $\lambda < \lambda_{k-1}$. The inequality $\lambda < \lambda_{k+1}$ is proved similarly. □

4.2.2 Some lemmas

Note that in the case $\lambda \leq \min\{\lambda_{k-1}, \lambda_{k+1}\}$, the basis $\{u_0, u_1\}$ defined in (4.3) exists. Indeed, if $\lambda < \lambda_{k+1}$, then as u_0 we can take the solution of the problem $\mathcal{L}_\lambda u = 0$, $u \in \mathcal{E}$, $u^{(k-1)}(l) = 0$, $(-1)^k u^{(k)}(l) = 1$. Then $u^{(n-k-1)}(0) > 0$ by Lemma 3.1. If $\lambda = \lambda_{k+1}$, then as u_0 we take the eigenfunction $\mathcal{L}_\lambda u = 0$, $B_{k+1} u = 0$. We construct u_1 similarly. So, we have the following lemma.

Lemma 4.1. *Suppose $\lambda \leq \min\{\lambda_{k-1}, \lambda_{k+1}\}$. Then problem* (4.1) *is uniquely solvable.*

Lemma 4.2. *If $\lambda < \lambda_{k+1}$ or ($\lambda = \lambda_{k+1}$ and $k \leq n-2$), then $u_0(x) > 0$, $x \in (0, l)$. If $\lambda < \lambda_{k+1}$, then $(-1)^k u_0^{(k)}(l) > 0$.*

If $\lambda < \lambda_{k-1}$ or ($\lambda = \lambda_{k-1}$ and $k \geq 2$), then $u_1(x) > 0$, $x \in (0, l)$. If $\lambda < \lambda_{k-1}$, then $u_1^{(n-k)}(0) > 0$.

Proof. Consider, for example, u_0. If $\lambda < \lambda_{k+1}$, in the vector $B_{k+1} u_0$, all components are zero, except $u_0^{(k)}(l)$. This last component cannot be zero, because otherwise u_0 would be a solution to the homogeneous problem $\mathcal{L}_\lambda u = 0$, $B_{k+1} u = 0$. By Lemma 3.1, $u_0(x) > 0$, $x \in (0, l)$.

If $\lambda = \lambda_{k+1}$, then $k+1 \leq n-1$, and u_0 is positive by Lemma 3.2. □

Corollary 4.1. *Let $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$ or ($n > 2$ and $\lambda = \min\{\lambda_{k-1}, \lambda_{k+1}\}$).*

If $u(x) = c_1 u_0 + c_2 u_1$ and $c_1 c_2 > 0$, then $u(x) \neq 0$ on $(0, l)$. If $c_1 c_2 < 0$, then the function $u(x)$ changes sign.

Remark 4.6. In the case of $n = 2$, the statement of Corollary 4.1 is false, since both functions u_0 and u_1 may vanish on parts of the interval with non-empty intersection (see Example 5.1).

Lemma 4.3. *For any $\xi \in (0, l)$, the inequality $\lambda_{k,\xi} \geq \min\{\lambda_{k-1}, \lambda_{k+1}\}$ holds. If $2 \leq k \leq n-2$, the strict inequality $\lambda_{k,\xi} > \min\{\lambda_{k-1}, \lambda_{k+1}\}$ holds.*

Proof. Let $\lambda = \lambda_{k,\xi}$ and $\lambda_{k,\xi}$ do not satisfy the conditions of the lemma. Then, by Lemma 4.2, the basis functions u_0, u_1 are strictly positive on $(0, l)$. By Lemma 3.5, for $\lambda = \lambda_{k,\xi}$ there exists a non-negative non-trivial solution $\mathcal{L}_\lambda u = 0$, $B_{k,\xi} u = 0$. It is a linear combination $u = c_1 u_0 + c_2 u_1$. The equalities $u(\xi) = u'(\xi) = 0$ contradict Corollary 4.1. □

4.2.3 Sufficiency

Here, we prove that statements (1) and (2) of Theorem 4.1 follow from statement (3). From Lemma 4.2 we obtain the following theorem.

Theorem 4.6. *Let $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$. Then the solution $u(x)$ to BVP ($u \in \mathcal{E}$),*

$$\mathcal{L}_\lambda u = 0, \;\; u^{(n-k-1)}(0) = c_1 \geq 0, \;\; (-1)^{k-1} u^{(k-1)}(l) = c_2 \geq 0 \;\; (c_1 + c_2 > 0)$$

is $(n-k, k)$-positive.

Theorem 4.7 ($\mathcal{E}$-nonoscillation ($\mathcal{E}$-disconjugacy))**.** *If $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$, then any solution $u \in \mathcal{E}$ of the homogeneous equation $\mathcal{L}_\lambda u = 0$ has at most one simple zero in the interval $(0, l)$ (the total number of zeros in $[0, l]$ is at most $n-1$, counting multiplicities).*

Proof. By Lemma 4.3, $\lambda < \lambda_{k,\xi}$. Up to a factor, any solution of $\mathcal{L}_\lambda u = 0$ in $\mathcal{E}$ can be represented as

$$u(x) = -u_0(x) + C u_1(x),$$

where C is a constant. If $C \le 0$, then $u(x)$ has no zeros in $(0, l)$, since in this case $u(x) = -u_0(x) + Cu_1(x) \le -u_0(x) < 0$.

Now, let $C > 0$. Then

$$(-1)^{k-1} u^{(k-1)}(l) = C(-1)^{k-1} u_1^{(k-1)}(l) = C > 0,$$

but $u^{(n-k-1)}(0) = u_1^{(n-k-1)}(0) < 0$. Therefore, $u(x)$ has zeros in $(0, l)$. Let x_2 be the largest zero (first from the right) (see Fig. 1).

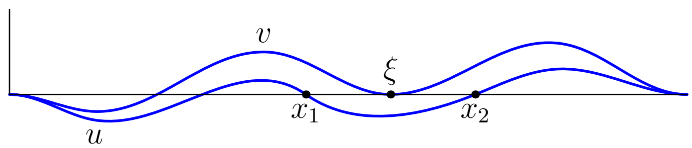

Figure 1: To Theorem 4.7.

This zero can be simple or multiple. First, consider the case of a simple zero, when $u'(x_2) > 0$. Let us show that in this case there are no other zeros. Assume, on the contrary, that they exist and that $x_1 < x_2$ is the one closest to x_2. Let $v(x) = u(x) + Du_1(x)$, where

$$D = \max_{x \in [x_1, x_2]} \Big(-\frac{u(x)}{u_1(x)} \Big) = -\frac{u(\xi)}{u_1(\xi)}, \quad \xi \in (x_1, x_2).$$

Then on $[x_1, x_2]$ we have $v(x) \ge 0$, $v(\xi) = v'(\xi) = 0$. So, $v(x)$ is a non-trivial solution to the problem $\mathcal{L}_\lambda v = 0$, $B_{k,\xi} v = 0$. But this contradicts $\lambda < \lambda_{k,\xi}$.

If x_2 is a multiple zero of u, then we put $\xi = x_2$, $v = u$. □

Lemma 4.4. *Let $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$, and the function $u(x)$ be the solution to the BVP*

$$\mathcal{L}_\lambda u = f, \quad B_k u = 0. \tag{4.4}$$

If $f(\cdot) \ge\ne 0$, then $u^{(n-k)}(0) < 0$ and $(-1)^k u^{(k)}(l) < 0$.

Proof. By Lemma 4.1, BVP (4.4) has a unique solution. It is also a non-zero solution to BVP (1.9) for $m = k-1$, with the conditions $u \in \mathcal{E}$, $u^{(n-k)}(0) = c$, where c is some number. Suppose $u^{(n-k)}(0) \ge 0$. By Lemma 3.1 (for $m = k-1$), $u(x) > 0$, $x \in (0, l)$, and $(-1)^{k-1} u^{(k-1)}(l) > 0$, which contradicts the boundary condition (4.4). The contradiction shows that $u^{(n-k)}(0) < 0$.

The inequality $(-1)^k u^{(k)}(l) < 0$ is proved similarly. □

Theorem 4.8. *If $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$, then BVP (4.4) is uniquely solvable and $G(x, s, \lambda)$ is $(n-k, k)$-negative.*

Proof. Let $u(x)$ be the solution to BVP (4.4) and $f \ge\ne 0$. By Lemma 4.4, $u(x) < 0$ in the neighborhoods of the points 0 and l.

Suppose that $u(x_0) \ge 0$ at some point $x_0 \in (0, l)$. We can assume that x_0 is a maximum point: $u(x_0) = \max\{u(x) : x \in I\}$. Let us construct a non-positive solution to problem (3.3), i.e., a solution that has a multiple zero at some point $\xi \in (0, l)$.

If $u(x_0) = 0$, then x_0 is a multiple of zero, since x_0 is a maximum point. In this case, u itself is the right solution. If $u(x_0) > 0$, then it is possible to construct a non-positive solution $\mathcal{L}_\lambda v = f$ with multiple zero (Fig. 2). Let $v(x) = u(x) - Cu_1(x)$ (we recall that $u_1(x) < 0$, $x \in (0, l)$) and

$$C = \max_{(0,l)} \frac{u(x)}{u_1(x)} = \frac{u(\xi)}{u_1(\xi)}, \quad \xi \in (0, l).$$

This maximum exists because $u(x_0) > 0$ and $u(x) < 0$ in some neighborhoods of the points $x = 0$, $x = l$. The function $v(x)$ is non-positive because

$$v(x) = u(x) - Cu_1(x) = u(x) - \frac{u(\xi)}{u_1(\xi)} u_1(x) \le u(x) - \frac{u(x)}{u_1(x)} u_1(x) = 0.$$

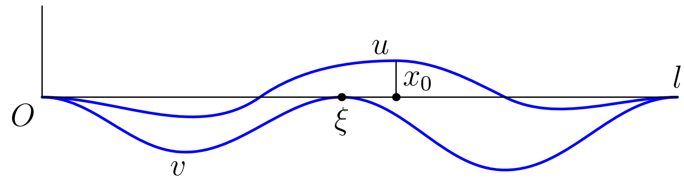

Figure 2: On Theorem 4.8.

So, $v(\xi) = v'(\xi) = 0$, and the function $v(x)$ is the solution to BVP (3.3).

By Lemma 4.3, $\lambda < \lambda_{k,\xi}$. By Lemma 3.4, $v(x) \geq 0$. But this contradicts inequality $v(x) \leq u(x)$ and the negativity of $u(x)$ in neighborhoods of the endpoints of the interval. The contradiction shows that $u(x) < 0$ in $(0, l)$.

The $(n-k, k)$-negativity follows from Lemma 4.4. □

4.2.4 Negativity of the Green function and positive solvability

Recall that the functions $g_0(x)$ and $g_1(x)$ are defined by equalities (2.8),(2.9).

Lemma 4.5. *Let $2 \leq k \leq n-2$, $G(x, s, \lambda) \leq 0$.*

If $g_0^{(k)}(l) \neq 0$, then $\lambda < \lambda_{k+1}$. If $g_0^{(k)}(l) = 0$, then $\lambda = \lambda_{k+1}$.

If $g_1^{(n-k)}(0) \neq 0$, then $\lambda < \lambda_{k-1}$. If $g_1^{(n-k)}(l) = 0$, then $\lambda = \lambda_{k-1}$.

Proof. It suffices to consider only one of these two cases. Limits (2.8) and (2.9) exist by Lemmas 2.4 and 2.6. Let $u(x) = -g_0(x)$ and let $g_0^{(k)}(l) \neq 0$. It follows from Lemma 2.4 and Theorem 3.1 (for $m = k+1$) that $\lambda < \lambda_{k+1}$.

If $g_0^{(k)}(l) = 0$, then $g_0(x)$ and $u(x)$ are the eigenfunctions of BVP $(-1)^{k+1}u^{(n)} - \lambda Qu = 0$, $B_{k+1}u = 0$. For $\varepsilon > 0$,

$$(-1)^{k+1}u^{(n)} - (\lambda - \varepsilon)Qu = \varepsilon Qu \geq\neq 0.$$

By Theorem 3.1, $\lambda - \varepsilon < \lambda_{k+1}$. But $\lambda_{k+1} \leq \lambda$ by definition of the number λ_m. From here, $\lambda = \lambda_{k+1}$. □

It follows from Lemma 2.5 and Theorem 3.2 that Lemma 4.5 remains true for strict inequalities in the case of $k = n-1$ under the additional condition $r(x, 0+) = r(x, 0)$, and, similarly, in the case of $k = 1$ under the additional condition $r(x, l-) = r(x, l)$. Namely, the following lemma is true.

Lemma 4.6. *Suppose $G(x, s, \lambda) \leq 0$.*

If $k = n-1$, $r(x, 0+) = r(x, 0)$, $g_0^{(n-1)}(l) \neq 0$, then $\lambda < \lambda_n$.

If $k = 1$, $r(x, l-) = r(x, l)$, $g_1^{(n-1)}(0) \neq 0$, then $\lambda < \lambda_0$.

Remark 4.7. For example, let $n = 3$, $k = 1$, and $\mathcal{L}_\lambda u = u''' - \lambda u(l)$. The Green function $G(x, s, \lambda)$ is the same as $H_k(x, s)$ and is therefore negative for any λ. At the same time, the BVP

$$u''' - \lambda u(l) = 0, \quad u(0) = u'(0) = u''(0) = 0$$

has a nonzero solution $u = x^3$ for $\lambda = 6/l^3$. The reason for this behavior is the inequality $r(x, l) \neq r(x, l-)$. This is a special situation that is not relevant for the ordinary equation $(-1)^{k+1}u^{(n)} - p(x)u = 0$.

From Lemmas 4.5 and 4.6, we directly obtain the following two theorems.

Theorem 4.9. *Let $G(x, s, \lambda) \leq 0$ and $g_0^{(k)}(l) \neq 0$, $g_1^{(n-k)}(0) \neq 0$. Then $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$.*

Theorem 4.10. *Suppose $\lambda < \min\{\lambda_{k-1}, \lambda_{k+1}\}$. Then $G(x, s, \lambda)$ is $(n-k, k)$-negative, and $g_0^{(k)}(l) \neq 0$, $g_1^{(n-k)}(0) \neq 0$.*

4.3 Negativity of the Green function in the extreme case

Lemma 4.7. *Let $k \le n-2$, $\mathcal{L}_\lambda u = f \ge\ne 0$, $B_k u = 0$, and $\lambda = \lambda_{k+1}$. Then $(-1)^k u^{(k)}(l) < 0$.*

Proof. The existence of the Green function follows from Lemma 4.1. It follows from the continuous dependence of $G(x,s,\lambda)$ on λ that $G(x,s,\lambda) \le 0$. Therefore, $u(x) \le 0$.

Suppose that $u^{(k)}(l) = 0$. By Lemma 3.2 for $m = k+1$, there exists a solution $u_0(x) > 0$, $x \in (0,l)$, of BVP $\mathcal{L}_\lambda u = 0$, $B_{k+1}u = 0$ and $u_0^{(n-k-1)}(0) > 0$, $(-1)^{k+1}u_0^{(k+1)}(l) > 0$.

Since $u^{(n-k-1)}(0) = 0$ and $u^{(k)}(l) = 0$, we have $-u(x) \le Cu_0(x)$ for some $C > 0$. The function $v = u + Cu_0$ satisfies Theorem 3.1 for $m = k+1$, due to which $\lambda < \lambda_{k+1}$. But this contradicts $\lambda = \lambda_{k+1}$. □

Theorem 4.11. *Let $2 \le k \le n-2$, $\lambda = \min\{\lambda_{k-1}, \lambda_{k+1}\}$.*

The Green function $G(x,s,\lambda)$ is $(n-k,k)$-negative.

Proof. The existence of the Green function follows from Lemma 4.1. Let $f \ge\ne 0$, $u = Gf$. By Lemma 4.7, the solution $u(x) \le 0$ and is negative near the ends of the segment. The presence of multiple zeros inside $(0,l)$ contradicts Lemma 4.3. □

5 Appendix

Let us give the previously promised example.

Example 5.1. Let $n = 2$, $k = 1$, $l = 2$, $0 < x_0 < 1 < x_1 < 2$, $x_0 + x_1 = 2$. We define the operator Q by the equality

$$Qu(x) = \begin{cases} u(x_0) & \text{if } x \in [0,1), \\ u(x_1) & \text{if } x \in [1,2]. \end{cases}$$

The equation $u'' - \lambda Qu = 0$ has solutions ($\lambda = 2/(x_0-1)^2$)

$$u_0(x) = \begin{cases} (x-1)^2 & \text{if } x \in [0,1), \\ 0 & \text{if } x \in [1,2], \end{cases}$$

$$u_1(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ (x-1)^2 & \text{if } x \in [1,2]. \end{cases}$$

These solutions are the eigenfunctions of boundary value problems with functionals B_2 and B_0, respectively. Therefore, $\lambda_2 = \lambda_0 = 1/(x_0-1)^2$. The linear combination $c_1u_0 + c_2u_1$ vanishes at the point $x = 1$. Thus the condition of Corollary 4.1 is essential.

5.1 Estimation of eigenvalues

Let us use Theorem 3.1 to estimate the eigenvalues λ_m. Consider only the case $1 \le m \le n-1$, $Qu(x) = q(x)u(x)$. We use Yu. Pokorny's representation (2.2). Substituting $u = x^{n-m}(l-x)^m$ into the integral inequality $u - \lambda H_m Qu = g \ge 0$ and dividing by u, we get

$$1 - \lambda \int_0^l \frac{1}{x(l-x)}\,(l-s)^{n-1}s^{n-1}\,\frac{G_{11}(x,s)\varphi(x,s)}{l^{n-2}(n-1)!}\,q(s)\,ds \ge 0. \tag{5.1}$$

The function $\varphi(x,s)$ is bounded from above,

$$\varphi(x,s) \le \max\{\varphi(0,s), \varphi(l,s)\} = C_{n-1}^{\sigma}, \quad \sigma = \min\{n-m, m\}.$$

Let

$$J(x) := \int_0^l \frac{1}{x(l-x)}\,(l-s)^{n-1}s^{n-1}G_{11}(x,s)q(s)\,ds = J_1(x) + J_2(x),$$

where

$$J_1(x) = \frac{1}{lx} \int\limits_0^x s^n (l-s)^{n-1} q(s)\, ds,$$

$$J_2(x) = \frac{1}{l(l-x)} \int\limits_x^l s^{n-1} (l-s)^n q(s)\, ds.$$

(5.1) implies that if for $0 \le x \le l$

$$\frac{1}{\lambda} \ge J(x)\, \frac{C^\sigma_{n-1}}{l^{n-2}(n-1)!}\,,$$

then $\lambda < \lambda_m$. Thus

$$\frac{1}{\lambda_m} < \max J(x)\, \frac{C^\sigma_{n-1}}{l^{n-2}(n-1)!}\,. \tag{5.2}$$

Lemma 5.1. *At the maximum point x_0 of $J(x)$ there takes place*

$$J(x_0) = \frac{l}{x_0}\, J_1(x_0) = \frac{l}{l-x_0}\, J_2(x_0). \tag{5.3}$$

Proof. Let $h(s) := s^{n-1}(l-s)^{n-1}q(s)/l$. Then

$$J_1(x) = \frac{1}{x} \int\limits_0^x s\, h(s)\, ds, \quad J_2(x) = \frac{1}{l-x} \int\limits_x^l (l-s) h(s)\, ds,$$

$$J'(x) = -\frac{1}{x^2} \int\limits_0^x s\, h(s)\, ds + \frac{1}{x}\, x\, h(x) + \frac{1}{(l-x)^2} \int\limits_x^l (l-s)h(s)\, ds - \frac{1}{l-x}\, (l-x)h(x)$$

$$= -\frac{1}{x^2} \int\limits_0^x s\, h(s)\, ds + \frac{1}{(l-x)^2} \int\limits_x^l (l-s)h(s)\, ds,$$

$$J'(x_0) = 0 \Longrightarrow \frac{1}{x_0}\, J_1(x_0) = \frac{1}{l-x_0}\, J_2(x_0).$$

This implies (5.3). □

From (5.2) and (5.3) follows

$$\frac{1}{\lambda_m} < \frac{C^\sigma_{n-1}}{l^{n-2}(n-1)!}\, \frac{1}{x_0^2} \int\limits_0^{x_0} s^n (l-s)^{n-1} q(s)\, ds.$$

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