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ON MULTIDIMENSIONAL SPACE-PERIODIC AND TIME-ANTIPERIODIC BOUNDARY VALUE PROBLEM FOR ONE CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Abstract. In this work, a multidimensional space-periodic and time-antiperiodic boundary value problem for one class of nonlinear partial differential equations is studied. The conditions imposed on the nonlinear term of the equation are found, the fulfillment of which allows us to prove theorems on the existence, uniqueness and nonexistence of solutions of this problem.

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1 Statement of the problem

In the Euclidean space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \dots, x_n)$ and t, consider the nonlinear partial differential equation of the type

$$L_f u := \frac{\partial^2 u}{\partial t^2} - \Delta^2 u + \lambda \Delta u + f(u) = F(x,t), \tag{1.1}$$

where f, F are given functions and u is an unknown functions, $\Delta := \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$, $\lambda = const$.

For equation (1.1), consider a space-periodic and time-antiperiodic boundary value problem: in the domain $D_T : \Omega \times (0,T)$, where $\Omega := \{x \in \mathbb{R}^n : \ 0 < x_i < l_i, \ i = 1, \dots, n\}$, find a solution $u(x,t)$ of equation (1.1) according to the boundary conditions

$$\frac{\partial^i u}{\partial x_j^i}\bigg|_{\partial D_T \cap \{x_j = 0\}} = \frac{\partial^i u}{\partial x_j^i}\bigg|_{\partial D_T \cap \{x_j = l_j\}}, \quad j = 1, \dots, n; \quad i = 0, 1, 2, 3, \tag{1.2}$$

$$u(x,0) = -u(x,T), \quad u_t(x,0) = -u_t(x,T), \quad x \in \Omega. \tag{1.3}$$

Here, equality (1.2) should be understood as the equality of values of the function u at the points x and $\tilde{x}$, respectively, where $x_i = \tilde{x}_i$ for $i \neq j$ and $x_j = 0$, $\tilde{x}_j = l_j$.

Remark 1.1. Note that space-periodic and time-antiperiodic boundary value problems posed for nonlinear partial differential equations with a structure, different from (1.1), have been studied in numerous works (see, e.g., [1–5,7,10,12,13,15,16,18–20] and the references therein). For equation (1.1) with one spatial variable, i.e., for $n = 1$, the space-periodic and time-antiperiodic problem is considered in [11], and the antiperiodic problem, with respect to both spatial and time variables, is studied in [9].

Let us consider the classical space $C^{2,4}(\overline{D}_T)$ of functions continuous in $\overline{D}_T$ and having continuous partial derivatives $\frac{\partial^i u}{\partial t^i}$, $i = 1, 2$, $\partial_x^\beta u$, where $\partial_x^\beta = \frac{\partial^{|\beta|}}{\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}}$, $\beta = (\beta_1, \dots, \beta_n)$, $|\beta| = \sum\limits_{i=1}^{n} \beta_i \leq 4$. Let

$$C_0^{2,4}(\overline{D}_T) := \bigg\{ u \in C^{2,4}(\overline{D}_T) : \ \frac{\partial^i u}{\partial x_j^i}\bigg|_{\partial D_T \cap \{x_j = 0\}} = \frac{\partial^i u}{\partial x_j^i}\bigg|_{\partial D \cap \{x_j = l_j\}}, \quad j = 1, \dots, n; \quad i = 0, 1, 2, 3;$$
$$u(x,0) = -u(x,T), \ \ u_t(x,0) = -u_t(x,T), \ \ x \in \Omega \bigg\}. \tag{1.4}$$

Denote by $W_0^{1,2}(D_T)$ the Hilbert space obtained by completion of the classical space $C^{2,4}(\overline{D}_T)$ with respect to the norm

$$\|u\|^2_{W_0^{1,2}(D_T)} = \int\limits_{D_T} \left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 + (\Delta u)^2 \right] dx\, dt. \tag{1.5}$$

From (1.5) it follows that if $u \in W_0^{1,2}(D_T)$, then $u \in W_2^1(D_T)$ and $\Delta u \in L_2(D_T)$. Here, $W_2^1(D_T)$ is the Sobolev space consisting of the elements of $L_2(D_T)$, having the first order generalized derivatives from $L_2(D_T)$.

Remark 1.2. Let $u \in C_0^{2,4}(\overline{D}_T)$ be a classical solution of problem (1.1)–(1.3). Multiplying the both sides of equation (1.1) by an arbitrary function $\varphi \in C_0^{2,4}(\overline{D}_T)$ and integrating the obtained equation by parts over the domain D_T, and also taking into account that the functions from $C_0^{2,4}(\overline{D}_T)$ satisfy the boundary conditions (1.2) and (1.3), we obtain

$$\int\limits_{D_T} \left[\frac{\partial u}{\partial t} \frac{\partial \varphi}{\partial t} + \Delta u \Delta \varphi + \lambda \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \right] dx\, dt - \int\limits_{D_T} f(u) \varphi \, dx\, dt = - \int\limits_{D_T} F \varphi \, dx\, dt \ \ \forall \varphi \in C_0^{2,4}(\overline{D}_T). \tag{1.6}$$

We take equality (1.6) as a basis for the definition of the weak generalized solution of problem (1.1)–(1.3) in the space $W_0^{1,2}(D_T)$. But for this we have to impose some conditions on the function f so that the integral $\int\limits_{D_T} f(u) \varphi \, dx\, dt$ exists.

Remark 1.3. Below we require that the function f from equation (1.1) satisfy the following conditions:

$$f \in C(\mathbb{R}), \quad |f(u)| \le M_1 + M_2|u|^\alpha \quad \forall\, u \in \mathbb{R}, \tag{1.7}$$

where $M_i = const \ge 0$, $i = 1, 2$, and

$$0 \le \alpha = const < \frac{n+1}{n-1}\,. \tag{1.8}$$

As is known, the embedding operator $I : W_2^1(D_T) \to L_q(D_T)$ represents a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, $n > 1$ [14]. At the same time, the Nemytski operator $N : L_q(D_T) \to L_2(D_T)$, acting by the formula $Nu = f(u)$, where $u \in L_q(D_T)$ and the function f satisfies conditions (1.7), (1.8), is continuous and bounded for $q \ge 2\alpha$ [8]. Thus, if $\alpha < \frac{n+1}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Therefore, the operator

$$N_0 = NI : W_2^1(D_T) \to L_2(D_T) \tag{1.9}$$

is continuous and compact. Whence, in particular, it follows that if $u \in W_2^1(D_T)$, then $f(u) \in L_2(D_T)$, and if $u_n \to u$ in the space $W_2^1(D_T)$, then $f(u_n) \to f(u)$ in the space $L_2(D_T)$.

Definition 1.1. Let $F \in L_2(D_T)$ and the function f satisfy conditions (1.7) and (1.8). The function $u \in \overset{\circ}{W}{}^{1,2}_0(D_T)$ is said to be a weak generalized solution of problem (1.1)–(1.3) if for any function $\varphi \in \overset{\circ}{W}{}^{1,2}_0(D_T)$ the integral equality (1.6) holds, i.e.,

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \Delta u \Delta\varphi + \lambda\sum_{i=1}^n \frac{\partial u}{\partial x_i}\,\frac{\partial \varphi}{\partial x_i}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \quad \forall\,\varphi \in \overset{\circ}{W}{}^{1,2}_0(D_T). \tag{1.10}$$

Note that, due to Remark 1.3, on the left-hand side of equality (1.10), the integral $\int\limits_{D_T} f(u)\varphi\,dx\,dt$ is defined correctly, since $u \in \overset{\circ}{W}{}^{1,2}_0(D_T)$ implies $f(u) \in L_2(D_T)$, and thereby $f(u)\varphi \in L_1(D_T)$, since $\varphi \in L_2(D_T)$.

It is not difficult to see that if the weak generalized solution u of problem (1.1)–(1.3) in the sense of Definition 1.1 belongs to the class $C_0^{2,4}(\overline{D}_T)$, then it will also be a classical solution of this problem.

2 Equivalent reduction of problem (1.1)–(1.3) to the nonlinear functional equation in the space $\overset{\circ}{W}{}^{1,2}_0(D_T)$

Under the following condition

$$\lambda > 0, \tag{2.1}$$

in the space $C_0^{2,4}(\overline{D}_T)$, endowed with the scalar product

$$(u,v)_0 = \int\limits_{D_T} \Big[uv + \frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \sum_{i=1}^n \frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i} + \Delta u \Delta v\Big]\,dx\,dt, \tag{2.2}$$

and with the norm $\|\cdot\|_0 = \|\cdot\|_{\overset{\circ}{W}{}^{1,2}_0(D_T)}$, defined by the right-hand side of equality (1.5), we introduce the following scalar product:

$$(u,v)_1 = \int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \Delta u \Delta v + \lambda\sum_{i=1}^n \frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\Big]\,dx\,dt \tag{2.3}$$

with the norm

$$\|u\|_1^2 = \int\limits_{D_T} \Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + (\Delta u)^2 + \lambda\sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\,dx\,dt, \tag{2.4}$$

where $u, v \in C_0^{2,4}(\overline{D}_T)$.

Let us show that under condition (2.1), the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent, i.e., the inequalities

$$c_1\|u\|_0 \le \|u\|_1 \le c_2\|u\|_0 \quad \forall\, u \in C_0^{2,4}(\overline{D}_T) \tag{2.5}$$

hold with the positive constants c_1 and c_2 independent of u.

Let us prove inequalities (2.5). If $u \in C_0^{2,4}(\overline{D}_T)$, then, due to (1.4), we have $u(x,0) = -u(x,T)$, $x \in \Omega$, and therefore, there exists at least one point $t_0 = t_0(x) \in [0,T]$ such that

$$u(x,t_0) = 0. \tag{2.6}$$

Indeed, if $u(x,0) = 0$, then condition (2.6) holds for $t_0 = 0$. In the case $u(x,0) \neq 0$, due to the equality $u(x,0) = -u(x,T)$, we have $u(x,0)u(x,T) < 0$ and, according to the Bolzano–Cauchy theorem, on the segment $\{(x,t):\ 0 \le t \le T\}$ there exists a point (x,t_0) for which equality (2.6) is valid. In view of (2.6) and the Newton–Leibnitz formula, we have

$$u(x,t) = \int\limits_{t_0}^{t} \frac{\partial u}{\partial t}(x,\tau)\, d\tau \quad \forall\, t \in [0,T],$$

whence, due to the Cauchy inequality, we obtain

$$\begin{aligned} |u(x,t)|^2 &= \Bigg[\int\limits_{t_0}^{t} 1 \cdot \frac{\partial u}{\partial t}(x,\tau)\, d\tau\Bigg]^2 \le \Bigg[\Bigg(\Bigg|\int\limits_{t_0}^{t} 1^2\, d\tau\Bigg|\Bigg)^{1/2} \Bigg(\Bigg|\int\limits_{t_0}^{t} \Big(\frac{\partial u}{\partial t}(x,\tau)\Big)^2 d\tau\Bigg|\Bigg)^{1/2}\Bigg]^2 \\ &= |t-t_0|\,\Bigg|\int\limits_{t_0}^{t} \Big(\frac{\partial u}{\partial t}(x,\tau)\Big)^2 d\tau\Bigg| \le T\int\limits_{0}^{T} \Big(\frac{\partial u}{\partial t}(x,\tau)\Big)^2 d\tau. \end{aligned} \tag{2.7}$$

Integrating both parts of inequality (2.7) over the domain D_T, we get

$$\begin{aligned} \int\limits_{D_T} u^2\, dx\, dt &\le \int\limits_{\Omega} dx \int\limits_{0}^{T} u^2(x,t)\, dt \le \int\limits_{\Omega} dx \int\limits_{0}^{T} \Bigg[\int\limits_{0}^{T} T\Big(\frac{\partial u}{\partial t}\,(x,\tau)\Big)^2 d\tau\Bigg] dt \\ &= \int\limits_{\Omega} dx\, T^2 \int\limits_{0}^{T} \Big(\frac{\partial u}{\partial t}\,(x,\tau)\Big)^2 d\tau = T^2 \int\limits_{D_T} \Big(\frac{\partial u}{\partial t}\,(x,\tau)\Big)^2 dx\, d\tau = T^2 \int\limits_{D_T} \Big(\frac{\partial u}{\partial t}\Big)^2 dx\, dt. \end{aligned} \tag{2.8}$$

In view of (2.1), inequalities (2.5) easily follow from (2.8).

Remark 2.1. In view of the (2.5), by completing the space $C_0^{2,4}(\overline{D}_T)$ with the norm (2.4), due to (1.5) and (2.2), we obtain the same Hilbert space $W_0^{1,2}(D_T)$ with equivalent scalar products (2.2) and (2.3).

Before considering the solvability of problem (1.1)–(1.3) in the nonlinear case, let us first consider this question for the linear problem, i.e., when in equation (1.1) the function $f = 0$. In this case, for $F \in L_2(D_T)$, we analogously introduce a notion of a weak generalized solution $u \in W_0^{1,2}(D_T)$ of this problem when in the integral equality (1.10) the function $f = 0$, i.e.,

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \Delta u \Delta\varphi + \lambda\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\,\frac{\partial \varphi}{\partial x_i}\Big]\, dx\, dt = -\int\limits_{D_T} F\varphi\, dx\, dt \quad \forall\, \varphi \in W_0^{1,2}(D_T). \tag{2.9}$$

According to (2.3), the integral equality (2.9) can be rewritten in the following form:

$$(u,\varphi)_1 = -\int\limits_{D_T} F\varphi\, dx\, dt \quad \forall\, \varphi \in W_0^{1,2}(D_T). \tag{2.10}$$

Taking into account (1.5), (2.2) and (2.5), from (2.10) we have

$$\Big| - \int\limits_{D_T} F\varphi\,dx\,dt \Big| \le \|F\|_{L_2(D_T)}\|\varphi\|_{L_2(D_T)} \le \|F\|_{L_2(D_T)}\|\varphi\|_0 \le c_1^{-1}\|F\|_{L_2(D_T)}\|\varphi\|_1. \tag{2.11}$$

According to Remark 2.1, (2.10) and (2.11), from the Riesz theorem it follows that there exists a unique function $u \in \overset{\circ}{W}{}^{1,2}_0(D_T)$ satisfying equality (2.10) for any $\varphi \in \overset{\circ}{W}{}^{1,2}_0(D_T)$ for which the estimate

$$\|u\|_1 \le c_1^{-1}\|F\|_{L_2(D_T)} \tag{2.12}$$

is valid.

In view of (2.5) and (2.12), it follows that

$$\|u\|_0 = \|u\|_{W_0^{1,2}(D_T)} \le c_1^{-1}\|u\|_1 \le c_1^{-2}\|F\|_{L_2(D_T)}. \tag{2.13}$$

Thus, introducing the notation $u = L_0^{-1}F$, we find that the linear problem corresponding to (1.1)–(1.3), i.e., for $f=0$, corresponds to the linear bounded operator $L_0^{-1} : L_2(D_T) \to W_0^{1,2}(D_T)$ and its norm satisfies the estimate

$$\|L_0^{-1}\|_{L_2(D_T)\to W_0^{1,2}(D_T)} \le c_1^{-2} \tag{2.14}$$

which holds true by virtue of (2.13).

Taking into account Definition 1.1 and (2.9), as well as the definition of the operator L_0^{-1}, we can rewrite equality (1.10), equivalent to problem (1.1)–(1.3), in the form of nonlinear functional equation

$$u = L_0^{-1}[f(u) - F] \tag{2.15}$$

in the Hilbert space $W_0^{1,2}(D_T)$.

3 The existence of a solution of problem (1.1)–(1.3)

As shown below, the fulfillment of conditions (1.7), (1.8) and (2.1) does not yet guarantee the existence of a solution of problem (1.1)–(1.3).

Let us consider the following condition imposed on the nonlinear function $f = f(u)$:

$$\lim_{|u|\to\infty} \sup \frac{f(u)}{u} \le 0. \tag{3.1}$$

Lemma 3.1. *Let $F \in L_2(D_T)$ and conditions (1.7), (1.8), (2.1) and (3.1) be fulfilled. Then for any weak generalized solution $u \in W_0^{1,2}(D_T)$ of problem (1.1)–(1.3), the a priori estimate*

$$\|u\|_0 = \|u\|_{W_0^{1,2}(D_T)} \le c_3\|F\|_{L_2(D_T)} + c_4 \tag{3.2}$$

is valid with the constants $c_3 > 0$ and $c_4 \ge 0$, independent of u and F.

Proof. Since $f \in C(\mathbb{R})$, it follows from (3.1) that for any $\varepsilon > 0$, there exists a number $M_\varepsilon \ge 0$ such that

$$uf(u) \le M_\varepsilon + \varepsilon u^2 \quad \forall\, u \in \mathbb{R}. \tag{3.3}$$

Substituting $\varphi = u \in W_0^{1,2}(D_T)$ in equality (1.10) and taking into account (2.2), (2.3) and (3.3), for any $\varepsilon > 0$ we obtain

$$\begin{aligned}\|u\|_1^2 &= \int\limits_{D_T} uf(u)\,dx\,dt - \int\limits_{D_T} Fu\,dx\,dt \\ &\le M_\varepsilon \operatorname{mes} D_T + \varepsilon \int\limits_{D_T} u^2\,dx\,dt + \int\limits_{D_T} \Big(\frac{1}{4\varepsilon}F^2 + \varepsilon u^2\Big)\,dx\,dt = \frac{1}{4\varepsilon}\,\|F\|^2_{L_2(D_T)} + M_\varepsilon \operatorname{mes} D_T + 2\varepsilon\|u\|_0^2.\end{aligned}$$

Due to (2.5), it follows that $c_1^2\|u\|_0^2 \le \|u\|_1^2 \le \frac{1}{4\varepsilon}\,\|F\|^2_{L_2(D_T)} + M_\varepsilon \operatorname{mes} D_T + 2\varepsilon\|u\|_0^2$, whence for $\varepsilon = \frac{1}{4}\,c_1^2$ we obtain

$$\|u\|_0^2 \le 2c_1^{-4}\|F\|^2_{L_2(D_T)} + 2c_1^{-2}M_\varepsilon \operatorname{mes} D_T. \tag{3.4}$$

From (3.4) follows (3.2) for $c_3^2 = 2c_1^{-4}$ and $c_4^2 = 2c_1^{-2}M_\varepsilon \operatorname{mes} D_T$, where $\varepsilon = \frac{1}{4}\,c_1^2$. □

Remark 3.1. By the definition of the space $W_0^{1,2}(D_T)$, it follows that it is continuously embedded into the Sobolev space $W_2^1(D_T)$, and due to Remark 1.3, the operator $N_1 = N_0 I_1 : W_0^{1,2}(D_T) \to L_2(D_T)$, where the continuous and compact operator $N_0 = NI$ is defined in (1.9) and $I_1 : W_0^{1,2}(D_T) \to W_2^1(D_T)$, is the embedding operator and is also continuous and compact. According to the above-said, we rewrite the functional equation (2.15) in the form

$$u = Ku := L_0^{-1}(N_1 u - F). \tag{3.5}$$

Taking into account (2.14), the linear operator $L_0^{-1} : L_2(D_T) \to W_0^{1,2}(D_T)$ is bounded, and by Remark 3.1, we conclude that the operator $K : W_0^{1,2}(D_T) \to W_0^{1,2}(D_T)$ from (3.5) is continuous and compact. At the same time, according to the scheme of proving the a priori estimate (3.2), where $c_3^2 = 2c_1^{-4}$ and $c_4^2 = 2c_1^{-2}M_\varepsilon \operatorname{mes} D_T$, $\varepsilon = \frac{1}{4}\,c_1^2$, it is clear that for any parameter $\tau \in [0,1]$ and for any solution $u \in W_0^{1,2}(D_T)$ of the equation $u = \tau K u$, the a priori estimate (3.2) is valid with the same constants $c_3 > 0$ and $c_4 \ge 0$, independent of u, F and τ. Therefore, by the Schaefer fixed point theorem [6], equation (3.5), and hence problem (1.1)–(1.3), has at least one weak generalized solution u in the space $W_0^{1,2}(D_T)$ in the sense of Definition 1.1. Thus the following theorem is valid.

Theorem 3.1. *Let conditions (1.7), (1.8), (2.1) and (3.1) be fulfilled. Then for any $F \in L_2(D_T)$, problem (1.1)–(1.3) has at least one weak generalized solution $u \in W_0^{1,2}(D_T)$ in the sense of Definition 1.1.*

4 Cases of absence of solution to problem (1.1)–(1.3)

Let us consider the condition imposed on the nonlinear function $f = f(u)$:

$$f(u) \le -|u|^\alpha, \;\; \alpha = const > 1, \tag{4.1}$$

under which problem (1.1)–(1.3) may have no solution. Obviously, if condition (4.1) is fulfilled, then condition (3.1) will be violated.

Theorem 4.1. *Let conditions (1.7), (1.8), (4.1) be fulfilled and $F = \beta F_0$, where $\beta = const > 0$, $F_0 \in L_2(D_T)$, $F_0|_{D_T} > 0$. Then there exists a number $\beta_0 = \beta_0(F_0, \alpha) > 0$ such that for $\beta > \beta_0$, problem (1.1)–(1.3) has no weak generalized solution in the space $W_0^{1,2}(D_T)$ in the sense of Definition 1.1.*

Proof. In the proof of this theorem we use the method of test functions [17]. As a test function we take a function φ satisfying the following conditions:

$$\begin{gathered} \varphi \in C^4(\overline{D}_T), \;\; \varphi\big|_{D_T} > 0, \;\; \frac{\partial^i \varphi}{\partial t^i}\bigg|_{t=0} = \frac{\partial^i \varphi}{\partial t^i}\bigg|_{t=T} = 0, \;\; i = 0,1, \\ \frac{\partial^i \varphi}{\partial x_j^i}\bigg|_{\partial D_T \cap \{x_j = 0\}} = \frac{\partial^i \varphi}{\partial x_j^i}\bigg|_{\partial D_T \cap \{x_j = l_j\}} = 0, \;\; j = 1,\dots,n; \;\; i = 0,1,2,3. \end{gathered} \tag{4.2}$$

It is obvious that when (4.2) is fulfilled, the function $\varphi \in C_0^{2,4}(\overline{D}_T) \subset W_0^{1,2}(D_T)$.

Let $u \in W_0^{1,2}(D_T)$ be a weak generalized solution of problem (1.1)–(1.3) satisfying the integral equality (1.10). Taking the function φ from (1.10) satisfying conditions (4.2), and integrating the integral terms of equality (1.10) by parts, we obtain

$$\int\limits_{D_T} u L_0 \varphi \, dx\, dt + \int\limits_{D_T} f(u)\varphi \, dx\, dt = \int\limits_{D_T} F\varphi \, dx\, dt, \tag{4.3}$$

where $L_0 := \frac{\partial^2}{\partial t^2} - \Delta^2 + \lambda\Delta$.

Since $\varphi|_{D_T} > 0$, due to (4.1), we have $-f(u)\varphi \geq |u|^\alpha \varphi$ and from (4.3) we obtain the following inequality:

$$\int\limits_{D_T} |u|^\alpha \varphi \, dx \, dt \leq \int\limits_{D_T} uL_0\varphi \, dx \, dt - \int\limits_{D_T} F\varphi \, dx \, dt = \int\limits_{D_T} uL_0\varphi \, dx \, dt - \beta \int\limits_{D_T} F_0\varphi \, dx \, dt. \tag{4.4}$$

If in the Young inequality with parameter $\varepsilon > 0$,

$$ab \leq \frac{\varepsilon}{\alpha} a^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} b^{\alpha'}, \quad a, b \geq 0, \quad \alpha' = \frac{\alpha}{\alpha - 1}$$

we take $a = |u|\varphi^{1/\alpha}$, $b = |L_0\varphi|/\varphi^{1/\alpha}$, then, taking into account that $\alpha'/\alpha = \alpha' - 1$, we get

$$|uL_0\varphi| = |u|\varphi^{1/\alpha} \frac{|L_0\varphi|}{\varphi^{1/\alpha}} \leq \frac{\varepsilon}{\alpha} |u|^\alpha \varphi + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \,. \tag{4.5}$$

From (4.4), (4.5) it follows that

$$\Big(1 - \frac{\varepsilon}{\alpha}\Big) \int\limits_{D_T} |u|^\alpha \varphi \, dx \, dt \leq \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt - \beta \int\limits_{D_T} F_0\varphi \, dx \, dt,$$

whence for $\varepsilon < \alpha$ we get

$$\int\limits_{D_T} |u|^\alpha \varphi \, dx \, dt \leq \frac{\alpha}{(\alpha - \varepsilon)\alpha' \varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt - \frac{\alpha\beta}{\alpha - \varepsilon} \int\limits_{D_T} F_0\varphi \, dx \, dt. \tag{4.6}$$

Taking into account the equalities $\alpha' = \frac{\alpha}{\alpha-1}$, $\alpha = \frac{\alpha'}{\alpha'-1}$, and $\min\limits_{0<\varepsilon<\alpha} \frac{\alpha}{(\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = 1$, which is achieved when $\varepsilon = 1$, from (4.6) we obtain

$$\int\limits_{D_T} |u|^\alpha \varphi \, dx \, dt \leq \int\limits_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt - \alpha'\beta \int\limits_{D_T} F_0\varphi \, dx \, dt. \tag{4.7}$$

It is not difficult to show the existence of a test function φ such that, together with (4.2), it satisfies the condition

$$\kappa_0 = \int\limits_{D_T} \frac{|L_0\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt < +\infty. \tag{4.8}$$

Indeed, as it can be easily verified, the function

$$\varphi(x, t) = \prod_{j=1}^{n} \big[x_j(l_j - x_j)t(T - t)\big]^m,$$

for a sufficiently large positive m, satisfies conditions (4.2) and (4.8).

Since $F_0 \in L_2(D_T)$, $F_0 > 0$, and $\varphi|_{D_T} > 0$, we have

$$0 < \kappa_1 = \int\limits_{D_T} F_0\varphi \, dx \, dt < +\infty. \tag{4.9}$$

Let $\chi(\beta)$ denote the right-hand side of inequality (4.7) which is a linear function with respect to β. Then, by (4.8) and (4.9), we have

$$\chi(\beta) < 0 \ \text{ for } \ \beta > \beta_0 \ \text{ and } \ \chi(\beta) > 0 \ \text{ for } \ \beta < \beta_0, \tag{4.10}$$

where $\chi(\beta) = \kappa_0 - \alpha'\beta\kappa_1$, $\beta_0 = \frac{\kappa_0}{\alpha'\kappa_1}$.

Due to (4.10), for $\beta > \beta_0$, the right-hand side of inequality (4.7) is negative, whereas the left-hand side of that inequality is nonnegative. The obtained contradiction proves the theorem. □

5 Uniqueness of the solution of problem (1.1)–(1.3)

Consider the following monotonicity condition imposed on function $f = f(u)$:

$$(f(u) - f(v))(u - v) \leq 0 \ \ \forall\, u, v \in \mathbb{R}. \tag{5.1}$$

Theorem 5.1. *Let conditions (1.7), (1.8), (2.1) and (5.1) be fulfilled. Then for any $F \in L_2(D_T)$, problem (1.1)–(1.3) cannot have more than one weak generalized solution in the space $\overset{\circ}{W}{}^{1,2}_0(D_T)$ in the sense of Definition 1.1.*

Proof. Let $F \in L_2(D_T)$, and let u_1 and u_2 be two weak generalized solutions of problem (1.1)–(1.3) from the space $\overset{\circ}{W}{}^{1,2}_0(D_T)$ in the sense of Definition 1.1. Then, due to (1.10), the following equalities

$$\int\limits_{D_T} \Big[\frac{\partial u_j}{\partial t}\,\frac{\partial \varphi}{\partial t} + \Delta u_j \Delta \varphi + \lambda \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i}\,\frac{\partial \varphi}{\partial x_i}\Big]\, dx\, dt$$
$$- \int\limits_{D_T} f(u_j)\varphi\, dx\, dt - \int\limits_{D_T} F\varphi\, dx\, dt \ \ \forall\, \varphi \in \overset{\circ}{W}{}^{1,2}_0(D_T), \ \ j = 1, 2, \tag{5.2}$$

are valid.

From (5.2), for the difference $v = u_2 - u_1$, we have

$$\int\limits_{D_T} \Big[\frac{\partial v}{\partial t}\,\frac{\partial \varphi}{\partial t} + \Delta v \Delta \varphi + \lambda \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\,\frac{\partial \varphi}{\partial x_i}\Big]\, dx\, dt = \int\limits_{D_T} (f(u_2) - f(u_1))\varphi\, dx\, dt \ \ \forall\, \varphi \in \overset{\circ}{W}{}^{1,2}_0(D_T). \tag{5.3}$$

Substituting $\varphi = v \in \overset{\circ}{W}{}^{1,2}_0(D_T)$ to equality (5.3), in view of (2.4), we obtain

$$\|v\|_1^2 = \int\limits_{D_T} (f(u_2) - f(u_1))(u_2 - u_1)\, dx\, dt. \tag{5.4}$$

From (2.5), (5.1) and (5.4), it follows that $c_1^2\|v\|_0^2 \leq \|v\|_1^2 \leq 0$, whence we find that $v = 0$, i.e., $u_2 = u_1$, and therefore, Theorem 5.1 is proved. □

From Theorems 3.1 and 5.1 follows

Theorem 5.2. *Let conditions (1.7), (1.8), (2.1), (3.1) and (5.1) be fulfilled. Then for any $F \in L_2(D_T)$, problem (1.1)–(1.3) has a unique weak generalized solution in the space $\overset{\circ}{W}{}^{1,2}_0(D_T)$ in the sense of Definition 1.1.*

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