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FRACTIONAL q-DIFFERENCE EQUATIONS WITH INTEGRAL AND ANTI-PERIODIC CONDITIONS

Abstract. In this paper, we investigate the existence of solutions for a class of fractional q-difference equations with integral and anti-periodic conditions involving the Caputo fractional q-derivative of order $\alpha \in]0,1]$. Existence results are obtained using the Mönch fixed point theorem and the technique of measures of noncompactness.

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1 Introduction

In recent years, fractional calculus has attracted considerable attention within the scientific community, establishing itself as a highly active and rapidly developing field of research. This heightened interest stems, both from its deep theoretical foundations and its extensive applicability across a broad spectrum of disciplines. In particular, fractional calculus has demonstrated remarkable utility in the development and analysis of mathematical models arising in technical sciences, physics, engineering, biophysics, and biomathematics. For a comprehensive treatment of the subject, the reader is referred to [13, 15, 16, 18, 21].

At the beginning of the twentieth century, Jackson initiated the development of quantum calculus, commonly known as q-difference calculus, by introducing the notion of the q-integral along with several other foundational constructs of the theory. His pioneering work provided the basis for subsequent developments in the field. For an in-depth exposition of quantum calculus, the reader is referred to [10, 14].

In the late 1960s, fractional q-difference calculus emerged as a natural extension of classical q-difference calculus, marking the inception of a new branch of analysis. This theoretical advancement is chiefly credited to the foundational contributions of Al-Salam [6] and Agarwal [2]. Since its introduction, fractional q-difference calculus has garnered considerable attention within the academic community due to its wide-ranging applicability in the modeling and analyzing complex phenomena across numerous scientific disciplines.

In recent years, the study of fractional q-difference equations involving the Caputo fractional q-derivative has attracted substantial scholarly interest. A variety of fixed point theorems have been employed by numerous researchers to establish existence and uniqueness results, leading to a number of important developments in this field. Noteworthy contributions in this context include the works of Abbas et al. [1] and Ahmad et al. [4].

In [8], M. Benchohra et al. studied the existence of solutions to the following boundary value problem:

$$\begin{gathered} {}^cD^{\alpha}u(t) = f(t, u(t)), \quad t \in I = [0, T], \quad 0 < \alpha < 1, \\ au(0) + bu(T) = c, \end{gathered}$$

where $T > 0$, ${}^cD^{\alpha}$ denotes the Caputo fractional derivative, $f \in C(I \times \mathbb{R}; \mathbb{R})$ and $a, b, c \in \mathbb{R}$ such that $a + b \neq 0$.

In [5], N. Allouch et al. applied some standard fixed point theorems and investigated the existence of solutions of fractional q-difference equations of the type

$$\begin{gathered} {}^cD_q^{\alpha}u(t) = f(t, u(t)), \quad t \in I = [0, T], \quad 0 < \alpha \leq 1, \\ au(0) + bu(T) = c, \end{gathered}$$

where $T > 0$, $q \in]0, 1[$, ${}^cD_q^{\alpha}$ denotes the Caputo fractional q-derivative, $f \in C(I \times \mathbb{R}; \mathbb{R})$ and $a, b, c \in \mathbb{R}$ such that $a + b \neq 0$.

In this paper, motivated by the works of W. Benhamida et al. [9], we establish the existence of solutions to the fractional q-difference equations of the type

$$ {}^cD_q^{\alpha}u(t) = f(t, u(t)), \quad t \in J = [0, T], \quad 0 < \alpha \leq 1, \tag{1.1}$$

$$ u(T) + u(0) = b \int_0^T u(s)\, ds, \quad bT \neq 2, \tag{1.2}$$

where ${}^cD_q^{\alpha}$ denotes the Caputo fractional q-derivative of order α, $(E, |\cdot|)$ is a Banach space, $f \in C(J \times E; E)$ and b is a real constant. The existence result is based on the Mönch's fixed point theorem.

The paper is organized as follows. In Section 2, we introduce some notations and definitions and recall preliminary facts of the fractional q-calculus which will be used throughout this paper. In Section 3, Mönch's fixed point theorem is employed to demonstrate the existence of solutions to problem (1.1), (1.2). In Section 4, we present an example to illustrate the applications of our main results.

2 Preliminaries

In this section, we will present some basic definitions and preliminary results of the fractional integrals and fractional derivatives which will be used throughout this paper.

For a fixed $T>0$, we denote $J:=[0,T]$. We consider the Banach space $C(J;\mathbb{R})$, consisting of all continuous functions from J into $\mathbb{R}$, equipped with the norm

$$\|u\|_\infty = \sup_{t\in J}|u(t)|.$$

Let $L^1(J;E)$ be the Banach space of Bochner integrable functions $u: J\to E$, equipped with the norm

$$\|u\|_{L^1} = \int_J |u(t)|\,dt,$$

and let $L^\infty(J;E)$ denote the Banach space of bounded measurable functions $u: J\to E$, with the norm

$$\|u\|_{L^\infty} = \inf\big\{c>0:\ \|u(t)\|\le c \text{ for a.e. } t\in J\big\}.$$

Now, we recall the fundamental definitions and some properties of the fractional q-calculus. For more details, see [10, 14].

We assume that $q\in\,]0,1[\,$. For all $a\in\mathbb{R}$, we set

$$[a]_q = \frac{1-q^a}{1-q}\,.$$

Let $a,b\in\mathbb{R}$. The q-analogue of $(a-b)^{(n)}$ is defined by

$$(a-b)^{(n)} = \begin{cases} 1 & \text{if } n=0,\\ \prod_{i=0}^{n-1}(a-bq^i) & \text{if } n\in\mathbb{N}^*. \end{cases}$$

If $\beta\in\mathbb{R}$, we have

$$(a-b)^{(\beta)} = a^\beta\prod_{i=0}^{\infty}\Big(\frac{a-bq^i}{a-bq^{i+\beta}}\Big),\quad a,b\in\mathbb{R}.$$

Note that if $b=0$, then $a^{(\beta)}=a^\beta$.

Definition 2.1 ([14])**.** The q-gamma function is defined by

$$\Gamma_q(\beta) = \frac{(1-q)^{(\beta-1)}}{(1-q)^{\beta-1}}\,,\quad \beta>0.$$

Notice that the q-gamma function satisfies $\Gamma_q(\beta+1)=[\beta]_q\Gamma_q(\beta)$.

Definition 2.2 ([14])**.** Let $f: J\to\mathbb{R}$. The q-derivative of order $n\in\mathbb{N}$ is defined by

$$(D_q^0 f)(t) = f(t),$$
$$(D_q^1 f)(t) = \frac{f(t)-f(qt)}{(1-q)t}\,,\quad t\neq 0,$$

and

$$(D_q^n f)(t) = (D_q^1 D_q^{n-1} f)(t),\quad t\in J,\quad n\in\mathbb{N}^*.$$

Definition 2.3 ([14])**.** Set $J_t=\{tq^n:\ n\in\mathbb{N}\}\cup\{0\}$. The q-integral of $f: J_t\to\mathbb{R}$ is given by

$$(I_q f)(t) = \int_0^t f(s)\,d_qs = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

under the assumption that the series converges.

It can be observed that $(D_qI_qf)(t) = f(t)$. Moreover, if f is continuous at 0, then

$$(I_qD_qf)(t) = f(t) - f(0).$$

Definition 2.4 ([2])**.** Given a function $f : J \to \mathbb{R}$, the Riemann–Liouville fractional q-integral of order $\alpha \geqslant 0$ is defined by

$$(I_q^\alpha f)(t) = \begin{cases} f(t) & \text{if } \alpha = 0, \\ \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)\, d_qs & \text{if } \alpha > 0. \end{cases}$$

Observe that when $\alpha = 1$, we have $(I_q^1 f)(t) = (I_qf)(t)$.

Lemma 2.1 ([20])**.** *For every $\alpha \geq 0$ and $\beta \in]-1, +\infty[$, we have*

$$(I_q^\alpha (t-a)^{(\beta)})(t) = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}\,(t-a)^{(\alpha+\beta)},\ \ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(\alpha+1)}\, t^{(\alpha)}\,.$$

Definition 2.5 ([19])**.** The Riemann–Liouville fractional q-derivative of order $\alpha \geq 0$ for a function $f : J \to \mathbb{R}$ is defined as follows:

$$(D_q^0 f)(t) = f(t) \ \text{ and } \ (D_q^\alpha f)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(t),\ \ t \in J,$$

where $[\alpha]$ denotes the integer part of α.

Definition 2.6 ([19])**.** Let $f : J \to \mathbb{R}$ and $\alpha \geq 0$. The Caputo fractional q-derivative of order α is defined by

$$(D_q^0 f)(t) = f(t) \ \text{ and } \ ({}^cD_q^\alpha f)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(t),\ \ t \in J,$$

where $[\alpha]$ denotes the integer part of α.

Lemma 2.2 ([19])**.** *Let $\alpha, \beta \geq 0$, and let $f : J \to \mathbb{R}$ be a given function. Then the following identities are satisfied:*

(i) $(I_q^\alpha I_q^\beta f)(t) = (I_q^{\alpha+\beta} f)(t)$;

(ii) $(D_q^\alpha I_q^\alpha f)(t) = f(t)$.

Lemma 2.3 ([19])**.** *Let $\alpha \geq 0$, and let f be a function defined on the interval J. Then the following equality is satisfied:*

$$(I_q^\alpha\, {}^cD_q^\alpha f)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(k+1)}\,(D_q^k f)(0).$$

For $\alpha \in (0,1)$, we have

$$(I_q^\alpha\, {}^cD_q^\alpha f)(t) = f(t) - f(0).$$

We now recall the definition of the Kuratowski measure of noncompactness and present a brief summary of some of its fundamental properties.

Definition 2.7 ([7])**.** Let E be a Banach space, and denote by Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the mapping $\mu : \Omega_E \to [0,\infty)$, defined for each $B \in \Omega_E$, as follows:

$$\mu(B) = \inf\Big\{\varepsilon > 0 :\ B \subset \bigcup_{i=1}^m B_i \text{ with } \operatorname{diam}(B_i) \leq \varepsilon \text{ for all } i\Big\}.$$

Properties ([7]). *The essential properties of the Kuratowski measure of noncompactness are listed below:*

1. $\mu(B) = 0$ *if and only if* B *is relatively compact.*
2. $\mu(B) = \mu(\overline{B})$, *where* $\overline{B}$ *denotes the closure of* B.
3. *If* $A \subseteq B$, *then* $\mu(A) \leq \mu(B)$.
4. $\mu(A + B) \leq \mu(A) + \mu(B)$.
5. $\mu(cB) = |c|\mu(B)$, *for any scalar* $c \in \mathbb{R}$.
6. $\mu(\mathrm{conv}(B)) = \mu(B)$, *where* $\mathrm{conv}(B)$ *is the convex hull of* B.
7. $\mu(B + x_0) = \mu(B)$, *for all* $x_0 \in E$.

Definition 2.8. A mapping $f : J \times E \to E$ is said to be of Carathéodory if it satisfies the following conditions:

1. For each fixed $u \in E$, the mapping $t \mapsto f(t, u)$ is measurable on J.
2. For almost every $t \in J$, the mapping $u \mapsto f(t, u)$ is continuous on E.

Let V be a given set of functions $v : J \to E$. For each $t \in J$, we define

$$V(t) = \{v(t) : \ v \in V, \ t \in J\},$$
$$V(J) = \{v(t) : \ v \in V, \ t \in J\}.$$

Mönch's fixed point theorem is stated as follows.

Theorem 2.1 ([3, 17]). *Let* D *be a bounded, closed, and convex subset of a Banach space* E, *with* $0 \in D$, *and let* $N : D \to D$ *be a continuous mapping. Suppose that for every subset* $V \subset D$, *the implication*

$$V = \overline{\mathrm{conv}} N(V) \ \text{ or } \ V = N(V) \cup \{0\} \implies \mu(V) = 0$$

holds, then N *has a fixed point in* D.

Lemma 2.4 ([12]). *Let* $V \subset C(J; E)$ *be a bounded and equicontinuous subset. Then:*

1. *The function* $t \mapsto \mu(V(t))$ *is continuous on* J.
2. *The following inequality holds:*

$$\mu\left(\left\{ \int_J y(t)\, dt : \ y \in V \right\}\right) \leq \int_J \mu(V(t))\, dt.$$

3 Results

This section deals with the existence of solutions for the fractional problem (1.1), (1.2).

Definition 3.1. A function $u \in C(J; E)$ is said to be a solution of the fractional problem (1.1), (1.2) if u satisfies the equation ${}^cD_q^\alpha u(t) = f(t, u(t))$ on J, and conditions (1.2).

In order to obtain the existence of solutions for the fractional problem (1.1), (1.2), we need the following lemma.

Lemma 3.1. *Let $0 < \alpha \leq 1$, $bT \neq 2$, and let $h : J \to E$ be a continuous function. The solution of the fractional q-difference problem*

$$^{c}D_q^{\alpha} u(t) = h(t), \quad t \in J = [0, T], \quad 0 < \alpha \leq 1, \tag{3.1}$$

$$u(T) + u(0) = b \int_0^T u(s)\, ds, \quad bT \neq 2, \tag{3.2}$$

is given by

$$u(t) = \int_0^T G(t, s) h(s)\, d_q s, \tag{3.3}$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \dfrac{b(T-qs)^{(\alpha)}}{(2-Tb)\Gamma_q(\alpha+1)} - \dfrac{(T-qs)^{(\alpha-1)}}{(2-Tb)\Gamma_q(\alpha)}, & 0 \leq s \leq t \leq T, \\ \dfrac{b(T-qs)^{(\alpha)}}{(2-Tb)\Gamma_q(\alpha+1)} - \dfrac{(T-qs)^{(\alpha-1)}}{(2-Tb)\Gamma_q(\alpha)}, & 0 \leq t \leq s \leq T. \end{cases} \tag{3.4}$$

Proof. Let us apply the Riemann–Liouville fractional q-integral of order α to both sides of equation (3.1) and, using Lemma 2.3, we have

$$u(t) = c_0 + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_q s.$$

Using conditions (3.2), we obtain

$$c_0 = \frac{b}{(2-Tb)} \int_0^T \frac{(T-qs)^{(\alpha)}}{\Gamma_q(\alpha+1)} h(s)\, d_q s - \frac{1}{(2-bT)} \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\, d_q s.$$

As a result, equation (3.3) is obtained, with the function G defined in equation (3.4). □

We proceed to establish an existence result for the boundary value problem (1.1), (1.2) by employing Mönch's fixed point theorem.

Theorem 3.1. *Let us assume that the following hypotheses are satisfied:*

(H1) *The function $f : J \times E \to E$ verifies the Carathéodory conditions.*

(H2) *There exists $p \in L^1(J, \mathbb{R}^+)$ such that for every $t \in J$ and for all $u \in E$, the following inequality holds:*

$$\|f(t,u)\| \leq p(t)\|u\|,$$

(H3) *For every $t \in J$ and for every bounded set $B \subset E$, the following inequality holds:*

$$\mu(f(t,B)) \leq p(t)\mu(B),$$

Then the fractional problem (1.1), (1.2) has at least one solution in the space $C(J;B)$, provided

$$\|I_q^{\alpha}(p)\|_{L^1} + \frac{|b|(I_q^{\alpha+1}p)(T)}{|2-Tb|} + \frac{(I_q^{\alpha}p)(T)}{|2-Tb|} < 1. \tag{3.5}$$

Proof. Transform the fractional problem (1.1), (1.2) into a fixed point problem. We consider the operator $N : C(J;E) \to C(J;E)$, where

$$N(u) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s))\, d_q s + \int_0^T \frac{b(T-qs)^{(\alpha)}}{(2-Tb)\Gamma_q(\alpha+1)} f(s,u(s))\, d_q s - \int_0^T \frac{(T-qs)^{(\alpha-1)}}{(2-Tb)\Gamma_q(\alpha)} f(s,u(s))\, d_q s.$$

It is clear that the fixed points of the operator N are solutions of the fractional problem (1.1), (1.2).

Let $R > 0$ and we consider

$$D_R = \big\{ u \in C(J;E) : \ \|u\|_\infty \le R \big\}.$$

It is evident that D_R is a closed, bounded, and convex subset of the Banach space $C(J;E)$. We now proceed to demonstrate that N satisfies the assumptions of Mönch's fixed point theorem.

The proof is structured in three steps.

Step 1: N is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J;E)$. Then, for each $t \in J$,

$$|(Nu_n)(t) - (Nu)(t)| \le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u_n(s)) - f(s,u(s))|\, d_q s + \int_0^T \frac{|b|(T-qs)^{(\alpha)}}{|2-Tb|\Gamma_q(\alpha+1)} |f(s,u_n(s)) - f(s,u(s))|\, d_q s + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{|2-Tb|\Gamma_q(\alpha)} |f(s,u_n(s)) - f(s,u(s))|\, d_q s.$$

Let $\rho > 0$ be a fixed constant such that

$$\|u_n\|_\infty \le \rho \ \text{ and } \ \|u\|_\infty \le \rho.$$

Then, by assumption (H2), it follows that

$$\|f(s,u_n(s)) - f(s,u(s))\| \le 2\rho p(s) := \sigma(s),$$

where $\sigma(s) \in L^1(J, \mathbb{R}^+)$. Since the function f satisfies the Carathéodory conditions, it follows from the Lebesgue Dominated Convergence Theorem that

$$\|N(u_n) - N(u)\|_\infty \to 0 \ \text{ as } \ n \to \infty.$$

Therefore, N is continuous on $C(J;E)$.

Step 2: N maps the set D_R into itself. Moreover, for any $u \in D_R$, it follows from condition (H2) and equation (3.5) that, for each $t \in J$,

$$\begin{aligned}|N(u)(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s))|\, d_q s \\ &\quad + \int_0^T \frac{|b|(T-qs)^{(\alpha)}}{|2-Tb|\Gamma_q(\alpha+1)} |f(s,u(s))|\, d_q s + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{|2-Tb|\Gamma_q(\alpha)} |f(s,u(s))|\, d_q s \\ &\le R\Big(\|I_q^\alpha(p)\|_{L^1} + \frac{|b|(I_q^{\alpha+1}p)(T)}{|2-Tb|} + \frac{(I_q^\alpha p)(T)}{|2-Tb|} \Big) \\ &\le R.\end{aligned}$$

Step 3: The set $N(D_R)$ is bounded and equicontinuous. From the result established in Step 2, we deduce that $N(D_R)$ is bounded. In order to demonstrate the equicontinuity of $N(D_R)$, we consider $t_1, t_2 \in J$ such that $t_1 < t_2$, and let $y \in D_R$. Then

$$
\begin{aligned}
|(Nu)(t_2) - (Nu)(t_1)| &\leq \frac{1}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}\right] |f(s, u(s))| \, d_q s \\
&\quad + \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)} |f(s, u(s))| \, d_q s \\
&\leq \frac{R}{\Gamma_q(\alpha)} \Bigg[\int_0^{t_1} \big((t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}\big) p(s) \, d_q s \\
&\quad + \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)} p(s) \, d_q s \Bigg].
\end{aligned}
$$

Since the right-hand side of the inequality tends to zero as $t_1 \to t_2$, it follows that the set $N(D_R)$ is equicontinuous.

Assume that $V \subset D_R$ satisfies $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$. The boundedness and equicontinuity of V ensure that the function $t \to v(t) = \mu(V(t))$ is continuous on J. Moreover, applying condition (H3), Lemma 2.4, and the properties of the measure μ, for each $t \in J$, we obtain

$$
v(t) \leq \mu(N(V)(t) \cup \{0\}) \leq \mu(N(V)(t)) \leq \mu(V(t)) \Big((I_q^{\alpha} p)(T) + \frac{|b|(I_q^{\alpha+1} p)(T)}{|2 - Tb|} + \frac{(I_q^{\alpha} p)(T)}{|2 - Tb|} \Big).
$$

Hence,

$$
\|v\|_{\infty} \left(1 - \left[\|I_q^{\alpha}(p)\|_{L^1} + \frac{|b|(I_q^{\alpha+1} p)(T)}{|2 - Tb|} + \frac{(I_q^{\alpha} p)(T)}{|2 - Tb|} \right] \right) \leq 0.
$$

From equation (3.5), we deduce that $\|v\|_{\infty} = 0$, and thus $v(t) = 0$ for every $t \in J$. As a consequence, the set $V(t)$ is relatively compact in E. It follows from the Ascoli–Arzelà theorem that V is relatively compact in D_R. Applying Theorem 2.1 ensures that N has a fixed point, which, in turn, provides a solution of problem (1.1), (1.2). □

4 An example

In this section, we present an example to illustrate our results.

Consider the fractional differential equation with the integral boundary conditions:

$$
{}^c D_{\frac{1}{3}}^{\frac{1}{2}} u(t) = \frac{t\sqrt{\pi}}{10} u(t), \quad \text{for a.e. } t \in J = [0, 1], \quad u \in \mathbb{R}^+, \tag{4.1}
$$

$$
u(1) + u(0) = \int_0^1 u(s) \, ds. \tag{4.2}
$$

Here, we take $\alpha = \frac{3}{2}$, $q = \frac{1}{3}$, $b = 1$, and define

$$
f(t, u) = \frac{t\sqrt{\pi}}{10} u(t).
$$

It can be easily verified that the function f satisfies hypotheses (H1)–(H3) of Theorem 3.1, with

$$
p(t) = \frac{t\sqrt{\pi}}{10} \in L^1([0, 1]; \mathbb{R}^+).
$$

It remains to verify condition (3.5) for this choice of p and the given boundary condition. A direct computation yields

$$\left(\|I_q^{\alpha}(p)\|_{L^1} + \frac{|b|(I_q^{\alpha+1}p)(T)}{|2-Tb|} + \frac{(I_q^{\alpha}p)(T)}{|2-Tb|}\right) = (I_{\frac{1}{3}}^{1/2}p)(1) + (I_{\frac{1}{3}}^{3/2}p)(1) + (I_{\frac{1}{3}}^{1/2}p)(1) = 0.2641 < 1.$$

Consequently, by Theorem 3.1, the fractional q-difference problem (4.1), (4.2) admits a solution on the interval $[0,1]$.

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References

[1] S. Abbas, M. Benchohra and J. Henderson, Existence and oscillation for coupled fractional q-difference systems. *J. Fract. Calc. Appl.* **12** (2021), no. 1, 143–155.

[2] R. P. Agarwal, Certain fractional q-integrals and q-derivatives. *Proc. Cambridge Philos. Soc.* **66** (1969), 365–370.

[3] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications.* Cambridge Tracts in Mathematics, 141. Cambridge University Press, Cambridge, 2001.

[4] B. Ahmad, S. K. Ntouyas and I. K. Purnaras, Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations. *Adv. Difference Equ.* **2012**, 2012:140, 15 pp.

[5] N. Allouch, S. Hamani and J. Henderson, Boundary value problem for fractional q-difference equations. *Nonlinear Dyn. Syst. Theory* **24** (2024), no. 2, 111–122.

[6] W. A. Al-Salam, Some fractional q-integrals and q-derivatives. *Proc. Edinburgh Math. Soc. (2)* **15** (1966/67), 135–140.

[7] J. Banaś, On measures of noncompactness in Banach spaces. *Comment. Math. Univ. Carolin.* **21** (1980), no. 1, 131–143.

[8] M. Benchohra, S. Hamani and S. K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions. *Nonlinear Anal.* **71** (2009), no. 7-8, 2391–2396.

[9] W. Benhamida, J. R. Graef and S. Hamani, Boundary value problems for fractional differential equations with integral and anti-periodic conditions in a Banach space. *Progr. Fract. Differ. Appl.* **4** (2018), no. 2, 65–70.

[10] G. Gasper and M. Rahman, *Basic Hypergeometric Series.* Encyclopedia of Mathematics and its Applications, 35. Cambridge University Press, Cambridge, 1990.

[11] A. Granas and J. Dugundji, *Fixed Point Theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[12] D. Guo, V. Lakshmikantham and X. Liu, *Nonlinear Integral Equations in Abstract Spaces.* Mathematics and its Applications, 373. Kluwer Academic Publishers Group, Dordrecht, 1996.

[13] R. Hilfer, *Applications of Fractional Calculus in Physics.* World Scientific, Singapore–New Jersey–London–Hong Kong, 2000.

[14] V. Kac and P. Cheung, *Quantum Calculus.* Universitext. Springer-Verlag, New York, 2002.

[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[16] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1993.

[17] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Anal.* **4** (1980), no. 5, 985–999.

[18] I. Podlubny, *Fractional differential equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications.* Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[19] P. M. Rajković, S. D. Marinković and M. S. Stanković, On q-analogues of Caputo derivative and Mittag–Leffler function. *Fract. Calc. Appl. Anal.* **10** (2007), no. 4, 359–373.

[20] P. M. Rajković, S. D. Marinković and M. S. Stanković, Fractional integrals and derivatives in q-calculus. *Appl. Anal. Discrete Math.* **1** (2007), no. 1, 311–323.

[21] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* Translated from the 1987 Russian original. Gordon and Breach Science Publishers, Yverdon, 1993.

[22] D. R. Smart, *Fixed Point Theorems.* Cambridge University Press, Cambridge, 1980.

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