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EXISTENCE OF POSITIVE SOLUTIONS TO PERTURBED SEMILINEAR STRONGLY DEGENERATE ELLIPTIC PROBLEMS INVOLVING CRITICAL GROWTH

Abstract. In this article, we consider the following perturbed semilinear equations involving strongly degenerate elliptic problem with critical growth:

$$\begin{gathered}
-\varepsilon^2\Delta_{\alpha_1,\beta_1}^{\alpha,\beta} u + V(X)u = f(X)|u|^{p-2}u + \frac{a}{a+b}\,K(X)|u|^{a-2}u|v|^b, \quad X \in \mathbb{R}^N, \\
-\varepsilon^2\Delta_{\alpha_1,\beta_1}^{\alpha,\beta} v + V(X)v = g(X)|v|^{p-2}v + \frac{b}{a+b}\,K(X)|u|^{a}|v|^{b-2}v, \quad X \in \mathbb{R}^N, \\
u(X),\ v(X) \to 0 \ \text{ as } \ |X| \to \infty,
\end{gathered}$$

where $\Delta_{\alpha_1,\beta_1}^{\alpha,\beta}$ is the subelliptic operator of the type

$$\begin{gathered}
\Delta_{\alpha_1,\beta_1}^{\alpha,\beta} := \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\left(|x|^{\alpha_1} + |y|^{\beta_1}\right)^2\Delta_z, \quad x \in \mathbb{R}^{N_1}, \quad y \in \mathbb{R}^{N_2}, \quad z \in \mathbb{R}^{N_3}, \\
N = N_1 + N_2 + N_3, \quad \alpha, \beta, \alpha_1, \beta_1 > 0, \quad X = (x, y, z).
\end{gathered}$$

Using variational methods, we prove the existence of positive solutions.

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1 Introduction

In this article, we discuss the following perturbed degenerate elliptic system involving critical growth:

$$\begin{aligned} -\varepsilon^2\Delta^{\alpha,\beta}_{\alpha_1,\beta_1}u + V(X)u &= f(X)|u|^{p-2}u + \frac{a}{a+b}K(X)|u|^{a-2}u|v|^b, \quad X\in\mathbb{R}^N,\\ -\varepsilon^2\Delta^{\alpha,\beta}_{\alpha_1,\beta_1}v + V(X)v &= g(X)|v|^{p-2}v + \frac{b}{a+b}K(X)|u|^{a}|v|^{b-2}v, \quad X\in\mathbb{R}^N,\\ u(X),\ v(X)&\to 0 \ \text{ as } \ |X|\to\infty, \end{aligned} \tag{1.1}$$

where $2<p<\widetilde{2}^*$, $a>1$, $b>1$ satisfy $a+b=\widetilde{2}^*$, $\widetilde{2}^*=2\widetilde{N}/(\widetilde{N}-2)(\widetilde{N}>2)$, $\widetilde{N}:=N_1+N_2+N_3(1+\alpha+\alpha_1+\beta+\beta_2)$ and $\Delta^{\alpha,\beta}_{\alpha_1,\beta_1}$ is the subelliptic operator of the type

$$\begin{aligned} \Delta^{\alpha,\beta}_{\alpha_1,\beta_1} &:= \Delta_x+\Delta_y+|x|^{2\alpha}|y|^{2\beta}\big(|x|^{\alpha_1}+|y|^{\beta_1}\big)^2\Delta_z, \quad x\in\mathbb{R}^{N_1},\ \ y\in\mathbb{R}^{N_2},\ \ z\in\mathbb{R}^{N_3},\\ N&=N_1+N_2+N_3, \quad \alpha,\beta,\alpha_1,\beta_1\ge 0, \quad X=(x,y,z). \end{aligned}$$

We assume that $V(X)$, $K(X)$, $f(X)$ and $g(X)$ satisfy the following conditions:

(A1) $V\in C(\mathbb{R}^N,\mathbb{R})$ satisfies $V(0)=\inf\limits_{X\in\mathbb{R}^N}V(X)=0$, and for any $M>0$,

$$\operatorname{Vol}\Big(\big\{X\in\mathbb{R}^N, V(X)\le M\big\}\Big)<\infty;$$

(A2) $K(X)\in C(\mathbb{R}^N,\mathbb{R})$,

$$0<\inf_{X\in\mathbb{R}^N}K(X)\le\sup_{X\in\mathbb{R}^N}K(X)<\infty;$$

(A3) $f(X)$, $g(X)$ are positive functions and

$$0<f_0=\inf_{X\in\mathbb{R}^N}f(X)\le\sup_{X\in\mathbb{R}^N}f(X)<\infty, \quad 0<g_0=\inf_{X\in\mathbb{R}^N}g(X)\le\sup_{X\in\mathbb{R}^N}g(X)<\infty.$$

Let $\lambda=\varepsilon^{-2}$. Then problem (1.1) can be rewritten as

$$\begin{aligned} -\Delta^{\alpha,\beta}_{\alpha_1,\beta_1}u + \lambda V(X)u &= \lambda f(X)|u|^{p-2}u + \frac{\lambda a}{a+b}K(X)|u|^{a-2}u|v|^b, \quad X\in\mathbb{R}^N,\\ -\Delta^{\alpha,\beta}_{\alpha_1,\beta_1}v + \lambda V(X)v &= \lambda g(X)|v|^{p-2}v + \frac{\lambda b}{a+b}K(X)|u|^{a}|v|^{b-2}v, \quad X\in\mathbb{R}^N,\\ u(X),\ v(X)&\to 0 \ \text{ as } \ |X|\to\infty. \end{aligned} \tag{1.2}$$

Since problem (1.1) and problem (1.2) are equivalent, we focus on system (1.2).

Theorem 1.1. *Assume* (A1)–(A3) *hold. Then for any $\sigma>0$, there is $\Lambda_\sigma>0$ such that if $\lambda>\Lambda_\sigma$, problem* (1.2) *has at least one positive solution (u_λ,v_λ) that satisfies*

$$\frac{p-2}{2p}\int\limits_{\mathbb{R}^N}\Big(|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}u_\lambda|^2+|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}v_\lambda|^2+\lambda V(X)\big(|u_\lambda|^2+|v_\lambda|^2\big)\Big)\,\mathrm{d}X\le\sigma\lambda^{1-\frac{\widetilde{N}}{2}},$$

where

$$\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}u:=\Big(\nabla_x u,\nabla_y u,|x|^\alpha|y|^\beta\big(|x|^{\alpha_1}+|y|^{\beta_1}\big)\nabla_z u\Big), \quad \mathrm{d}X:=\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Set $\alpha=\beta=\alpha_1=\beta_1=0$, $a=b$, $f(X)=g(X)$ and $u=v$. Then problem (1.1) can be rewritten as

$$\begin{aligned} -\varepsilon^2\Delta u+V(x)u &= f(x)|u|^{p-2}u+\frac{1}{2}K(x)|u|^{2^*-2}u, \quad x\in\mathbb{R}^N,\\ u(x)&\to 0 \ \text{ as } \ |x|\to\infty. \end{aligned} \tag{1.3}$$

Many studies on problem (1.3) can be found in the literature [1, 4, 5, 7–11, 16].

In the last years, many authors have studied (see [3, 13, 15, 17] and the references therein) the following semilinear degenerate elliptic equation in $\mathbb{R}^N$:

$$-\Delta_\gamma u + V(x)u := -\sum_{i=1}^{N} \partial_{x_i}(\gamma_i^2 \partial_{x_i} u) + V(x)u = f(x,u),$$

where the functions $\gamma_i : \mathbb{R}^N \to \mathbb{R}$, $\gamma_i \in C^1(\mathbb{R}^N)$ and $\gamma_i \neq 0$ in $\mathbb{R}^N \setminus \Pi$ for all $i = 1, 2, \dots, N$ (see [12]),

$$\Pi := \Big\{ x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : \ \prod_{i=1}^{N} x_i = 0 \Big\},$$

and γ_i are such that

(i) there exist a semigroup of dilations $\{\delta_t\}_{t>0}$,

$$\delta_t : \mathbb{R}^N \to \mathbb{R}^N,$$
$$(x_1, \dots, x_N) \mapsto \delta_t(x_1, \dots, x_N) := (t^{\varepsilon_1} x_1, \dots, t^{\varepsilon_N} x_N),$$

and the constants $1 = \varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_N$ such that γ_i is δ_t-homogeneous of degree $\varepsilon_i - 1$, i.e.,

$$\gamma_i(\delta_t(x)) = t^{\varepsilon_j - 1}\gamma_i(x) \ \text{ for all } \ x \in \mathbb{R}^N, \ \ t > 0, \ \ i = 1, \dots, N;$$

(ii) $\gamma_1(x) \equiv 1$ and for any $i = 2, \dots, N$, the functions $\gamma_i(x)$ depend on $x_1, x_2, \dots, x_{i-1}$;

(iii) there exists a constant $\rho \geq 0$ such that

$$0 \leq x_k \partial_{x_k} \gamma_i(x) \leq \rho \gamma_i(x) \ \text{ for all } \ k \in \{1, 2, \dots, i-1\}, \ \ i = 2, \dots, N,$$

and for every $x \in \overline{\mathbb{R}}^N_+$, where $\overline{\mathbb{R}}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : \ x_i \geq 0, \ \forall\, i = 1, 2, \dots, N\}$;

(iv) the equalities $\gamma_i(x) = \gamma_i(x^*)$ $(i = 1, 2, \dots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where

$$x^* = \big(|x_1|, \dots, |x_N|\big) \ \text{ if } \ x = (x_1, x_2, \dots, x_N).$$

The operator $\Delta^{\alpha,\beta}_{\alpha_1,\beta_1}$ for

$$\gamma = \Big(\underbrace{1, 1, \dots, 1}_{N_1 + N_2\text{-times}}, \underbrace{|x|^\alpha |y|^\beta \big(|x|^{\alpha_1} + |y|^{\beta_1}\big)}_{N_3\text{-times}} \Big),$$

does not satisfy condition (i). Moreover, to the known of our knowledge, no studies were conducted on the existence of semiclassical solutions to problem (1.1) in $\mathbb{R}^N$. In this paper, we study system (1.1) in the whole space involving the critical growth. The main difficulty of this problem is the lack of compactness of the Sobolev embedding.

The structure of our paper is as follows. In Section 2, we prove some embedding theorems for the weighted Sobolev spaces associated with the operator and Palais–Smale condition. In Section 3, we prove the main result.

2 Embedding theorem and Mountain Pass Theorem

2.1 Embedding theorem

Definition 2.1. Let $S^p_{\alpha,\beta,\alpha_1,\beta_1}(\mathbb{R}^N)$ $(1 \leq p < +\infty)$ be the Sobolev space obtained as completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{S^p_{\alpha,\beta,\alpha_1,\beta_1}(\mathbb{R}^N)} = \bigg(\int\limits_{\mathbb{R}^N} \big(|u|^p + |\nabla^{\alpha,\beta}_{\alpha_1,\beta_1} u|^p\big) \, \mathrm{d}X \bigg)^{\frac{1}{p}}.$$

If $p=2$, we can also define the scalar product in $S^2_{\alpha,\beta,\alpha_1,\beta_1}(\mathbb{R}^N)$ as follows:

$$(u,v)_{S^2_{\alpha,\beta,\alpha_1,\beta_1}(\mathbb{R}^N)} = (u,v)_{L^2(\mathbb{R}^N)} + \big(\nabla^{\alpha,\beta}_{\alpha_1,\beta_1} u, \nabla^{\alpha,\beta}_{\alpha_1,\beta_1} v\big)_{L^2(\mathbb{R}^N)},$$

where

$$\nabla^{\alpha,\beta}_{\alpha_1,\beta_1} u := \Big(\nabla_x u, \nabla_y u, |x|^\alpha |y|^\beta \big(|x|^{\alpha_1}+|y|^{\beta_1}\big)\nabla_z u\Big).$$

Define

$$S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N) = \Big\{u \in S^2_{\alpha,\beta,\alpha_1,\beta_1}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} \big(|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1} u|^2 + \lambda V(X) u^2\big)\,\mathrm{d}X < +\infty\Big\}$$

with $V(X)$ satisfying condition (A1), then $S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)$ is a Hilbert space with the norm

$$\|u\|_{S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)} = \Big(\int\limits_{\mathbb{R}^N} \big(|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1} u|^2 + \lambda V(X) u^2\big)\,\mathrm{d}X\Big)^{\frac{1}{2}}.$$

By (A1), the embedding $S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N) \hookrightarrow S^2_{\alpha,\beta,\alpha_1,\beta_1}(\mathbb{R}^N)$ is continuous. From an embedding inequality in [2] and Hölder's inequality, we have

$$S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \ \text{ for } \ 2 \le q \le \widetilde{2}^*.$$

Moreover, we have

Lemma 2.1. *Let* (A1) *be satisfied. Then the embedding map from* $S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)$ *into* $L^q(\mathbb{R}^N)$ *is compact for* $2 \le q < \widetilde{2}^*$.

Proof. Let $\{u_n\}_{n=1}^\infty \subset S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)$ be a bounded sequence such that $u_n \rightharpoonup u$ weakly in $S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)$. Then, by the Sobolev embedding theorem, $u_n \to u$ strongly in $L^p_{loc}(\mathbb{R}^N)$ for $2 \le q < \widetilde{2}^*$. We claim that

$$u_n \to u \ \text{ strongly in } \ L^2(\mathbb{R}^N). \tag{2.1}$$

To prove (2.1), we only need to prove that $\nu_n := \|u_n\|^2_{L^2(\mathbb{R}^N)} \to \|u\|^2_{L^2(\mathbb{R}^N)}$, since the space $L^2(\mathbb{R}^N)$ is uniformly convex. Assume, up to a subsequence, that $\nu_n \to \nu$.

Put

$$\begin{aligned} B_R &:= \big\{X \in \mathbb{R}^N : \ |X| < R\big\}, \\ \mathbb{R}^N_{M,\lambda V(X),R} &:= \big\{X \in \mathbb{R}^N \setminus B_R : \ \lambda V(X) \ge M\big\}, \\ \mathscr{C}\mathbb{R}^N_{M,\lambda V(X),R} &:= \big\{X \in \mathbb{R}^N \setminus B_R : \ \lambda V(X) < M\big\}, \end{aligned}$$

then

$$\begin{aligned} \int\limits_{\mathbb{R}^N_{M,\lambda V(X),R}} |u_n|^2\,\mathrm{d}x &\le \int\limits_{\mathbb{R}^N_{M,\lambda V(X),R}} \frac{\lambda V(X)}{M}|u_n|^2\,\mathrm{d}X \\ &\le \frac{1}{M}\int\limits_{\mathbb{R}^N} \big(|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1} u_n|^2 + \lambda V(X) u_n^2\big)\,\mathrm{d}X \le \frac{\|u_n\|^2_{S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)}}{M}. \end{aligned}$$

Choose $\tau \in (1, \frac{\widetilde{N}}{\widetilde{N}-2})$ and τ' such that $\frac{1}{\tau}+\frac{1}{\tau'}=1$, then, applying Hölder's inequality, we have

$$\begin{aligned} \int\limits_{\mathscr{C}\mathbb{R}^N_{M,\lambda V(X),R}} |u_n|^2\,\mathrm{d}X &\le \Big(\int\limits_{\mathscr{C}\mathbb{R}^N_{M,\lambda V(X),R}} |u_n|^{2\tau}\Big)^{\frac{1}{\tau}} \big(\mathrm{Vol}(\mathscr{C}\mathbb{R}^N_{M,\lambda V(X),R})\big)^{\frac{1}{\tau'}} \\ &\le C\|u_n\|^2_{S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)} \big(\mathrm{Vol}(\mathscr{C}\mathbb{R}^N_{M,\lambda V(X),R})\big)^{\frac{1}{\tau'}}. \end{aligned}$$

Since $\{\|u_n\|_{S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)}\}_{n=1}^{\infty}$ is bounded and condition (A1) holds, we can choose R, M large enough such that the quantities $\frac{\|u_n\|^2_{S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)}}{M}$ and $(\mathrm{Vol}(\mathscr{C}\mathbb{R}^N_{M,\lambda V(X),R}))^{\frac{1}{\tau'}}$ are small enough. Hence, for all $\varepsilon > 0$, we have

$$\int\limits_{\mathbb{R}^N\setminus B_R} |u_n|^2 \,\mathrm{d}X = \int\limits_{\mathbb{R}^N_{M,\lambda V(X),R}} |u_n|^2 \,\mathrm{d}X + \int\limits_{\mathscr{C}\mathbb{R}^N_{M,\lambda V(X),R}} |u_n|^2 \,\mathrm{d}X < \varepsilon.$$

Thus

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^N)} &= \|u\|^2_{L^2(B_R)} + \|u\|^2_{L^2(\mathbb{R}^N\setminus B_R)}\\ &\ge \lim_{n\to\infty}\|u_n\|^2_{L^2(B_R)} = \lim_{n\to\infty}\left(\|u_n\|^2_{L^2(\mathbb{R}^N)} - \|u\|^2_{L^2(\mathbb{R}^N\setminus B_R)}\right) \ge \nu^2 - \varepsilon.\end{aligned}$$

On the other hand, let Ω be an arbitrary domain in $\mathbb{R}^N$, then

$$\int\limits_{\Omega} |u_n|^2 \,\mathrm{d}X \le \int\limits_{\mathbb{R}^N} |u_n|^2 \,\mathrm{d}X \to \nu^2,$$

hence $\|u\|_{L^2(\mathbb{R}^N)} \le \nu$. By the arbitrariness of ε, we have $\nu = \|u\|_{L^2(\mathbb{R}^N)}$. So, (2.1) is proved.

Finally, we prove that $u_n \to u$ in $L^q(\mathbb{R}^N)$ for $2 \le q < \widetilde{2}^*$. In fact, if $q \in (2, \widetilde{2}^*)$, there is a number $\theta \in (0,1)$ such that $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{\widetilde{2}^*}$. Then, by Hölder's inequality,

$$\|u_n - u\|^q_{L^q(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N} |u_n - u|^{\theta p}|u_n - u|^{(1-\theta)q} \,\mathrm{d}X \le \|u_n - u\|^{\theta q}_{L^2(\mathbb{R}^N)}\|u_n - u\|^{(1-\theta)q}_{L^{\widetilde{2}^*}(\mathbb{R}^N)}.$$

Since u_n is bounded in $L^{\widetilde{2}^*}(\mathbb{R}^N)$ and $\|u_n - u\|_{L^2(\mathbb{R}^N)} \to 0$, we have $u_n \to u$ in $L^q(\mathbb{R}^N)$. □

Let $\mathbb{H} = S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N) \times S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)$ be the Hilbert space with the norm

$$\|(u,v)\|_{\mathbb{H}} = \left(\int\limits_{\mathbb{R}^N}\left(|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}u|^2 + \lambda V(X)u^2 + |\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}v|^2 + \lambda V(X)v^2\right)\mathrm{d}X\right)^{\frac{1}{2}}$$

for any $(u,v) \in \mathbb{H}$. We will show the existence of nontrivial solutions of problem (1.2) by searching for critical points of the functional associated to problem (1.2),

$$\begin{aligned}\Phi(u,v) = \frac{1}{2}\int\limits_{\mathbb{R}^N}&\left(|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}u|^2 + \lambda V(X)u^2 + |\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}v|^2 + \lambda V(X)v^2\right)\mathrm{d}X\\ &- \frac{\lambda}{p}\int\limits_{\mathbb{R}^N}\left(f(X)|u|^p + g(X)|v|^p\right)\mathrm{d}X - \frac{\lambda}{a+b}\int\limits_{\mathbb{R}^N} K(X)|u|^a|v|^b \,\mathrm{d}X.\end{aligned}$$

In fact, the critical points of the functional Φ are the weak solutions of problem (1.2). Recall that the weak solution (u,v) of problem (1.2) satisfies

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\left(\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}u\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}\varphi + \lambda V(X)u\varphi + \nabla^{\alpha,\beta}_{\alpha_1,\beta_1}v\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}\psi + \lambda V(x)v\psi\right)\mathrm{d}X\\ &\quad= \lambda\int\limits_{\mathbb{R}^N}\left(f(X)|u|^{p-2}u\varphi + g(X)|v|^{p-2}v\psi\right)\mathrm{d}X\\ &\qquad+ \frac{\lambda a}{a+b}\int\limits_{\mathbb{R}^N} K(X)|u|^{a-2}u|v|^b\varphi \,\mathrm{d}X + \frac{\lambda a}{a+b}\int\limits_{\mathbb{R}^N} K(X)|u|^a|v|^{b-2}v\psi \,\mathrm{d}X\end{aligned}$$

for all $(\varphi, \psi) \in \mathbb{H}$. Based on the assumptions of Theorem 1.1, we can show that $\Phi \in C^1(\mathbb{H}, \mathbb{R})$ (see [13]).

By the Sobolev inequality found in [14], we let $C_{a,b}$ be the best Sobolev embedding constant defined as

$$C_{a,b} := \inf_{u,v \in S^p_{\alpha,\beta,\alpha_1,\beta_1}(\mathbb{R}^N)} \frac{\int\limits_{\mathbb{R}^N} (|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1} u|^2 + |\nabla^{\alpha,\beta}_{\alpha_1,\beta_1} v|^2)\, \mathrm{d}X}{\Big(\int\limits_{\mathbb{R}^N} |u|^a |v|^b \,\mathrm{d}X\Big)^{\frac{2}{2^*}}}.$$

2.2 Mountain Pass Theorem

Definition 2.2. Let $\mathbb{B}$ be a real Banach space with its dual space $\mathbb{B}^*$ and $J \in C^1(\mathbb{B}, \mathbb{R})$. For $c \in \mathbb{R}$, we say that J satisfies the $(PS)_c$ condition if for any sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{B}$ with

$$J(x_n) = c + o(1) \ \text{ and } \ J'(x_n) = o(1) \ \text{ strongly in } \ \mathbb{B}^*, \ \ o(1) \to 0 \ \text{ as } \ n \to \infty,$$

there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ that converges strongly in $\mathbb{B}$. If J satisfies the $(PS)_c$ condition for all $c > 0$, then we say that J satisfies the Palais–Smale condition.

We will use the following version of the Mountain Pass Theorem.

Lemma 2.2 (see [18]). *Let $\mathbb{B}$ be a real Banach space and let $J \in C^1(\mathbb{B}, \mathbb{R})$ satisfy the $(PS)_c$ condition for any $c \in \mathbb{R}$, $J(0) = 0$ and*

(i) *there exist the constants $\rho, \alpha > 0$ such that $J(u) \geq \alpha$, $\forall\, u \in \mathbb{B}$, $\|u\|_{\mathbb{B}} = \rho$;*

(ii) *there exists $u_1 \in \mathbb{B}$, $\|u_1\|_{\mathbb{B}} \geq \rho$ such that $J(u_1) \leq 0$.*

Then $\beta := \inf\limits_{\lambda \in \Lambda} \max\limits_{0 \leq t \leq 1} J(\lambda(t)) \geq \alpha$ is a critical value of J, where $\Lambda := \{\lambda \in C([0;1], \mathbb{B}) \colon \lambda(0) = 0,\ \lambda(1) = u_1\}$.

3 Proof of Theorem 1.1

We prove Theorem 1.1 by verifying that all conditions of Lemma 2.2 are satisfied. First, we check the Palais–Smale condition in the following lemma.

Lemma 3.1. *Assume (A1)–(A3) hold and the sequence $\{(u_n, v_n)\}_{n=1}^\infty \subset \mathbb{H}$ is a $(PS)_c$ sequence for Φ. Then we have $c \geq 0$, $\{(u_n, v_n)\}_{n=1}^\infty$ is bounded in the space $\mathbb{H}$ and there exists a subsequence $\{(u_{n_j}, v_{n_j})\}_{j=1}^\infty$ such that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for any $r \geq r_\varepsilon$,*

$$\limsup_{j \to \infty} \int\limits_{B_j \setminus B_r} \big(|u_{n_j}|^q + |v_{n_j}|^q\big)\, \mathrm{d}X \leq \varepsilon,$$

where $2 \leq q < \widetilde{2}^*$.

Proof. Let $\{(u_n, v_n)\}_{n=1}^\infty \subset \mathbb{H}$ be a $(PS)_c$ sequence:

$$\Phi(u_n, v_n) \to c \ \text{ and } \ \Phi'(u_n, v_n) \to 0 \ \text{ in } \ \mathbb{H}. \tag{3.1}$$

From (A3), we obtain

$$\begin{aligned}
&\Phi(u_n, v_n) - \frac{1}{p} \Phi'(u_n, v_n)(u_n, v_n) \\
&\quad = \frac{1}{2} \|(u_n, v_n)\|^2_{\mathbb{H}} - \frac{\lambda}{p} \int\limits_{\mathbb{R}^N} \big(f(X)|u_n|^p + g(X)|v_n|^p\big)\, \mathrm{d}X - \frac{\lambda}{a+b} \int\limits_{\mathbb{R}^N} K(X)|u_n|^a |v_n|^b \,\mathrm{d}X \\
&\qquad - \frac{1}{p} \Bigg[\|(u_n, v_n)\|^2_{\mathbb{H}} - \lambda \int\limits_{\mathbb{R}^N} \big(f(X)|u_n|^p + g(X)|v_n|^p\big)\, \mathrm{d}X - \lambda \int\limits_{\mathbb{R}^N} K(X)|u_n|^a |v_n|^b \,\mathrm{d}X \Bigg]
\end{aligned}$$

$$= \Big(\frac{1}{2} - \frac{1}{p}\Big)\|(u_n, v_n)\|_{\mathbb{H}}^2 + \Big(\frac{1}{p} - \frac{1}{a+b}\Big)\lambda \int\limits_{\mathbb{R}^N} K(X)|u_n|^a |v_n|^b \,\mathrm{d}X. \quad (3.2)$$

By $2 < p < \widetilde{2}^*$, we have

$$\Phi(u_n, v_n) - \frac{1}{p}\,\Phi'(u_n, v_n)(u_n, v_n) \geq \Big(\frac{1}{2} - \frac{1}{p}\Big)\|(u_n, v_n)\|_{\mathbb{H}}^2.$$

Due to (3.1), the sequence $\{(u_n, v_n)\}_{n=1}^{\infty}$ is bounded in $\mathbb{H}$. Taking the limit in (3.2) shows that $c \geq 0$. In view of the above result, without loss of generality, we can suppose that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } \mathbb{H} \text{ as } n \to \infty,$$
$$u_n \to u, \quad v_n \to v \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty,$$
$$(u_n, v_n) \to (u, v) \text{ in } L^q_{loc}(\mathbb{R}^N) \times L^q_{loc}(\mathbb{R}^N) \text{ as } n \to \infty, \quad 2 \leq q < \widetilde{2}^*.$$

For each $j \in \mathbb{N}$, we have

$$\int\limits_{B_j} \big(|u_n|^q + |v_n|^q\big)\,\mathrm{d}X \longrightarrow \int\limits_{B_j} \big(|u|^q + |v|^q\big)\,\mathrm{d}X.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$\int\limits_{B_j} \big(|u_n|^q + |v_n|^q - |u|^q - |v|^q\big)\,\mathrm{d}X < \frac{1}{j}$$

for all $n \geq n_0 + 1$. Without loss of generality, we choose $n_j = n_0 + j$ such that

$$\int\limits_{B_j} \big(|u_{n_j}|^q + |v_{n_j}|^q - |u|^q - |v|^q\big)\,\mathrm{d}X < \frac{1}{j}\,.$$

It is easy to show that there is r_ε satisfying

$$\int\limits_{\mathbb{R}^N \setminus B_r} \big(|u|^q + |v|^q\big)\,\mathrm{d}X < \varepsilon \text{ for all } r \geq r_\varepsilon.$$

Since

$$\int\limits_{B_j \setminus B_r} \big(|u_{n_j}|^q + |v_{n_j}|^q\big)\,\mathrm{d}X < \frac{1}{j} + \int\limits_{\mathbb{R}^N \setminus B_r} \big(|u|^q + |v|^q\big)\,\mathrm{d}X + \int\limits_{B_r} \big(|u|^q - |u_{n_j}|^q + |v|^q - |v_{n_j}|^q\big)\,\mathrm{d}X,$$

in connection with $(u_n, v_n) \to (u, v)$ in $L^q_{loc}(\mathbb{R}^N) \times L^q_{loc}(\mathbb{R}^N)$, the lemma follows. □

Let $\chi : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\chi(\xi) \equiv 1$ for $\xi \leq 1$, $\chi(\xi) \equiv 0$ for $\xi \geq 2$. Define

$$\widetilde{u}_j(X) = \chi\Big(\frac{2|X|}{j}\Big)u(X) \text{ and } \widetilde{v}_j(X) = \chi\Big(\frac{2|X|}{j}\Big)v(X).$$

Clearly,

$$(\widetilde{u}_j, \widetilde{v}_j) \to (u, v) \text{ in } \mathbb{H} \text{ as } j \to \infty. \quad (3.3)$$

Lemma 3.2. *We have*

$$\lim_{j\to\infty} \Big| \int\limits_{\mathbb{R}^N} f(x)\Big(|u_{n_j}|^{p-2}u_{n_j} - |u_{n_j} - \widetilde{u}_j|^{p-2}(u_{n_j} - \widetilde{u}_j) - |\widetilde{u}_j|^{p-2}\widetilde{u}_j\Big)\varphi\,\mathrm{d}X \Big| = 0,$$
$$\lim_{j\to\infty} \Big| \int\limits_{\mathbb{R}^N} g(x)\Big(|v_{n_j}|^{p-2}v_{n_j} - |v_{n_j} - \widetilde{v}_j|^{p-2}(v_{n_j} - \widetilde{v}_j) - |\widetilde{v}_j|^{p-2}\widetilde{v}_j\Big)\psi\,\mathrm{d}X \Big| = 0$$

uniformly in $(\varphi, \psi) \in \mathbb{H}$ *with* $\|(\varphi, \psi)\|_{\mathbb{H}} \leq 1$.

Proof. The proof is similar to that of [10, Lemma 3.4], so we omit it. □

Lemma 3.3. *One has along a subsequence*

$$\Phi(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n) \to c - \Phi(u,v) \ \ as \ \ n \to \infty,$$
$$\Phi'(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n) \to 0 \ \ in \ \ \mathbb{H}^* \ \ as \ \ n \to \infty.$$

Proof. From $(u_n, v_n) \rightharpoonup (u,v)$ and $(\widetilde{u}_n, \widetilde{v}_n) \to (u,v)$ in $\mathbb{H}$ as $n \to \infty$, we have

$$\begin{aligned}\Phi(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n) &= \Phi(u_n, v_n) - \Phi(\widetilde{u}_n, \widetilde{v}_n) \\ &\quad + \frac{\lambda}{\widetilde{2}^*}\int_{\mathbb{R}^N} K(X)\Big(|u_n|^a|v_n|^b - |u_n-\widetilde{u}_n|^a|v_n-\widetilde{v}_n|^b - |\widetilde{u}_n|^a|\widetilde{v}_n|^b\Big)\,\mathrm{d}X \\ &\quad + \frac{\lambda}{p}\int_{\mathbb{R}^N} f(X)\big(|u_n|^p - |u_n-\widetilde{u}_n|^p - |\widetilde{u}_n|^p\big)\,\mathrm{d}X \\ &\quad + \frac{\lambda}{p}\int_{\mathbb{R}^N} g(X)\big(|v_n|^p - |v_n-\widetilde{v}_n|^p - |\widetilde{v}_n|^p\big)\,\mathrm{d}X + o(1).\end{aligned}$$

Using (3.3) and following the proof of the Brézis–Lieb lemma (see, e.g., [6]), it is not difficult to check that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(X)\Big(|u_n|^a|v_n|^b - |u_n-\widetilde{u}_n|^a|v_n-\widetilde{v}_n|^b - |\widetilde{u}_n|^a|\widetilde{v}_n|^b\Big)\,\mathrm{d}X = 0,$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(X)\big(|u_n|^p - |u_n-\widetilde{u}_n|^p - |\widetilde{u}_n|^p\big)\,\mathrm{d}X = 0,$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N} g(X)\big(|v_n|^p - |v_n-\widetilde{v}_n|^p - |\widetilde{v}_n|^p\big)\,\mathrm{d}X = 0.$$

On the other hand, we get

$$\Phi(u_n, v_n) \to c \ \text{ and } \ \Phi(\widetilde{u}_n, \widetilde{v}_n) \to \Phi(u,v) \ \text{ as } \ n \to \infty,$$

hence

$$\Phi(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n) \to c - \Phi(u,v) \ \text{ as } \ n \to \infty.$$

In addition, for any $(\varphi, \psi) \in \mathbb{H}$, we obtain

$$\begin{aligned}\Phi'(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n)(\varphi,\psi) &= \Phi'(u_n, v_n)(\varphi,\psi) - \Phi'(\widetilde{u}_n, \widetilde{v}_n)(\varphi,\psi) \\ &+ \frac{\lambda a}{\widetilde{2}^*}\int_{\mathbb{R}^N} K(x)\Big(|u_n|^{a-2}u_n|v_n|^b - |u_n-\widetilde{u}_n|^{a-2}(u_n-\widetilde{u}_n)|v_n-\widetilde{v}_n|^b - |\widetilde{u}_n|^{a-2}\widetilde{u}_n|\widetilde{v}_n|^b\Big)\varphi\,\mathrm{d}X \\ &+ \frac{\lambda b}{\widetilde{2}^*}\int_{\mathbb{R}^N} K(x)\Big(|u_n|^a|v_n|^{b-2}v_n - |u_n-\widetilde{u}_n|^a|v_n-\widetilde{v}_n|^{b-2}(v_n-\widetilde{v}_n) - |\widetilde{u}_n|^a|\widetilde{v}_n|^{b-2}\widetilde{v}_n\Big)\psi\,\mathrm{d}X \\ &+ \lambda\int_{\mathbb{R}^N} f(x)\Big(|u_n|^{p-2}u_n - |u_n-\widetilde{u}_n|^{p-2}(u_n-\widetilde{u}_n) - |\widetilde{u}_n|^{p-2}\widetilde{u}_n\Big)\varphi\,\mathrm{d}X \\ &+ \lambda\int_{\mathbb{R}^N} g(x)\Big(|v_n|^{p-2}v_n - |v_n-\widetilde{v}_n|^{p-2}(v_n-\widetilde{v}_n) - |\widetilde{v}_n|^{p-2}\widetilde{v}_n\Big)\psi\,\mathrm{d}X.\end{aligned}$$

It follows again from the standard argument that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\Big(|u_n|^{a-2}u_n|v_n|^b - |u_n-\widetilde{u}_n|^{a-2}(u_n-\widetilde{u}_n)|v_n-\widetilde{v}_n|^b - |\widetilde{u}_n|^{a-2}\widetilde{u}_n|\widetilde{v}_n|^b\Big)\varphi\,\mathrm{d}X = 0,$$

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} K(x)\Big(|u_n|^a|v_n|^{b-2}v_n - |u_n-\widetilde{u}_n|^a|v_n-\widetilde{v}_n|^{b-2}(v_n-\widetilde{v}_n) - |\widetilde{u}_n|^a|\widetilde{v}_n|^{b-2}\widetilde{v}_n\Big)\psi\,\mathrm{d}X = 0$$

uniformly in $\|(\varphi,\psi)\|_E \le 1$. By Lemma 3.2 and $\Phi'(u_n,v_n)\to 0$ as $n\to\infty$, we complete the proof. □

Put

$$\psi_n := u_n - \widetilde{u}_n \text{ and } \nu_n := v_n - \widetilde{v}_n.$$

Hence

$$u_n - u = \psi_n + (\widetilde{u}_n - u) \text{ and } v_n - v = \nu_n + (\widetilde{v}_n - v).$$

Then $(u_n,v_n)\to(u,v)$ in $\mathbb{H}$ as $n\to\infty$ if and only if $(\psi_n,\nu_n)\to(0,0)$ in $\mathbb{H}$ as $n\to\infty$. We obtain

$$\Phi(\psi_n,\nu_n) - \frac{1}{2}\,\Phi'(\psi_n,\nu_n)(\psi_n,\nu_n) = \Big(\frac{1}{2}-\frac{1}{a+b}\Big)\lambda\int\limits_{\mathbb{R}^N} K(X)|\psi_n|^a|\nu_n|^b\,\mathrm{d}X$$
$$+\Big(\frac{1}{2}-\frac{1}{p}\Big)\lambda\int\limits_{\mathbb{R}^N}\big(f(X)|\psi_n|^p + g(X)|\nu_n|^p\big)\,\mathrm{d}X \ge \frac{\lambda}{\widetilde{N}}K_0\int\limits_{\mathbb{R}^N}|\psi_n|^a|\nu_n|^b\,\mathrm{d}X, \quad (3.4)$$

where $K_0 = \inf\limits_{x\in\mathbb{R}^N} K(X) > 0$. From Lemma 3.3 and (3.4), it follows that

$$\int\limits_{\mathbb{R}^N}|\psi_n|^a|\nu_n|^b\,\mathrm{d}X \le \frac{\widetilde{N}(c-\Phi(u,v))}{\lambda K_0} + o(1). \quad (3.5)$$

From (A2) and (A3), for any $M>0$, there is a constant $C_M>0$ such that

$$\int\limits_{\mathbb{R}^N}\Big(K(X)|\psi_n|^a|\nu_n|^b + f(X)|\psi_n|^p + g(X)|\nu_n|^p\Big)\,\mathrm{d}X$$
$$\le M\big(\|\psi_n\|^2_{L^2(\mathbb{R}^N)} + \|\nu_n\|^2_{L^2(\mathbb{R}^N)}\big) + C_M\int\limits_{\mathbb{R}^N}|\psi_n|^a|\nu_n|^b\,\mathrm{d}X.$$

Let $V_M(X) := \max\{V(X),M\}$, where M is the positive constant in the assumption (A1). Since $\mathrm{Vol}(\{X\in\mathbb{R}^N, V(X)\le M\})<\infty$ and $(\psi_n,\nu_n)\to(0,0)$ in $L^2_{loc}(\mathbb{R}^N)\times L^2_{loc}(\mathbb{R}^N)$, we obtain

$$\int\limits_{\mathbb{R}^N} V(X)\big(|\psi_n|^2+|\nu_n|^2\big)\mathrm{d}X = \int\limits_{\mathbb{R}^N} V_M(X)\big(|\psi_n|^2+|\nu_n|^2\big)\,\mathrm{d}X + o(1). \quad (3.6)$$

Lemma 3.4. *Under the assumptions of Lemma 3.1, there is a constant $C_0>0$ independent of λ such that for any $(PS)_c$-sequence $\{(u_n,v_n)\}_{n=1}^{\infty}$, for Φ with $(u_n,v_n)\rightharpoonup(u,v)$, either*

$$(u_n,v_n)\to(u,v) \text{ in } \mathbb{H} \text{ as } n\to\infty \text{ or } c-\Phi(u,v)\ge C_0\lambda^{1-\frac{\widetilde{N}}{2}}.$$

Proof. Assume

$$(u_n,v_n)\nrightarrow(u,v) \text{ in } \mathbb{H} \text{ as } n\to\infty.$$

Then

$$\liminf_{n\to\infty}\|(\psi_n,\nu_n)\|_{\mathbb{H}} > 0 \text{ and } c-\Phi(u,v)>0.$$

By Lemma 2.1 and (3.6), we have

$$\begin{aligned}
C_{a,b}\Big(\int_{\mathbb{R}^N}|\psi_n|^a|\nu_n|^b\,\mathrm{d}X\Big)^{\frac{2}{a+b}} &\le \int_{\mathbb{R}^N}\big(|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}\psi_n|^2+|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}\nu_n|^2\big)\,\mathrm{d}X\\
&= \int_{\mathbb{R}^N}\Big(|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}\psi_n|^2+|\nabla^{\alpha,\beta}_{\alpha_1,\beta_1}\nu_n|^2+\lambda V(X)|\psi_n|^2+\lambda V(X)|\nu_n|^2\Big)\,\mathrm{d}X\\
&\qquad -\int_{\mathbb{R}^N}\lambda V(X)\big(|\psi_n|^2+|\nu_n|^2\big)\,\mathrm{d}X\\
&= \lambda\int_{\mathbb{R}^N}\Big(K(X)|\psi_n|^a|\nu_n|^b+f(X)|\psi_n|^p+g(X)|\nu_n|^p\Big)\,\mathrm{d}X\\
&\qquad -\lambda\int_{\mathbb{R}^N}V_M(X)\big(|\psi_n|^2+|\nu_n|^2\big)\,\mathrm{d}X+o(1)\\
&\le \lambda C_M\int_{\mathbb{R}^N}|\psi_n|^a|\nu_n|^b\,\mathrm{d}X+o(1).
\end{aligned}$$

From (3.5), we get

$$\begin{aligned}
C_{a,b} &\le \lambda C_M\Big(\int_{\mathbb{R}^N}|\psi_n|^a|\nu_n|^b\Big)^{1-\frac{2}{a+b}}\,\mathrm{d}X+o(1)\\
&\le \lambda C_M\Big(\frac{\widetilde{N}(c-\Phi(u,v))}{\lambda K_0}\Big)^{\frac{2}{\widetilde{N}}}+o(1)=\lambda^{1-\frac{2}{\widetilde{N}}}C_M\Big(\frac{\widetilde{N}}{K_0}\Big)^{\frac{2}{\widetilde{N}}}(c-\Phi(u,v))^{\frac{2}{\widetilde{N}}}+o(1).
\end{aligned}$$

Set $C_0:=C_{a,b}^{\frac{\widetilde{N}}{2}}C_M^{-\frac{\widetilde{N}}{2}}\widetilde{N}^{-1}K_0$. This implies

$$C_0\lambda^{1-\frac{\widetilde{N}}{2}}\le c-\Phi(u,v)+o(1).$$

The proof is complete. □

In particular, we obtain the following

Lemma 3.5. *Let* (A1)–(A3) *be satisfied. Then* $\Phi(u,v)$ *satisfies the* $(PS)_c$ *condition for all* $c<C_0\lambda^{1-\frac{\widetilde{N}}{2}}$.

Lemma 3.6. *Assume that* (A1)–(A3) *are satisfied and* $\lambda\ge 1$. *Then there exist* $\eta_\lambda>0$ *and* $\kappa_\lambda>0$ *such that*

$$\Phi(u,v)>0 \ \text{ if } \ 0<\|(u,v)\|_{\mathbb{H}}<\kappa_\lambda \ \text{ and } \ \Phi(u,v)\ge\eta_\lambda \ \text{ if } \ \|(u,v)\|_{\mathbb{H}}=\kappa_\lambda.$$

Proof. From Lemma 2.1, for each $p\in[2,\widetilde{2}^*]$, we have that there is C_p such that if $\lambda\ge 1$, then

$$\|u\|_{L^p(\mathbb{R}^N)}\le C_p\|u\|_{S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N)} \ \text{ for all } \ u\in S^2_{\alpha,\beta,\alpha_1,\beta_1,\lambda V(X)}(\mathbb{R}^N).$$

By the Young inequality, we have

$$|u|^a|v|^b\le\frac{a}{a+b}\,|u|^{a+b}+\frac{b}{a+b}\,|v|^{a+b}.$$

Furthermore, we obtain

$$\int_{\mathbb{R}^N}K(X)|u|^a|v|^b\,\mathrm{d}X\le C_1\big(\|u\|^{\widetilde{2}^*}_{L^{\widetilde{2}^*}(\mathbb{R}^N)}+\|v\|^{\widetilde{2}^*}_{L^{\widetilde{2}^*}(\mathbb{R}^N)}\big)\le C_1C_{\widetilde{2}^*}\|(u,v)\|^{\widetilde{2}^*}_{\mathbb{H}}. \tag{3.7}$$

Combining (A3) and (3.7), there is a constant C_δ such that

$$\Phi(u,v) \geq \frac{1}{4}\,\|(u,v)\|_{\mathbb{H}}^2 - C_\delta\|(u,v)\|_{\mathbb{H}}^{\widetilde{2}^*} = \frac{1}{4}\,\|(u,v)\|_{\mathbb{H}}^2\big(1 - 4C_\delta\|(u,v)\|_{\mathbb{H}}^{\widetilde{2}^*-2}\big).$$

Set $\kappa_\lambda = (\frac{1}{8C_\delta})^{\frac{1}{\widetilde{2}^*-2}}$, this implies that

$$\Phi(u,v) \geq \frac{1}{8}\,\kappa_\lambda^2 =: \eta_\lambda > 0 \ \text{ if } \ \|(u,v)\|_{\mathbb{H}} = \kappa_\lambda.$$

The proof is complete. □

Lemma 3.7. *Assume that* (A1)–(A3) *are satisfied. Then for any finite-dimensional subspace* $\mathbb{E} \subset \mathbb{H}$*, we have*

$$\Phi(u,v) \to -\infty \ \text{ as } \ \|(u,v)\|_{\mathbb{H}} \to \infty \ \text{ for } \ (u,v) \in \mathbb{E}.$$

Proof. From assumptions (A2) and (A3), it follows that

$$\Phi(u,v) \leq \frac{1}{2}\,\|(u,v)\|_{\mathbb{H}}^2 - \lambda\widetilde{C}_0\|(u,v)\|_{L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)}^p \ \text{ for all } \ (u,v) \in \mathbb{E},$$

where $\widetilde{C}_0 := \frac{\min\{f_0,g_0\}}{p}$. Since all norms in a finite-dimensional space are equivalent and $p > 2$, it is easy to obtain the desired conclusion. □

Lemma 3.8. *Assume that* (A1)–(A3) *are satisfied. Then for any* $\sigma > 0$*, there is* $\Lambda_\sigma > 0$ *such that for each* $\lambda \geq \Lambda_\sigma$*, there exists* $\overline{e}_\lambda \in \mathbb{H}$ *with* $\|\overline{e}_\lambda\|_{\mathbb{H}} > \kappa_\lambda$ *such that* $\Phi(\overline{e}_\lambda) \leq 0$ *and*

$$\max_{t\geq 0} \Phi(t\overline{e}_\lambda) \leq \sigma\lambda^{1-\frac{\widetilde{N}}{2}},$$

where κ_λ *is defined in Lemma 3.6.*

Proof. Define the functionals

$$I(u,v) = \frac{1}{2}\int\limits_{\mathbb{R}^N} \Big(|\nabla_{\alpha_1,\beta_1}^{\alpha,\beta} u|^2 + \lambda V(X)|u|^2 + |\nabla_{\alpha_1,\beta_1}^{\alpha,\beta} v|^2 + \lambda V(X)|v|^2\Big)\,\mathrm{d}X - \lambda\widetilde{C}_0\int\limits_{\mathbb{R}^N} \big(|u|^p + |v|^p\big)\,\mathrm{d}X,$$

$$J(u,v) = \frac{1}{2}\int\limits_{\mathbb{R}^N} \Big(|\nabla_{\alpha_1,\beta_1}^{\alpha,\beta} u|^2 + |\nabla_{\alpha_1,\beta_1}^{\alpha,\beta} v|^2 + V(\lambda^{-\frac{1}{2}}X)\big(|u|^2 + |v|^2\big)\Big)\,\mathrm{d}X - \widetilde{C}_0\int\limits_{\mathbb{R}^N} \big(|u|^p + |v|^p\big)\,\mathrm{d}X.$$

We obtain that $I \in C^1(\mathbb{H},\mathbb{R})$ and $\Phi(u,v) \leq I(u,v)$ for all $(u,v) \in \mathbb{H}$. Observe that

$$\inf\Big\{\int\limits_{\mathbb{R}^N} |\nabla_{\alpha_1,\beta_1}^{\alpha,\beta}\phi|^2\,\mathrm{d}X : \ \ \phi \in C_0^\infty(\mathbb{R}^N,\mathbb{R}), \ \ \|\phi\|_{L^p(\mathbb{R}^N)} = 1\Big\} = 0.$$

For any $\delta > 0$, there are $\phi_\delta, \psi_\delta \in C_0^\infty(\mathbb{R}^N,\mathbb{R})$ with $\|\phi_\delta\|_{L^p(\mathbb{R}^N)} = \|\psi_\delta\|_{L^p(\mathbb{R}^N)} = 1$ such that

$$\operatorname{supp}(\phi_\delta,\psi_\delta) \subset B_{r_\delta}(0) \ \text{ and } \ \|\nabla_{\alpha_1,\beta_1}^{\alpha,\beta}\phi_\delta\|_{L^2(\mathbb{R}^N)}^2 < \delta, \ \ \|\nabla_{\alpha_1,\beta_1}^{\alpha,\beta}\psi_\delta\|_{L^2(\mathbb{R}^N)}^2 < \delta.$$

Let $e_\lambda(X) = (\phi_\delta(\sqrt{\lambda}X), \psi_\delta(\sqrt{\lambda}X))$, then $\operatorname{supp} e_\lambda \subset B_{\lambda^{-\frac{1}{2}}r_\delta}(0)$. Furthermore,

$$I(te_\lambda) = \lambda^{1-\frac{\widetilde{N}}{2}} J(t\phi_\delta, t\psi_\delta).$$

It is clear that

$$\max_{t\geq 0} J(t\phi_\delta, t\psi_\delta) \leq \frac{p-2}{2p(p\widetilde{C}_0)^{\frac{2}{p-2}}}\Big\{\int\limits_{\mathbb{R}^N} \big(|\nabla_{\alpha_1,\beta_1}^{\alpha,\beta}\phi_\delta|^2 + V(\lambda^{-\frac{1}{2}}X)|\phi_\delta|^2\big)\,\mathrm{d}X\Big\}^{\frac{p}{p-2}}$$
$$+ \frac{p-2}{2p(p\widetilde{C}_0)^{\frac{2}{p-2}}}\Big\{\int\limits_{\mathbb{R}^N} \big(|\nabla_{\alpha_1,\beta_1}^{\alpha,\beta}\psi_\delta|^2 + V(\lambda^{-\frac{1}{2}}X)|\psi_\delta|^2\big)\,\mathrm{d}X\Big\}^{\frac{p}{p-2}}.$$

Combining $V(0) = 0$ and $\mathrm{supp}(\phi_\delta, \psi_\delta) \subset B_{r_\delta}(0)$, there is $\Lambda_\delta > 0$ such that for all $\lambda \geq \Lambda_\delta$, we have

$$\max_{t\geq 0} I(t\phi_\delta, t\psi_\delta) \leq \lambda^{1-\frac{N}{2}} \frac{(p-2)}{p(p\widetilde{C}_0)^{\frac{2}{p-2}}} (2\delta)^{\frac{p}{p-2}}.$$

Thus, for all $\lambda \geq \Lambda_\delta$,

$$\max_{t\geq 0} \Phi(te_\lambda) \leq \lambda^{1-\frac{\widetilde{N}}{2}} \frac{(p-2)}{p(p\widetilde{C}_0)^{\frac{2}{p-2}}} (2\delta)^{\frac{p}{p-2}}. \tag{3.8}$$

For any $\sigma > 0$, we can choose $\delta > 0$ small enough such that

$$\frac{(p-2)}{p(p\widetilde{C}_0)^{\frac{2}{p-2}}} (2\delta)^{\frac{p}{p-2}} \leq \sigma$$

and $e_\lambda(X) = (\phi_\delta(\sqrt{\lambda}x), \psi_\delta(\sqrt{\lambda}X))$. Taking $\Lambda_\delta = \Lambda_\sigma$, there is $\bar{t}_\lambda > 0$ such that $\|\bar{t}_\lambda e_\lambda\|_{\mathbb{H}} > \kappa_\lambda$ and $\Phi(te_\lambda) \leq 0$ for all $t \geq \bar{t}_\lambda$. By (3.8), $\bar{e}_\lambda = \bar{t}_\lambda e_\lambda$ satisfies the requirements. □

Proof of Theorem 1.1. Define

$$c_\lambda := \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} \Phi(\gamma(t)),$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], \mathbb{H}) : \ \gamma(0) = 0, \ \gamma(1) = \bar{e}_\lambda\}$. In addition, for any $\sigma > 0$ with $\sigma < \widetilde{C}_0$, there is $\Lambda_\sigma > 0$ such that $\lambda \geq \Lambda_\sigma$. We can take c_λ satisfying $c_\lambda \leq \sigma\lambda^{1-\frac{\widetilde{N}}{2}}$.

From the above results, the functional Φ satisfies the $(PS)_{c_\lambda}$ condition and Lemma 2.2 if $c_\lambda \leq \sigma\lambda^{1-\frac{\widetilde{N}}{2}}$. Hence, there is $(u_\lambda, v_\lambda) \in \mathbb{H}$ such that

$$\Phi(u_\lambda, v_\lambda) = c_\lambda \ \text{ and } \ \Phi'(u_\lambda, v_\lambda) = 0.$$

Therefore, (u_λ, v_λ) is a weak solution of problem (1.2). Similar to the arguments in [10], we also obtain that (u_λ, v_λ) is a positive least energy solution. Furthermore,

$$\Phi(u_\lambda, v_\lambda) = \Phi(u_\lambda, v_\lambda) - \frac{1}{p}\Phi'(u_\lambda, v_\lambda)(u_\lambda, v_\lambda) \geq \Big(\frac{1}{2} - \frac{1}{p}\Big)\|(u_\lambda, v_\lambda)\|^2_{\mathbb{H}}.$$

Hence

$$\frac{p-2}{2p}\|(u_\lambda, v_\lambda)\|^2_{\mathbb{H}} \leq \Phi(u_\lambda, v_\lambda) = c_\lambda \leq \sigma\lambda^{1-\frac{\widetilde{N}}{2}}.$$

The proof is complete. □

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